XVI. On the aberration of the concave grating, when used as an objective spectroscope

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by J. J. Thomson* and by me†. I have shown that the normal cathode drop of potential in the glowing discharge (glow current) is equal to the ionizing potential of the positive ion relative to the cathode metal. Further, J. A. McClelland‡ has already made an observation which is similar to the phenomenon studied by Townsend, but more simple. There are produced by heating positive and negative ions at the surface of a wire which is placed in the axis of a metal cylinder; the positive or negative ions are driven by an electric field through the rarefied gas to the inner side of the cylinder; if the freely passed potential-difference is large enough, they are able to ionize by collision the gas at the inner surface of the cylinder. I have shown (*Ann. d. Phys.* viii. p. 826, 1902) that in a set of measurements of McClelland it is the positive ion which produces new ions by collision.

I remain, Gentlemen,

Yours very truly,

Göttingen, 17 April, 1903.

J. STARK.

XVI. *On the Aberration of the Concave Grating, when used as an Objective Spectroscope.* By F. L. O. WADSWORTH§.

THE question of the aberration of concave gratings was first investigated by Rowland|| and Glazebrook¶. The results obtained by them refer only to the case in which the ruled surface is spherical in form, and is so mounted that both it and the source of light lie on the circumference of a circle whose diameter is ρ , the radius of curvature of the spherical surface. When so ruled and mounted the aberration is symmetrical in form, and is so small that for gratings of the angular aperture usually employed we may use resolving-powers as high as 4,000,000 before the definition begins to be affected. Since the resolving-powers ordinarily employed in the largest concave-grating spectroscopes are only from one to five per cent. of this limiting value, the effect of aberration can be safely neglected, even when the angular aperture of the grating is increased from two to three times.

* J. J. Thomson, *Phil. Mag.* (5) l. p. 282 (1900).

† J. Stark, *Ann. d. Phys.* pp. 431, 435, 923 (1902), viii. p. 826 (1902).

‡ J. A. McClelland, *Proc. Cambr. Phil. Soc.* xi. p. 296 (1901).

§ Communicated by the Author.

|| *Phil. Mag.* [5] vol. xvi. p. 197.

¶ *Ibid.* vol. xv. p. 414; vol. xvi. p. 337. See also article on the "Wave Theory," by Lord Rayleigh, *Enc. Brit.* vol. xxiv. p. 439.

The same conclusion, however, does not hold when the method of mounting the grating is changed. In 1895 the writer suggested and investigated several new methods of using the concave grating as a direct objective spectroscope, with a parallel incident beam of light. At that time a number of mountings were designed and constructed for the purpose of testing the proposed new methods, and the results obtained were very satisfactory*. The grating used in these tests was, however, a comparatively small one of only about 3 cm. aperture, and 152 cm. radius of curvature. Used in the ordinary way (Rowland mounting) its semi-angular aperture measured at the centre of curvature would therefore be about $\beta = 0.01$. The effect of aberration on gratings of larger angular aperture than this was computed, and it was concluded at that time† that semi-angular apertures as large as $\beta = 0.05$ could be used if the resolving-power did not exceed 30,000 units. Since this paper was written the concave grating has been used in the way described by Poor and Mitchell at Johns Hopkins, and later by Mitchell at Yerkes Observatory, in the study of stellar spectra; and by Frost, Mohler and Daniel, Jewell and Humphreys, and the writer, in photographing the flash spectrum during a solar eclipse. The gratings used by Poor and Mitchell‡ were 100 cm. radius of curvature and about 5 cm. and 14.6 cm. aperture, ruled respectively with 5905 and 2842 lines per cm. The resolving-powers in the first order spectra were therefore about 30,000 and 41,500, and the semi-angular apertures about $\beta = 0.025$ and $\beta = 0.073$ respectively. The results obtained on stellar spectra were considered very good.

The results obtained by eclipse observers have all been much less satisfactory. Frost§ used a concave grating of 4.4 cm. aperture and 150 cm. radius of curvature, ruled with 5684 lines per cm. The resolving-power in the first spectrum was therefore about 25,000, and the semi-angular aperture about $\beta = 0.015$. The field obtained was small, but this is ascribed by Prof. Frost to be in part due to the fact that flat plates were used instead of curved plates. The grating used by Mohler and Daniel|| had an aperture of 10 cm., a radius of curvature of 305 cm. and a ruling of 5684 lines per cm. In the first order the resolving-power was about 57,000 and the semi-angular aperture, β , was about 0.016. The definition

* Astrophysical Journal, vol. iii. pp. 54-60, Jan. 1896.

† Ibid. pp. 60-61.

‡ Ibid. vii. p. 167, March 1898; and vol. x. p. 29, June 1899.

§ Ibid. xii. p. 85; p. 307, December 1900.

|| Ibid. vol. xii. p. 361.

was unsatisfactory. The authors of the paper suggest that this might have resulted from any one or more of the following causes:—(1) an error of setting the plane of dispersion in the correct position-angle, (2) errors of following, (3) errors of focussing, (4) astigmatism. The grating used by Jewell and Humphreys* at the Sumatra eclipse was still larger than that used by Mohler and Daniel, having a ruled surface of 20 cm. aperture and a radius of curvature of 915 cm. The ruling was defective on one side, and in actual use only 0.6 of the whole aperture was employed, so that the effective semi-angular aperture β was about 0.0065. It is stated that in this case also the definition was unsatisfactory, the cause ascribed being an error of focussing.

The grating used by the writer in his eclipse work at Union Springs in May 1900 was one ruled especially by Prof. Rowland for the Allegheny Observatory. It has an aperture of 13.2 cm., a radius of curvature of 175.3 cm., and an unusually long ruling, 8 cm. The number of lines per cm. is 1421. Hence the resolving-power in the first-order spectrum is about 18,750, and the semi-angular aperture at the centre of curvature is about 0.04. This grating gives good definition at the centre of curvature, but when it was used as an objective spectroscope at Union Springs the definition was greatly impaired, both visually and photographically, so much so that I do not consider the spectrum plates taken with the instrument mounted in the manner there used good enough for accurate measurement of the wave-lengths of the spectral lines. In this case the lack of definition cannot be ascribed to errors of focussing or mal-adjustment of the apparatus, and I have therefore been led to investigate more rigorously than was at first done the effect of aberration of a concave grating at its principal focus. The results of this investigation show very clearly that the field and range of action of the concave grating, when used as an objective spectroscope with parallel incident light, is very much more restricted than when it is mounted in the usual manner (Rowland method); and that in order to avoid injuring the definition in the former case the angles of incidence and diffraction must never exceed certain limiting values, which are rapidly varying functions both of the resolving-power and the angular aperture of the grating.

We will first consider the case in which the ruled surface is spherical, the usual form adopted for concave gratings. For a parallel incident beam of light the optical conditions

* *Ibid.* vol. xvi. p. 313, June 1902.

in the primary plane are shown in fig. 1, in which i = the angle pac = the angle of incidence; θ = angle cao' = angle of diffraction; $u = ao'$ = focal distance from grating to primary

Fig. 1.

image; $\rho = ca$ = radius of curvature of the spherical surface, and β = angle bca = semi-angular aperture of surface at the centre of curvature.

In investigating the aberration at the primary focal point o' we shall follow, with some slight modifications, the method adopted by Lord Rayleigh in his paper "Aberration of Lenses"*, and also followed by Glazebrook in the paper already referred to. Let ah , $b'h'$ be instantaneous sections of the plane wave-front incident at the central point a , and the extreme left-hand edge of the grating aperture. Then the relative retardations of the central with reference to the extreme edge rays at the focal point o' will be

$$\begin{aligned} Z_1 &= u - (bo' + bh) \text{ for the right-hand half,} \\ Z_2 &= u - (b'o' - ah') \text{ ,, left ,,} \end{aligned}$$

From the geometry of the triangles bao' , bah , and bac we find for bo' and bh

$$\begin{aligned} bo' &= \sqrt{u^2 + 4\rho^2 \sin^2 \frac{\beta}{2} - 4\rho u \sin \frac{\beta}{2} \sin \left(\frac{\beta}{2} + \theta \right)} \\ &= \sqrt{(u - \rho \sin \beta \sin \theta)^2 - \rho \sin^2 \beta \cos \theta (u - \rho \cos \theta) + \frac{\rho}{4} \sin^4 \beta (\rho - u \cos \theta)}, \quad (1) \end{aligned}$$

* Phil. Mag. vol. ix. p. 40, Jan. 1880.

and

$$\begin{aligned} bh &= 2\rho \sin \frac{\beta}{2} \sin \left(i - \frac{\beta}{2} \right) \\ &= \rho \sin \beta \sin i - \frac{\rho}{2} \sin^2 \beta \cos i - \frac{\rho}{8} \sin^4 \beta \cos i. \quad (2) \end{aligned}$$

Expanding (1) by the binomial theorem in terms containing successive powers of β , and neglecting as before those powers higher than the fourth, we obtain

$$\left. \begin{aligned} bo' &= u - \rho \sin \beta \sin \theta \\ &\quad - \frac{\rho}{2u} \sin^2 \beta \cos \theta (u - \rho \cos \theta) \\ &\quad - \frac{\rho^2}{2u^2} \sin^3 \beta \sin \theta \cos \theta (u - \rho \cos \theta) \\ &\quad - \frac{\rho^3}{2u^3} \sin^4 \beta \sin^2 \theta \cos \theta (u - \rho \cos \theta) \\ &\quad + \frac{\rho}{8u} \sin^4 \beta (\rho - u \cos \theta) \\ &\quad - \frac{\rho^2}{8u^3} \sin^4 \beta \cos^2 \theta (u - \rho \cos \theta)^2 \\ &\quad + \dots \end{aligned} \right\} \dots \dots \quad (3)$$

Combining this value with that for bh from (2), and subtracting the same from u , we find for Z_1

$$\left. \begin{aligned} Z_1 &= \rho \sin \beta (\sin \theta - \sin i) &= M \\ &+ \frac{\rho}{2} \sin^2 \beta (\cos \theta + \cos i - \frac{\rho}{u} \cos^2 \theta) &= N \\ &+ \frac{\rho^2}{2u^2} \sin^3 \beta [\sin \theta \cos \theta (u - \rho \cos \theta)] &= O \\ &+ \frac{\rho^2}{2u^3} \sin^4 \beta \left(\frac{3}{2} \rho \sin^2 \theta \cos \theta - \frac{1}{2} \rho \cos \theta - \frac{u}{4} \sin^2 \theta \right. \\ &\quad \left. - \frac{\rho^2}{u} \sin^2 \theta \cos^2 \theta + \frac{\rho^2}{4u} \cos^4 \theta \right) &= P \\ &+ \frac{\rho}{8} \sin^4 \beta (\cos \theta + \cos i) &= Q \end{aligned} \right\} \quad (4)$$

The expression for Z_1 can also be derived at once from Glazebrook's paper by putting u in equation (3) equal to ∞ , and substituting for ϕ , ψ , and ω the corresponding values i , θ , and β .

For the left-hand half of the surface we have similarly

$$Z_2 = M + N - O + P + Q, \quad \dots \dots \quad (5)$$

which is the same as the expression for Z_1 , with the exception of the sign of the term O.

From the general theory of the grating* we always have for any given spectral image

$$\rho \sin \beta (\sin \theta - \sin i) = mn\lambda.$$

Also for parallel incident light†

$$u(\cos \theta + \cos i) - \rho \cos^2 \theta = 0. \quad \dots \quad (6)$$

The first two terms of equations (4) and (5) are therefore always satisfied when the spectrum is in focus. The remaining terms express the aberration at the primary focal point o' for any given values of θ , i , and β .

From (6) we have for u

$$u = \frac{\rho \cos^2 \theta}{\cos \theta + \cos i} \cdot \dots \quad (7)$$

Substituting this value of u in the terms O, P, and Q of equations (4) and (5) and reducing, we obtain as the general expression for the primary aberration of a concave grating used with parallel incident light

$$Z_1 = -\frac{\rho}{2} \sin^3 \beta \sin \theta \cos i \frac{(\cos i + \cos \theta)}{\cos^2 \theta} \\ + \frac{\rho}{8} \sin^4 \beta \frac{(\cos i + \cos \theta)}{\cos^2 \theta} [\cos^2 i - (\cos \theta + 2 \cos i)^2 \tan^2 \theta], \quad (8)$$

and for Z_2 the same expression, save for the opposite sign of the first term in $\sin^3 \beta$. The corresponding general expression for the primary aberration of a concave grating mounted in the usual Rowland manner, as deduced by Glazebrook and Rayleigh, is

$$Z_0 = (u + u' = \text{central ray}) - (\text{lateral ray}) \ddagger \\ = -\frac{1}{8} \rho \sin^4 \beta (\sin \theta \tan \theta + \sin i \tan i). \quad (9)$$

Comparing (8) and (9) we see that in general the effect of aberration is not only larger, but is of a more prejudicial character, when the grating is used as an objective spectro-scope than when it is used as originally proposed by Rowland. In the former case the aberration is unsymmetrical on account of the term in $\sin^3 \beta$; in the latter case this term is eliminated

* See, for example, Rowland, *Phil. Mag.* vol. xvi. (1883).

† See *Astrophysical Journal*, vol. iii. p. 55.

‡ In Glazebrook and Rayleigh's papers the aberration is expressed as the difference, lateral ray—central ray, and is therefore of the opposite sign.

by the conditions of the mounting. It will also be noted that the signs of the terms in $\sin^4\beta$ are opposite in the two cases. This is a point of special significance, the importance of which will be developed later.

The general equation (8) reduces to a simpler form in several special cases corresponding to different forms of mounting. These will now be considered.

(A) Let the grating be mounted so that the eyepiece or the centre of the photographic plate lies on the axis of the grating, *i. e.* on the normal to the centre of the ruled surface. This is the form of mounting described and illustrated on pp. 54-57, and on Plates VII. and IX. of my previous paper, and is the one that has been most commonly used in subsequent work. In this case $\theta=0$ at the centre of the field, and we have for the aberration

$$Z_1' = Z_2' = Z_A = + \frac{\rho}{8} \sin^4\beta \cos^2i(1 + \cos i). \quad (10)$$

For the usual Rowland mounting in which this condition is also fulfilled we have similarly from (9)

$$Z_0 = - \frac{\rho}{8} \sin^4\beta \frac{\sin^2i}{\cos i} \dots \dots \dots (11)$$

Equations (10) and (11) show that with this form of objective grating mounting the aberration decreases with increasing angles of incidence, and becomes vanishingly small for values of i near 90° . With the Rowland mounting the reverse is the case.

Since the factor $\frac{\rho}{8} \sin^4\beta$ is common to both (10) and (11) the relative amount of aberration for any given grating with the two mountings will be given by the two trigonometrical functions

$$f_1(i) = \cos^2i(1 + \cos i) = a \quad \dots \dots (12)$$

$$f_2(i) = \frac{\sin^2i}{\cos i} = a_0. \quad \dots \dots (13)$$

For convenience the values of these coefficients, a and a_0 , have been computed for values of i from 0° to 90° , and are tabulated in Table I. columns 2 and 3.

(B) In using the grating photographically we cannot satisfy the condition $\theta=0$ for all parts of the field simultaneously. If we bend the plate to conform to the focal

curve [as given by (6)]*, and place it with its centre on the optical axis, the aberration at any other point of the field near the axis, *i. e.* for small values of θ , will be

$$Z_B = \mp \frac{\rho}{2} \sin^3 \beta \tan \theta \cos i (1 + \cos i) + \frac{\rho}{8} \sin^4 \beta (1 + \cos i) [\cos^2 i - (1 + 2 \cos i)^2 \tan^2 \theta] \sec^2 \theta. \quad (14)$$

For the Rowland mounting it will be similarly

$$Z_0 = -\frac{\rho}{8} \sin^4 \beta \left[\frac{\sin^2 i}{\cos i} + \frac{\sin^2 \theta}{\cos \theta} \right]. \quad (15)$$

This expression (14) may be put in the form

$$Z_B = \mp \frac{\rho}{2} \sin^3 \beta \frac{\tan \theta}{\cos i} (1 + \cos i) \cos^2 i + \frac{\rho}{8} \sin^4 \beta (1 + \cos i) \cos^2 i \left[1 - \left(\frac{1 + 2 \cos i}{\cos i} \right)^2 \tan^2 \theta \right] \sec^2 \theta, \quad (16)$$

in which the same functions of i appear as in (10). The only new function is

$$f_3(i) = \left(\frac{1 + 2 \cos i}{\cos i} \right)^2 = \left(2 + \frac{1}{\cos i} \right)^2 = b. \quad (17)$$

The values of $f_1(i) \div \cos i = a_1$ and of $f_3(i) = b$ have been computed for the same values of i assumed before, and are given in columns 4 and 5 of Table I.

(C) Let the grating be mounted so that the angles of incidence and diffraction are equal but opposite in sign, *i. e.* so that $\theta = -i$. This is the form of mounting discussed and illustrated on pp. 58, 59, and on Plate VII. fig. 9 of my former paper. As there shown, the central spectral image in the field always lies on a circle whose centre is on the axis of the grating, and whose radius is $\frac{\rho}{4}$. The mechanical mounting which fulfils this condition is very simple, but it

* For small values of θ the equation of the curve may be written

$$u = \frac{\rho}{1 + \cos i} \cos \theta, \quad (6a)$$

which is the equation of a circle with its centre on the axis of the grating and a radius equal to one-half u . The statement made by Poor and Mitchell in their paper (*Astrophysical Journal*, vol. vii. p. 159) that "The spectra . . . are brought to a focus on a circle whose radius is u ," is an error probably due to an oversight in considering $\cos^2 \theta$ to be of the same order as $\cos \theta$.

has not yet been used, to my knowledge, except in the small trial instrument there illustrated.

In this case we have for the aberration in the central image

$$Z_C = \pm \rho \sin^3 \beta \sin i + \frac{\rho}{4} \sin^4 \beta (1 - 9 \tan^2 i) \cos i. \quad (18)$$

The values of the function

$$f_4(i) = (1 - 9 \tan^2 i) \cos i = \cos i - 9a_0 = c \quad (19)$$

are tabulated in column 6, Table I.

(D) When the grating is mounted as just considered, and is used for photographic work, we have to consider not only the line at the centre of the field $\theta = -i$, but also lines at a distance κ each side of the centre. For any given value of i we have for θ_D

$$\theta_D = -i \pm \kappa. \quad (20)$$

The aberration in this case is most easily computed directly from the general equation (8) by substituting for θ and i the values determined by the relation (20).

(E) Let the grating be mounted so that the incident light is always normal to the surface. This case has been investigated for the Rowland mounting*, but not for the O.S. mounting. In the latter we have from (7) for $i=0$

$$u = \frac{\rho}{1 + \cos \theta} \cos^2 \theta, \quad (21)$$

which is the equation of the curve on which *all* focal images lie, *i. e.* the focal curves for both central and lateral images coincide.

The aberration for any image on the focal curve is

$$\begin{aligned} Z_E = & \mp \frac{\rho}{2} \sin^3 \beta \sin \theta \left(\frac{1 + \cos \theta}{\cos^2 \theta} \right) \\ & + \frac{\rho}{8} \sin^4 \beta \frac{1 + \cos \theta}{\cos^2 \theta} [1 - (2 + \cos \theta)^2 \tan^2 \theta]. \quad (22) \end{aligned}$$

The last term of (22) may be put in the form

$$\frac{\rho}{8} \sin^4 \beta \frac{1 + \cos \theta}{\cos^2 \theta} \sin \theta \left[\frac{\cos^2 \theta}{\sin \theta} - 4 \frac{1 + \cos \theta}{\cos^2 \theta} \sin \theta \right].$$

* "Fixed Arm Concave Grating Spectroscopes," *Astrophysical Journal*, vol. ii. p. 370. Forms of mounting fulfilling this condition are shown on plates xiii. (for $i=0$) and xv.

The functions

$$f_5(\theta) = \frac{\cos^2 \theta}{\sin \theta} = d, \quad . \quad . \quad . \quad . \quad . \quad (23)$$

$$\left. \begin{aligned} f_6(\theta) &= \frac{1 + \cos \theta}{\cos^2 \theta} = e \\ f_7(\theta) &= e \sin \theta = e_1 \end{aligned} \right\} \quad . \quad . \quad . \quad . \quad (24)$$

which appear in (21) and (22) have been computed for the same values of θ as have already been used for i in $f_1(i) \dots f_4(i)$. The results are given in columns 7, 8, and 9 of Table I.

(F) When θ is very small expressions (21) and (22) reduce respectively to

$$u = \frac{\rho}{2} \cos \theta, \quad . \quad . \quad . \quad . \quad . \quad (25)$$

and $Z_F = \mp \rho \sin^3 \beta \tan \theta$

$$+ \frac{\rho}{4} \sin^4 \beta \sec^2 \theta [1 - 9 \tan^2 \theta]. \quad . \quad . \quad (26)$$

These expressions may also be obtained from (6a)* and (14) by putting $i=0$. It is very nearly the same as (18) for small values of i . When the angle of incidence is nearly normal to the grating Cases B and C may therefore be considered as practically identical with Case F.

(G) Finally, let $\theta = +i$. This is the case of specular reflexion. The aberration at the central point in the field, *i. e.* in the undeviated image of the source of light, will be given by the expression

$$Z_G = \mp \rho \sin^3 \beta \sin \theta + \frac{\rho}{4} \sin^4 \beta (1 - 9 \tan^2 \theta) \cos \theta, \quad . \quad (27)$$

which is identical with (10) with the exception of the reversed sign of the first term. For the special case $\theta = i = 0$ we have

$$Z_G = \frac{\rho}{4} \sin^4 \beta = \frac{u}{2} \sin^4 \beta, \quad . \quad . \quad . \quad . \quad (28)$$

the well-known expression for the aberration at the principal focus of a spherical mirror.

(H) When the spectral image is situated at a distance κ from the undeviated image the aberration will be found, as in Case D, by substituting in (8) the value

$$\theta = i \pm \kappa.$$

* See p. 126, footnote.

Making use of the quantities $a, a_0, a_1, b, b_1, c, e$, and e_1 , already computed, and putting the quantity

$$\frac{1}{8} \rho \sin^4 \beta = K,$$

a constant for any given grating, we may write the expressions $Z_0, \dots, Z_H$ as follows:—

[illegible]

[illegible]

$$Z_{B_0} = -K(a_0 + \sin^2 \kappa) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (15a)$$

$$Z_B = K \left[-4a_1 \frac{\tan \kappa}{\sin \beta} + a(1 - b \tan^2 \kappa) \sec^2 \kappa \right]. \quad (16a)$$

$$Z_c = K \left(+8 \frac{\sin i}{\sin \beta} + 2c \right) \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot (18a)$$

$$Z_E = K \left[-\frac{4e_1}{\sin \beta} + e_1(a_0 - 4e_1) \right] \quad . \quad . \quad . \quad . \quad (22a)$$

$$Z_F = K \left(-\frac{8 \tan \kappa}{\sin \beta} + \frac{2c}{\cos^2 \kappa} \right) \quad . \quad . \quad . \quad . \quad . \quad (26a)$$

$$Z_G = K \left(-8 \frac{\sin \theta}{\sin \beta} + 2c \right) \quad . \quad . \quad . \quad . \quad . \quad . \quad (27a)$$

For Cases D and H, in which we must use the general equation (8), we may write

$$Z = K \left\{ \begin{aligned} & - \frac{4 \sin \theta \cos i}{\sin \beta} \cdot \frac{\cos i + \cos \theta}{\cos^2 \theta} \\ & + \frac{\cos i + \cos \theta}{\cos^2 \theta} [\cos^2 i - (\cos \theta + 2 \cos i)^2 \tan^2 \theta] \end{aligned} \right\} \quad (8a)$$

By means of the general formulæ (8a) ... (27a) and the aid of Table I. we can readily compute the aberration of the concave grating for any form of mounting, and for any given point in the spectral field. In order to compare the efficiency of the same grating mounted in the different ways above discussed, the aberrations $Z_A \dots Z_H$ have been computed for twelve different points in the spectral field for a grating of the semi-angular aperture ($\beta=0.1$) generally used. Four of these points correspond to the images formed on the axis of the grating for $i=5^\circ$, $i=15^\circ$, $i=30^\circ$, and $i=60^\circ$, of mounting A (or A_0), and are designated by the symbols S_0 , S_0'' , S_0''' , and S_0^{iv} respectively. Eight others correspond to the spectral images which lie 5° on each side ($\kappa=+5^\circ$) of the

first four, and these are designated by S'_+ , S'_{-} , S''_+ , S''_{-} , S'''_+ , S'''_{-} , and S^{iv}_+ , S^{iv}_{-} respectively.

TABLE I.

i	a $(1 + \cos i) \cos^2 i$	a_0 $\frac{\sin^2 i}{\cos i}$	a_1 $(1 + \cos i) \cos i$	b $\left(2 + \frac{1}{\cos i}\right)^2$	c $(1 - 9 \tan^2 i) \times \cos i$	d $\frac{\cos^2 i}{\sin i}$	e $\frac{1 + \cos i}{\cos^2 i}$	e_1 $e \sin i$
0 ...	2.000	0.000	2.000	9.000	1.000	∞	2.000	.000
5 ...	1.981	.0076	1.988	9.023	+ .928	11.35	2.011	.1754
10 ...	1.925	.0306	1.954	9.093	+ .709	5.58	2.046	.355
15 ...	1.834	.069	1.899	9.213	+ .341	3.61	2.107	.545
20 ...	1.713	.125	1.823	9.389	— .181	2.58	2.192	.750
25 ...	1.566	.197	1.728	9.631	— .867	1.94	2.320	.981
30 ...	1.399	.289	1.616	9.952	— 1.732	1.50	2.483	1.244
35 ...	1.220	.402	1.490	10.373	— 2.794	1.17	2.711	1.555
40 ...	1.036	.539	1.353	10.93	— 4.086	.913	3.010	1.935
45 ...	.854	.707	1.207	11.66	— 5.657	.707	3.414	2.414
50 ...	.678	.913	1.056	12.64	— 7.573	.539	3.976	3.045
55 ...	.518	1.170	.902	14.01	— 9.955	.402	4.783	3.919
60 ...	.375	1.500	.750	16.00	— 13.000	.289	6.001	5.196
65 ...	.254	1.944	.601	19.06	— 17.07	.197	7.965	7.218
70 ...	.157	2.582	.459	24.24	— 22.89	.124	11.47	10.78
75 ...	.084	3.515	.326	34.39	— 32.87	.067	18.79	18.15
80 ...	.035	5.585	.203	60.20	— 50.26	.031	38.92	38.33
85 ...	.008	11.386	.095	181.5	— 102.4	.008	143.2	142.6
90 ...	.000	∞	.000	∞	∞	.000	∞	∞

In order that the points S' to S^{iv} shall correspond to the same spectral images for each of the various mountings we must choose the values of θ and i so that we always have

$$(\sin \theta - \sin i) = \text{constant} \quad . \quad . \quad . \quad (29)$$

for any given point S . We therefore have

For the points ..	S'	S''	S'''	S^{iv}
Case A— A_0 ...	$i_A = 5^\circ$	15°	30°	60°
	$\theta_A = 0^\circ$	0°	0°	0°
Case E_0	$\theta_{E_0} = 5^\circ$	15°	30°	60°
	$i_{E_0} = 0^\circ$	0°	0°	0°

For Case C..... $\theta = -i$

$$\therefore (\sin \theta_A - \sin i_A) = -2 \sin i_C,$$

and we therefore have for i_C [to be used in (18a)]

$$\left. \begin{matrix} i_C \\ -\theta_C \end{matrix} \right\} \begin{matrix} 2^\circ 30', & 7^\circ 26', & 14^\circ 29', & 25^\circ 40'. \end{matrix}$$

For the lateral images $S_{\pm}$ to the right and left of the central ones we have

For $B_0 - B$	$S'_{\pm}$	$S''_{\pm}$	$S'''_{\pm}$	$S^{iv}_{\pm}$
$i_B =$	5°	15°	30°	60°
$\theta = \kappa = \pm$	5°	$\pm 5^{\circ}$	$\pm 5^{\circ}$	$\pm 5^{\circ}$

For Case D we must have

$$\sin i_A \mp \sin 5^{\circ} = \sin i_C + [\sin (i_0 \mp \kappa) = \sin (-\theta_D)],$$

from which we get for θ_D to be used in (8a)

	$S'_{\pm}$	$S''_{\pm}$	$S'''_{\pm}$	$S^{iv}_{\pm}$
$i_D = i_C =$	$2^{\circ} 30'$	$7^{\circ} 26'$	$14^{\circ} 29'$	$25^{\circ} 40'$
$\kappa_+ = \theta_{+D} =$	$2^{\circ} 30'$	$-(2^{\circ} 25')$	$-(9^{\circ} 22')$	$-(31^{\circ} 21')$

For the lateral images of Case E we must have

$$\sin i_A \mp \sin 5^{\circ} = \sin (\theta + \kappa) = \sin \theta_E,$$

and therefore for θ_E to be used in (8a)

	S'	S''	S'''	S^{iv}
for $\theta_E =$	0°	$9^{\circ} 53'$	$24^{\circ} 23'$	$51^{\circ} 9'$
	$10^{\circ} 2'$	$20^{\circ} 15'$	$35^{\circ} 58'$	$72^{\circ} 24'$
$i_E =$	0°	0°	0°	0°

The values of S'_+ and S'_0 are likewise the values of the aberration at the centre and edges of a 10° field for Case F, which therefore requires no separate consideration.

Finally, for Case H we have a number of identical fields, $S' \dots S^{iv}$, each corresponding to a particular value of i_A . Only one set of these is considered, *i. e.* the set S' for which $i = 5^{\circ}$. The centres of these will lie at points θ_H defined by the relation

$$\sin 5^{\circ} = \sin i_H - \sin \theta_H.$$

Hence for $i_H =$	5°	15°	30°	60°
$\theta_H =$	0°	$9^{\circ} 53'$	$24^{\circ} 23'$	$51^{\circ} 9'$

Also the two edges of this set S' will lie at points

$$\theta_H = \sin^{-1}(\sin i_H - \sin 5^{\circ} \pm \sin 5^{\circ}),$$

$\therefore$ for $i_H =$	5°	15°	30°	60°
$\theta_{+H} =$	5°	15°	30°	60°
$\theta_{-H} =$	-5°	$+ 4^{\circ} 51'$	$19^{\circ} 1'$	$43^{\circ} 46'$

Each of these fields lies to the left of the geometrical (undeviated) image of the source which always corresponds to S_+ . The first of these, S'_1 , for $i_H = 5^\circ$ evidently corresponds exactly to the corresponding one for Case A-B. For each field lying to the left there is a corresponding field lying to the right, which it is also of interest to consider in this special case. The centres of these right-hand fields lie also at points defined by the relation

$$\theta_{H_0} = \sin^{-1}(\sin i_H + \sin 5^\circ),$$

or for	$i_H = 5^\circ$	15°	30°	60°
(right)	$\theta_H = 10^\circ 2'$	$20^\circ 15'$	$35^\circ 58'$	$72^\circ 24'$

The edges of the right-hand fields similarly lie at points

$$\theta_H = \sin^{-1}(\sin i_H + \sin 5^\circ \mp \sin 5^\circ),$$

or for different values of i_H

(right)	$\theta_{+H} = 4^\circ$	15°	30°	60°
	$\theta_{-H} = 15^\circ 10'$	$25^\circ 40'$	$42^\circ 24'$	—

For θ_{+H} the aberrations may be computed by (27a); for all other points they must be computed by the general formula (8).

The values of the relative aberrations Z_A/K Z_H/K for a grating of semi-angular aperture $\beta = 0.1$, and for the different fields S'_1 S'_v above considered, are tabulated in Table II. For each point there are in general two values of Z , Z_1 , and Z_2 , as already defined, which are tabulated separately, although in most cases we are concerned only with the maximum (numerical) values of Z , which for positive values of β (right hand of grating) is always Z_2 except for a few cases which may be noted in the table.

Comparing the relative aberrations in any field for these different cases we see that the most advantageous type of mounting for an objective concave-grating spectroscope is A, for which $\theta = 0$ for S_0 , the centre of the field. This form is particularly advantageous for large values of $Nm\lambda$, *i. e.* for large angles of diffraction, because, as shown clearly both by (10) and (16) and by the table, the aberration both at the centre of the field and at the edges *decreases* with increasing values of i . With all of the other mountings, and with the Rowland mounting as well, the aberration *increases* with increasing values of i .

Comparing the aberrations of the concave grating when used as an objective spectroscope with the aberrations of the

TABLE II.

Objective Grating Spectroscope (O.S.) Mounting.									
Field.	Rowland A_0-B_0 mounting.	Case A-B.		Case C-D.		Case E (F special).		Case H, S' only (G special).	
		$Z_1/K.$	$Z_2/K.$	$Z_1/K.$	$Z_2/K.$	$\theta_{\text{av.}}$	$Z_1/K.$	$Z_2/K.$	
$S_0' \dots \left\{ \begin{array}{l} S_+ \\ S_0 \\ S_- \end{array} \right.$	$\begin{array}{l} -.02 \\ -.01 \\ -.02 \end{array}$	$\begin{array}{l} -.677 \\ +1.98 \\ +71.4 \end{array}$	$\begin{array}{l} +71.4 \\ -.329 \\ +36.8 \end{array}$	$\begin{array}{l} -.329 \\ +36.8 \\ -.329 \end{array}$	$\begin{array}{l} +36.8 \\ -.329 \\ -.1041 \end{array}$	$\begin{array}{l} 0^\circ 0' \\ 5^\circ 0' \\ 10^\circ 2' \end{array}$	$\begin{array}{l} +1.86 \\ -.682 \\ -140.1 \end{array}$	$\begin{array}{l} +1.86 \\ +71.8 \\ +143.0 \end{array}$	$\begin{array}{l} = 5^\circ \\ \left\{ \begin{array}{l} -.678 \\ +1.98 \\ +71.5 \end{array} \right. \end{array}$
$S_0'' \dots \left\{ \begin{array}{l} S_+ \\ S_0 \\ S_- \end{array} \right.$	$\begin{array}{l} -.08 \\ -.07 \\ -.08 \end{array}$	$\begin{array}{l} -.647 \\ +1.83 \\ +68.2 \end{array}$	$\begin{array}{l} +68.2 \\ +35.3 \\ +105.2 \end{array}$	$\begin{array}{l} +35.3 \\ -.315 \\ -.1019 \end{array}$	$\begin{array}{l} -.315 \\ -.1019 \\ -.1763 \end{array}$	$\begin{array}{l} 9^\circ 53' \\ 15^\circ 0' \\ 20^\circ 15' \end{array}$	$\begin{array}{l} -.1379 \\ -.217 \\ -.305 \end{array}$	$\begin{array}{l} +1.408 \\ +219 \\ +304 \end{array}$	$\begin{array}{l} \left\{ \begin{array}{l} -.206 \\ -.132 \\ -.628 \end{array} \right. \end{array}$
$S_0''' \dots \left\{ \begin{array}{l} S_+ \\ S_0 \\ S_- \end{array} \right.$	$\begin{array}{l} -.30 \\ -.29 \\ -.30 \end{array}$	$\begin{array}{l} -.553 \\ +1.40 \\ +57.9 \end{array}$	$\begin{array}{l} +57.9 \\ +128 \\ +201 \end{array}$	$\begin{array}{l} +128 \\ -.125 \\ -.199 \end{array}$	$\begin{array}{l} -.125 \\ -.199 \\ -.282 \end{array}$	$\begin{array}{l} 24^\circ 23' \\ 30^\circ 0' \\ 35^\circ 58' \end{array}$	$\begin{array}{l} -.384 \\ -.502 \\ -.659 \end{array}$	$\begin{array}{l} +381 \\ +493 \\ +642 \end{array}$	$\begin{array}{l} \left\{ \begin{array}{l} -.404 \\ -.308 \\ -.229 \end{array} \right. \end{array}$
$S_0^{IV} \dots \left\{ \begin{array}{l} S_+ \\ S_0 \\ S_- \end{array} \right.$	$\begin{array}{l} -.51 \\ -.50 \\ -.51 \end{array}$	$\begin{array}{l} -.259 \\ +.38 \\ +26.6 \end{array}$	$\begin{array}{l} +26.6 \\ +260 \\ +345 \end{array}$	$\begin{array}{l} +260 \\ -.261 \\ -.348 \end{array}$	$\begin{array}{l} -.261 \\ -.348 \\ -.456 \end{array}$	$\begin{array}{l} 51^\circ 9' \\ 60^\circ 0' \\ 72^\circ 24' \end{array}$	$\begin{array}{l} -.1349 \\ -.2185 \\ -.6196 \end{array}$	$\begin{array}{l} +1.267 \\ +1972 \\ +4712 \end{array}$	$\begin{array}{l} \left\{ \begin{array}{l} -.719 \\ -.457 \\ -.330 \end{array} \right. \end{array}$
									$\begin{array}{l} +667 \\ +435 \\ +318 \end{array}$
									$\begin{array}{l} -.719 \\ -.1819 \\ +1528 \end{array}$

same grating as mounted and used by Rowland, we have the following results.

In the field S' :—

(1) For mounting A-B the aberration at the centre of the field is nearly 200 times as great as with the Rowland mounting; at the edges of the 10° field the aberrations (maximum) are about thirty-five hundred times larger.

(2) For mounting C-D the aberration at the centre of the field (maximum) is about thirty-seven hundred times as great as with A_0 (R. mounting). At one edge of the 10° field it is about sixteen hundred times larger; at the other it is over five thousand times larger.

(3) With mounting E the aberration at the centre of the field is seven thousand times larger than with the R. mounting. At one edge it is only twenty-five times larger, but at the other it is again about seven thousand times as great.

(4) With mounting H the aberrations at the centres of the various fields S' vary from the same values already given for the A-B type to a maximum value nearly *two hundred thousand* times as great as that for the Rowland mounting.

For the fields S'' , S''' , and S^iv the comparisons are more favourable to the O.S. types of mounting, both for the centres and for the edges of the fields. In the field S'' for example the ratios

$$\frac{Z_{1-2} \text{ (O.S. mountings)}}{Z_0 \text{ (Rowland mountings)}} = Q$$

run as follows

$$(1) \text{ For the A-B type } Q_0'' \text{ (centre)} \simeq 26.0$$

$$Q_{\pm\kappa}'' \text{ (edges)} \simeq 830.0$$

$$(2) \text{ For the C-D type } Q_0'' \text{ (max.)} \simeq 1502.0$$

$$Q_{+\kappa}'' \text{ (, ,)} \simeq 441.0$$

$$Q_{-\kappa}'' \text{ (, ,)} \simeq 2233.0$$

$$(3) \text{ For the E type } Q_0'' \text{ (max.)} \simeq 3132.0$$

$$Q_{+\kappa}'' \text{ (, ,)} \simeq 1760.0$$

$$Q_{-\kappa}'' \text{ (, ,)} \simeq 3810.0$$

For the extreme field S^{iv} the ratios run

(1) A-B type	$Q_0^{iv} \simeq 0.25$
	$Q_{\pm\kappa}^{iv} \simeq 18.7$
(2) C-D type	$Q_0^{iv} \simeq 232.0$
	$Q_{+\kappa}^{iv} \simeq 174.0$
	$Q_{-\kappa}^{iv} \simeq 304.0$
(3) E type	$Q_0^{iv} \simeq 1456.0$
	$Q_{+\kappa}^{iv} \simeq 893.0$
	$Q_{-\kappa}^{iv} \simeq 4100.0$

The more favourable showing for the O.S. types of mounting for large angles of diffraction is due, in the case of the A-B type, to a real improvement in definition already noted; in the other case the improvement is only apparent and comparative, and not real. For type C-D the total aberrations are about ten times as large for field S^{iv} as for field S' , and for type E, about thirty times as large for S^{iv} as for S' .

These comparisons, particularly those for S' and S'' , the fields most frequently used with the O.S. types, show very clearly the great inferiority of the concave-grating objective spectroscope, as an optical instrument, to the ordinary form of concave-grating spectroscope, and are a sufficiently complete explanation of the disappointing results that have been secured by its use in solar eclipse work.

Poor and Mitchell considered the star-spectrum plates which they obtained (using the A-B and the F types of mounting) satisfactory, but it must be remembered that such plates afford no such refined tests of definition as plates of solar spectra. Their results are considered more at length later.

Although we are thus forced to the general conclusion that the concave objective grating spectroscope is unsuited to the class of work that can be performed by the same gratings when used on Rowland mountings, the advantages of the former type of instruments are so great in certain cases that it becomes desirable to investigate the question as to whether there is not some way of reducing the aberration in these instruments.

In dealing with a general problem of this nature, there are always three expedients that may be adopted with a greater or less degree of success :—1st, altering the focus of the instrument ; 2nd, diaphragming the optical surface of the instrument or limiting the field ; 3rd, altering the form of the reflected wave-front, generally by altering the form of the optical surface itself. Let us take the two most favourable types of O.S. mounting, the A-B and the F forms, and see what can be accomplished in the way of improving the definition by any one or a combination of these means.

1st. Effect of alterations of focus. By examining the Table we find that in every type of O.S. mounting save one, A, the numerical values of Z_1 and Z_2 , the aberrations of the two halves of the grating-surface, are different. Since these aberrations (with the single exception noted), are in large part due to the unsymmetrical action of the lateral elements b , b' , fig. 1, and are of opposite sign, it follows that a slight improvement in definition may be secured by shifting the focal point, O' , by a small quantity du such that

$$du = \frac{1}{2} [\text{longitudinal aberration due to } Z_1 \\ - \text{longitudinal aberration due to } Z_2] \quad \dots \quad (31)$$

$$= \frac{K}{2} (Z_1 - Z_2) \frac{3}{\beta_0^2} * \\ = \frac{1}{16} \rho \sin^4 \beta (Z_1 - Z_2) \frac{3u^2}{(\rho \sin \beta \cos \theta)^2} \\ \simeq 0.002\rho(Z_1 - Z_2) \frac{\cos^2 \theta}{\cos \theta + \cos i} \quad \dots \quad (31a)$$

in which the values of Z_1 and Z_2 are to be taken without regard to sign. In most cases the numerical differences, $Z_1 - Z_2$, are so small compared with $Z_{(\max)}$ that no very great improvement can be effected by departing from the focal curve defined by (6). In any case the departures will be very small. Thus in case A-B the numerical differences $Z_1 - Z_2$ at the edges of the field are always less than 0.04. Hence we have for the maximum values of du ($i = 60^\circ$).

$$du_{\max.} \simeq 0.00005\rho$$

or, for a Rowland grating of the usual radius of curvature,

$$du_{\max.} = \frac{1}{3} \text{ mm.}$$

Although no great improvement can be effected by a

* Astrophysical Journal, vol. xvii. p. 1, January 1903.

readjustment of the focus u in the objective type of mounting, it is interesting to note in passing that since the aberration at the centre of the field ($\theta=0$) is of the opposite sign for the regular Rowland mounting and the A type of O.S. mounting, there is a certain value of u , which we will call u' , intermediate between the values ρ (for the Rowland mounting) and the value

$$u = \frac{\rho}{1 + \cos \theta}$$

for the A-O.S. mounting, for which the aberration will vanish. We can find this value if we desire by equating the last term of Glazebrook's general equation (3)* to zero. This together with the second general equation for the focal curve of the grating for this case †, *i. e.*,

$$u' = \frac{\rho}{1 + \cos i - \frac{\rho}{\nu} \cos^2 i} \quad . \quad . \quad . \quad (32)$$

gives us two equations from which to determine u' and ν , the two conjugate spectral focal distances, so as to satisfy the given condition of zero aberration at the point u' for any given value of i . This solution is not of general, but may become of special interest in some individual cases.

2nd. Effect of diaphragming a portion of the optical surface or limiting the field of the instrument.

Several effects may be here discriminated, but they may all be best investigated by considering the character of the aberration in the lateral fields more in detail. This may be most conveniently done by expressing the aberration at a point $S_{\pm \kappa}$ in terms of the aberration Z_0 at the centre S_0 . For cases A-B and F we see from equations (10), (16), and (17)

$$Z_B = Z_A \left[-4 \tan \kappa \frac{1}{\sin \beta \cos i} + \sec^2 \kappa (1 - b \tan^2 \kappa) \right] \quad (33)$$

or

$$\frac{Z_B}{Z_A} = R.$$

By aid of the Tables already computed for b and $\frac{1}{\cos i}$ (Table I.) the values of the ratio R has been determined for seven intermediate points ($\kappa=5'$ to $\kappa=3^\circ$) in each of the

* *Loc. cit.* p. 379.

† See *Astrophysical Journal*, vol. iii. p. 55.

fields S' , S'' , S''' , and S^{iv} of cases A-B, and field S of case F. This has been done for three gratings, one having the same angular aperture ($\beta_1 = .01$) as already used, the others having angular apertures of $\beta_2 = .025$ and $\beta_3 = .05$. The results are given in Table III.

TABLE III.—Values of R_1 and R_2 .

κ .	For $\beta = 0.01$.									
	$i = 0^\circ$.		$i = 5^\circ$.		$i = 15^\circ$.		$i = 30^\circ$.		$i = 60^\circ$.	
	R_1 .	R_2 .	R_1 .	R_2 .	R_1 .	R_2 .	R_1 .	R_2 .	R_1 .	R_2 .
0°										
0'	+ 1.00	+ 1.00	+ 1.00	+ 1.00	+ 1.00	+ 1.00	+ 1.00	+ 1.00	+ 1.00	+ 1.00
5'	+ 0.44	+ 1.56	+ 0.44	+ 1.56	+ 0.42	+ 1.58	+ 0.35	+ 1.65	- 0.12	+ 2.12
10'	- 0.16	+ 2.16	- 0.16	+ 2.16	- 0.20	+ 2.20	- 0.34	+ 2.34	- 1.32	+ 3.32
15'	- 0.77	+ 2.76	- 0.77	+ 2.77	- 0.82	+ 2.82	- 1.03	+ 3.03	- 2.52	+ 4.52
30'	- 2.48	+ 4.48	- 2.49	+ 4.49	- 2.60	+ 4.60	- 3.02	+ 5.02	- 5.96	+ 7.96
1°	- 5.96	+ 7.96	- 5.99	+ 7.98	- 6.21	+ 8.20	- 7.04	+ 9.04	- 12.92	+ 14.92
2°	- 12.97	+ 14.95	- 13.03	+ 15.01	- 13.46	+ 15.44	- 15.13	+ 17.11	- 26.94	+ 28.90
3°	- 19.98	+ 21.94	- 20.07	+ 22.02	- 20.72	+ 22.67	- 23.23	+ 25.18	- 40.96	+ 42.88
5°	- 34.06	+ 35.94	- 34.20	+ 36.08	- 35.29	+ 37.16	- 39.49	+ 41.36	- 69.11	+ 70.88
For $\beta = 0.025$.										
0°										
0'	+ 1.00	+ 1.00	+ 1.00	+ 1.00	+ 1.00	+ 1.00	+ 1.00	+ 1.00	+ 1.00	+ 1.00
5'	+ 0.78	+ 1.22	+ 0.77	+ 1.23	+ 0.77	+ 1.23	+ 0.74	+ 1.26	+ 0.55	+ 1.45
10'	+ 0.54	+ 1.46	+ 0.53	+ 1.47	+ 0.52	+ 1.48	+ 0.46	+ 1.54	+ 0.07	+ 1.93
15'	+ 0.30	+ 1.70	+ 0.29	+ 1.71	+ 0.27	+ 1.73	+ 0.19	+ 1.81	- 0.41	+ 2.41
30'	- 0.39	+ 2.40	- 0.40	+ 2.40	- 0.44	+ 2.44	- 0.61	+ 2.61	- 1.78	+ 3.78
1°	- 1.79	+ 3.78	- 1.80	+ 3.79	- 1.86	+ 3.88	- 2.22	+ 4.21	- 4.57	+ 6.56
2°	- 4.59	+ 6.59	- 4.62	+ 6.60	- 4.82	+ 6.80	- 5.46	+ 7.44	- 10.19	+ 12.15
3°	- 9.41	+ 9.36	- 7.44	+ 9.39	- 7.70	+ 9.66	- 8.71	+ 10.66	- 15.81	+ 17.73
5°	- 13.06	+ 14.94	- 13.12	+ 15.00	- 13.56	+ 15.43	- 15.24	+ 17.10	- 27.12	+ 28.89
For $\beta = 0.05$.										
0°										
0'	+ 1.00	+ 1.00	+ 1.00	+ 1.00	+ 1.00	+ 1.00	+ 1.00	+ 1.00	+ 1.00	+ 1.00
5'	+ 0.89	+ 1.11	+ 0.89	+ 1.11	+ 0.88	+ 1.12	+ 0.87	+ 1.12	+ 0.78	+ 1.22
10'	+ 0.77	+ 1.23	+ 0.77	+ 1.23	+ 0.76	+ 1.24	+ 0.73	+ 1.27	+ 0.54	+ 1.46
15'	+ 0.65	+ 1.35	+ 0.65	+ 1.35	+ 0.64	+ 1.36	+ 0.59	+ 1.41	+ 0.30	+ 1.70
30'	+ 0.30	+ 1.70	+ 0.30	+ 1.70	+ 0.28	+ 1.72	+ 0.20	+ 1.80	- 0.39	+ 2.39
1°	- 0.40	+ 2.39	- 0.40	+ 2.40	- 0.44	+ 2.44	- 0.61	+ 2.60	- 1.79	+ 3.78
2°	- 1.80	+ 3.78	- 1.81	+ 3.79	- 1.90	+ 3.88	- 2.23	+ 4.21	- 4.60	+ 6.57
3°	- 3.21	+ 5.17	- 3.23	+ 5.19	- 3.36	+ 5.32	- 3.86	+ 5.82	- 7.42	+ 9.34
5°	- 6.06	+ 7.94	- 6.09	+ 7.97	- 6.31	+ 8.18	- 7.15	+ 9.01	- 13.12	+ 14.89

As before there are two values of R for each value of Z_B , corresponding to the two sides, right and left, of the grating-surface. These are designated as before by R_1 and R_2 . For positive values of κ , R_1 is negative and R_2 is positive. For negative values of κ these quantities are simply interchanged, *i. e.*

$$\begin{aligned} R_1 \text{ (for } -\kappa) &= R_2 \text{ (for } +\kappa) \\ R_2 \text{ (, , ,)} &= R_1 \text{ (, , ,)}. \end{aligned}$$

In general, in considering the prejudicial effect of aberration when the whole surface of the grating is used we must take the maximum numerical values of Z and R . But inspection of equation (33) and of the values in Table III. shows that for each *half* of the grating-surface there is a point κ_0 in the field for which the aberration for that half is zero. In the case of the right-hand half (for which the aberration is Z_1) of grating β_1 this point lies between $\kappa = +10'$ and $\kappa = +5'$ for values of i from 0° to 30° , and between $\kappa = +5'$ and $\kappa = 0$ for $i = 60^\circ$. For gratings β_2 and β_3 the point κ_0 is farther from the centre of the field.

The exact location of the point κ_0 for any grating and any value of i is easily found by solving the equation

$$Z_B = 0,$$

or from (16a) or (33)

$$\pm \frac{4}{\sin \beta \cos i} \tan \kappa = \frac{1 - b \tan^2 \kappa}{\cos^2 \kappa}. \quad \dots (34)$$

Put $\frac{4}{\sin \beta \cos i} = f$. Then since κ is by assumption always small, we may readily obtain from (34)

$$\begin{aligned} \tan \kappa &= \mp f/2b \left(1 \mp \sqrt{1 + \frac{b}{4f^2}} \right) \\ &= \pm \frac{1}{4} \sin \beta \cos i \left(1 - \frac{b}{4f^2} \right). \quad \dots (35) \end{aligned}$$

For grating $\beta_1 = .01$ we find for κ_0

$$\begin{aligned} \kappa_0 &= \pm (8' 40'') \text{ for } i = 0^\circ, \\ \kappa_0 &= \pm (7' 30'') \text{ , } i = 30^\circ, \\ \kappa_0 &= \pm (4' 20'') \text{ , } i = 60^\circ. \end{aligned}$$

For $\beta_3 = .05$ we similarly find

$$\begin{aligned} \kappa_0 &= (38' 40'') \text{ for } i = 0^\circ, \\ \kappa_0 &= 37' + \text{ , } i = 30^\circ, \\ \text{and } \kappa_0 &= 21' 30'' \text{ , } i = 60^\circ. \end{aligned}$$

If therefore we diaphragm off the left-hand half of the grating-surface, the spectral image at the point $+\kappa_0$ will be practically free from aberration; if we diaphragm off the right-hand half, the spectral image at the point $-\kappa_0$ will likewise be sharply defined. With gratings of the usual angular apertures we can therefore always secure sharp definition with the A-B-F types of O.S. mounting on any given spectral line λ_0 by suitably choosing the angle i such that

$$\rho \sin \beta \{ \sin [\tan^{-1} (\pm \sin \beta \cos i)] - \sin i \} = mn \lambda_0, \quad . \quad (36)$$

and then covering the right- or the left-hand half of the ruled surface.

For gratings of large angular aperture, such as β_3 , this device gives less satisfactory results. The aberration of the centre and extreme elements, a , b , or a , b' (fig. 1) is of course eliminated at the point κ_0 , defined by (34) and (35), but the aberration for the intermediate points (which correspond to different values of β) remains. In the case of grating β_3 for example, for which $\kappa_0 = \pm 37'$ at $i = 30^\circ$, the aberration due to the element intermediate between elements a and b will be

$$\begin{aligned} & \frac{1}{8} \rho \sin^4 \beta_2 [-6.46 \tan 37' + .035 (1 - 9.95 \tan^2 37') \sec^2 37'] \\ &= \frac{1}{8} \rho \sin^4 \beta_1 (2.5)^3 (-3.4) \cong 53 \text{ K} ; \end{aligned}$$

i. e. the aberration of the intermediate element of the grating β_3 is nearly as great as that of the extreme elements of the grating β_1 at the limit of the 5° field (see Table II. $S''_{+\kappa}$) and over thirty times as great as that of the entire surface of β_1 at the centre of the field S_0'' .

Examination of the formulæ (16), (16a), (26), (26a), and (33), or of Table III., shows that for very small values of κ , *i. e.*, very near the axis of the grating, the symmetrical aberration, which is expressed by the second term of these equations, is of the greater numerical value; for larger values, the unsymmetrical aberration expressed by the first term becomes by far the most important. Since the first varies in absolute amount as $\sin^4 \beta$ and the second only as $\sin^3 \beta$ it at once follows that the *relative* differences over a given field, $\pm \kappa$, become *smaller* as the value of β increases, *i. e.* while the absolute amount of aberration is greater for all parts of the field, the definition, or rather lack of definition, is more uniform.

In considering the effect of a given aberration on the performance of an optical instrument, it is usually assumed

that the "definition" (*sic*) begins to deteriorate when the aberration exceeds one fourth of the wave-length of light. Definition is itself rather of an indefinite term in a physical sense, although it has a well-founded physiological basis. Of this point it will be necessary to speak later. Assuming for the present the limit assumed above, we can determine on that basis the maximum angular aperture $\beta_{\max.}$ and the maximum field $\kappa_{\max.}$ that can be utilized in any given case by the equation

$$Z \leq \frac{1}{4}\lambda. \quad . \quad . \quad . \quad . \quad . \quad (37)$$

For the case A of the O.S. spectroscope this equation becomes

$$\rho \sin^4 \beta = \frac{2\lambda}{a}$$

or

$$\sin^3 \beta = 2 \frac{\lambda}{W a} = 4 \frac{\alpha_0}{a}, \quad . \quad . \quad . \quad . \quad (38)$$

where W is the semi-ruled surface of the grating, and α_0 the angular resolving-power of a telescope of an aperture equal to that of the grating.

Equation (38) determines the maximum permissible semi-angular aperture $\beta_{\max.}$ which can be used for any given value of i . This varies, as will be seen, inversely as the cube root of the *linear* aperture, $2W$, of the grating, and also inversely as the coefficient a . For a grating of the usual linear aperture $W \cong 72$ mm. In the position $i = 15^\circ$ we have for $\beta_{\max.}$ in the visual region of the spectrum ($\lambda = 5500$ tenths-metres)

$$\beta_{\max.} = \sqrt[3]{\frac{.0011}{1.834 \times 72}} = .020.$$

For the same grating used in the position $i = 60^\circ$

$$\beta_{\max.} = .034, \quad . \quad . \quad . \quad . \quad . \quad (39)$$

or since 60° is about the maximum permissible angle of diffraction, (39) is about the maximum permissible semi-angular aperture of a spherical grating of 15 cm. aperture used as an objective spectroscope.

Equation (37) also enables us to determine for any given grating (whose semi-angular aperture is *less* than $\beta_{\max.}$), the maximum permissible field of good definition κ_m . To find this we put

$$Z_{B(\max.)} = + \frac{\lambda}{4}$$

since the maximum aberration is always positive.

From (33) with the same notation above used we get

$$4a_1 \tan \kappa_m + a \sin \beta (1 - b \tan^2 \kappa_m) = 2 \frac{\lambda}{W \sin^2 \beta},$$

whence

$$\tan \kappa_{\pm m} = \pm \frac{1}{2a_1} \left(\frac{\lambda}{W \sin^2 \beta} - \frac{a \sin \beta}{2} \right) \left[1 + \frac{1}{8} \frac{ab}{a_1^2} \left(\frac{\lambda}{W \sin^2 \beta} - \frac{a \sin \beta}{2} \right) \right]. \quad (40)$$

The usual Rowland grating is of from 70 to 75 millimetres semi-aperture and 650 to 660 centimetres radius of curvature. For such a grating $\beta \cong \beta_1 = 0.11$, and we find for κ_m

$$\left. \begin{aligned} \kappa_{+m} &\cong 0^\circ.8 \cong 50' \text{ for } i = 15^\circ \\ \kappa_{+m} &= 2^\circ 20' \quad \quad \quad \text{,, } i = 60^\circ \end{aligned} \right\} \quad (41)$$

As the value of β increases the available field κ_{+m} of good definition decreases and becomes zero, *i. e.* is confined to the central line of the field, at the point

$$\frac{\lambda}{W \sin^2 \beta} = \frac{a \sin \beta}{2}$$

which gives us the same condition as expressed in (38).

The only way in which we can increase the value of κ_m and β_m is to decrease W . The maximum permissible angular aperture varies inversely as the cube root of the linear aperture, while the maximum field varies very nearly inversely as the first power of the linear aperture. The maximum *spectral* field of good definition varies therefore very nearly as the inverse square of the angular aperture.

3rd. Effect of changing form of diffracted wave-front. The results of the preceding investigations show that the ordinary Rowland concave grating, whose ruled surface is a portion of a sphere, has an extremely restricted field of good definition when used as an objective spectroscope, and that although this field may be somewhat increased by slight readjustments of foci and in some cases by diaphragming off a portion of the ruled surface, the improvement resulting from such changes is small. It remains now to determine whether any better results can be secured by altering the form of the ruled surface or the character of the ruling. To do this we must first derive a general expression similar to that for Z , equation (4), which will express the relative retardation of the extreme ray bo' with reference to the central ray ao' when the light is incident on a surface of any form. This may be done in one of two ways.

Thus, in fig. 2, let ab be the spherical surface which we

Fig 2.

$$\begin{aligned}\Delta Z &= (hb + bo') - (hg + go') = kg + gl \\ &= bg(\cos \angle cak + \cos \angle cgl) \\ &= dr \left[\cos(i - \beta) + \cos \left(\tan^{-1} \frac{u \sin \theta - \rho \sin \beta}{u \cos \theta - 2\rho \sin^2 \frac{\beta}{2}} + \beta \right) \right]. \quad (42)\end{aligned}$$
$$r = \rho + dr$$

$$= \text{const.} + f(\beta, \theta, i). \quad . \quad . \quad . \quad (43)$$

The method above indicated of determining the form of

grating-surfaces which shall have certain predetermined amounts of aberration is a very useful and general one, and is of special value to the optician because it gives at once the variation dr of the required surface from the true spherical surface; the latter being the usual basis of operations in the final correction and testing of optical work. The mathematical transformations and developments required in the expansion of (42) are, however, rather tedious and cumbersome, and for that reason will be omitted from this paper. A more simple and elegant, although somewhat less general, solution of the problem is found by expressing the difference in path, $ao' - (bh + hb)$, in terms of the rectangular coordinates of the points b and o' , instead of polar coordinates as before. This second method is the one adopted by Plummer*, who has recently extended the investigations of Glazebrook and Rowland to the case of gratings ruled on surfaces other than spherical. Assuming the origin of coordinates at a , the centre of the grating, and expressing the coordinates of the point b by x, y , and of o' by x', y' , we have in general

$$ao' - bo' = u - \sqrt{(u \cos \theta - x)^2 + (u \sin \theta - y)^2}. \quad (44)$$

Similarly we have for hb (for parallel incident light),

$$hb = y \sin i - x \cos i,$$

and therefore for any grating whose surface is represented by the equation $x = f(y)$ we have

$$Z' = f(y) \cos i - y \sin i + u - \sqrt{u^2 + y^2 + [f(y)]^2 - 2uy \sin \theta - 2uf(y) \cos \theta}. \quad (45)$$

If we assume that the grating-surface may be represented by the equation

$$x = f(y) = ly^2 + my^3 + ny^4,$$

equation (45) may be at once reduced to the form

$$\begin{aligned} Z' = & y(\sin \theta - \sin i) \\ & + y^2 \left[l(\cos i + \cos \theta) - \frac{\cos^2 \theta}{2u} \right] \\ & + y^3 \left[m(\cos i + \cos \theta) + \frac{l \sin \theta \cos \theta}{u} - \frac{\sin \theta \cos^2 \theta}{2u^2} \right] \\ & + y^4 \left[n(\cos i + \cos \theta) + \frac{m \sin \theta \cos \theta}{u} - \frac{l^2}{2u} \right. \\ & \left. + \left(\frac{\cos^2 \theta}{8u^3} - \frac{l \cos \theta}{4u^2} \right) (1 - 2lu \cos \theta - 5 \sin^2 \theta) \right]^\dagger. \quad (46) \end{aligned}$$

* Dr. Plummer's paper, "Note on the Concave Grating" appeared in the *Astrophysical Journal*, vol. xvi. Sept. 1902, while the writer was engaged in this investigation.

† This expression may be readily deduced from that given by Plummer by putting $\phi = 0$, $\psi = i$, and $u' = \infty$.

To satisfy the condition that there shall be a spectral image formed at the point o' we must have as before

$$u = \frac{\cos^2 \theta}{2l (\cos i + \cos \theta)} \cdot \cdot \cdot \cdot \quad (47)$$

Substituting this value of u in (46), and reducing, we obtain for the aberration of the right-hand side, ab , of the surface

$$\begin{aligned} Z_1' = & y^3 \left(\frac{\cos i + \cos \theta}{\cos^2 \theta} \right) (m \cos^2 \theta - 2l^2 \sin \theta \cos i) \\ & + y^4 \left(\frac{\cos i + \cos \theta}{\cos^2 \theta} \right) \{ n \cos^2 \theta + 2lm \sin \theta \cos \theta - l^3 [\sin^2 i \\ & + 4 \tan^2 \theta \cos i (\cos i + \cos \theta)] \}, \cdot \cdot \cdot \quad (48) \end{aligned}$$

and for Z_2' a similar expression, save that the signs of the terms in l^2 and lm are reversed.

When the grating-surface is spherical, as previously assumed, its equation with reference to the point a is

$$\left. \begin{aligned} x &= \frac{y^2}{2\rho} + \frac{y^4}{8\rho^3} \\ \text{or} \quad l &= \frac{1}{2\rho}, \quad m=0, \quad \text{and} \quad n = \frac{1}{8\rho^3} \end{aligned} \right\}, \cdot \cdot \cdot \quad (49)$$

and (48) reduces at once to the form

$$\begin{aligned} & -\frac{1}{2} \frac{y^3}{\rho^2} \sin \theta \cos i \frac{(\cos i + \cos \theta)}{\cos^2 \theta} \\ & + \frac{1}{8} \frac{y^4}{\rho^3} \frac{(\cos i + \cos \theta)}{\cos^2 \theta} [\cos^2 i - (\cos \theta + 2 \cos i)^2 \tan^2 \theta], \quad (50) \end{aligned}$$

which is identical with (8).

The only other form of surface that has thus far been used for gratings is the parabola of revolution. The equation of the curve ab in this case is simply

$$\left. \begin{aligned} x &= \frac{y^2}{2\rho}, \\ \text{i. e.,} \quad m &= 0 \quad \text{and} \quad n = 0. \end{aligned} \right\} \cdot \cdot \cdot \quad (51)$$

It has been shown qualitatively by Rowland and Rayleigh, and quantitatively by Plummer, that the parabolic surface is distinctly *inferior* to a spherical surface for a grating as mounted and used according to the Rowland method. Let us see whether this is true for the O. S. type of mountings.

For these we have from (48) and (51)

$$Z_1' = \frac{1}{2} \frac{y^3}{\rho^2} \frac{\cos \theta + \cos i}{\cos^2 \theta} \sin \theta \cos i \\ - \frac{1}{8} \frac{y^4}{\rho^3} \frac{\cos \theta + \cos i}{\cos^2 \theta} [\sin^2 i + 4 \tan^2 \theta (\cos \theta + \cos i) \cos i]. \quad (52)$$

Comparing this with (50) or (8), we see that the unsymmetrical aberration is just the same in amount and sign for the two surfaces (spherical and parabolic). The symmetrical aberration is, however, *less* for the parabolic surface. This may be more readily seen by putting the second term of (52) in the form

$$\frac{1}{8} \frac{y^4}{\rho^3} \frac{\cos \theta + \cos i}{\cos^2 \theta} [\cos^2 i - (\cos \theta + 2 \cos i)^2 \tan^2 \theta - \cos^2 \theta]. \quad (53)$$

The parabolic surface is therefore in general *better* than a spherical surface for a concave grating when the O. S. type of mounting is used. The degree of improvement effected by the change in the form of surface depends on the type of mounting used. For all types for which θ has any appreciable value, the first term of (52) will, as we have already seen, be *large* compared with the second term, and the improvement in definition will be slight. On the other hand, for those images for which θ is zero or very small, the improvement may be very considerable.

As before, the type of mounting which will be most advantageous will be the type A-B, for which $\theta=0$ for the centre of the field. For these types we have for the parabolic grating

$$Z_A = \frac{1}{8} \frac{y^4}{\rho^3} (1 + \cos i) (\cos^2 i - 1) \\ = K [a - (1 + \cos i)] = -K a \tan^2 i. \quad . \quad . \quad (54)$$

Comparing this with (10) and (10 *a*), we see that the alteration in the form of the surface has not only decreased the aberration but has altered its sign. The relative values of $\frac{Z_A}{K}$ and $\frac{Z_A''}{K}$ for the different fields, S_0 , S_0' , S_0'' , S_0''' , and S_0^{iv} , used in previous comparisons, run as follows:—

For	S_0	S_0'	S_0''	S_0'''	S_0^{iv}	
	$\frac{Z_A}{K}$	$\frac{Z_A'}{K}$	$\frac{Z_A''}{K}$	$\frac{Z_A'''}{K}$	$\frac{Z_A^{iv}}{K}$	
	= 2.00	+ 1.98	+ 1.83	+ 1.40	+ .38 = a	
	= 0.00	- .015	- 0.13	- 0.47	- 1.17 = $-a \tan^2 i$,	
also	$\frac{Z_{A_0}}{K}$	= 0.00	- .01	- .07	- .29	- 1.5 = a_0 .

For the very centre of the field, therefore, the definition with the parabolic grating is much better than with the spherical grating for small angles of incidence, but inferior to it for large angles of incidence. It is very nearly the same throughout as with a spherical grating on the Rowland mounting.

Substituting for a the value $a \tan^2 i$ in (38), we find for the limiting value of β ,

$$\beta_{\max.} = \sqrt[3]{\frac{2\lambda}{W a \tan^2 i}}.$$

Hence, for the grating previously considered ($W=73$ mm.),

$$\left. \begin{aligned} \beta'_{\max.} &= 0.100, & \text{for } i &= 5^\circ \\ \beta'_{\max.} &= 0.063, & \text{,, } i &= 10^\circ \\ \beta'_{\max.} &= 0.049, & \text{,, } i &= 15^\circ \\ \beta'_{\max.} &= 0.024, & \text{,, } i &= 60^\circ \end{aligned} \right\} \quad . \quad . \quad (56)$$

which are likewise very near the limiting values for $\beta_{\max.}$ for the Rowland mounting.

When we consider the effect of the unsymmetrical aberration in parts of the field away from the axis of the grating the results are much less satisfactory. The general expression for Z_B for small values of $\kappa (= \theta)$ becomes

$$\begin{aligned} Z'_B &= -\frac{1}{2} \frac{y^3}{\rho^2} \tan \kappa \cos i (1 + \cos i) \\ &+ \frac{1}{8} \frac{y^4}{\rho^3} (i + \cos i) \{ [\cos^2 i - (1 + 2 \cos i)^2 \tan^2 \kappa] \sec^2 \kappa - 1 \}. \end{aligned} \quad (57)$$

As in the case of (33), this may be put in the form

$$\begin{aligned} Z'_B &= Z_A \left[-\frac{4 \tan \kappa}{\cos i \sin \beta} + \sec^2 \kappa (1 - b \tan^2 \kappa) - \frac{1}{\cos^2 i} \right] \\ &= \frac{Z'_A}{\tan^2 i} \left(\frac{1}{\cos^2 i} - R \right), \quad . \quad . \quad . \quad . \quad . \quad . \quad (58) \end{aligned}$$

and therefore

$$R' = \frac{Z'_B}{Z'_A} = \left(\frac{1}{\cos^2 i} - R \right) \cot^2 i. \quad . \quad . \quad . \quad (59)$$

In order to compare directly the definition of the parabolic and spherical gratings at a given point in the field, it is also desirable to find the ratio between the maximum values of R and R' . Since the maximum value of Z_B is always positive (for positive values of κ), and the maximum value of Z'_B is

always negative (for the same values), we have

$$R'' = \frac{Z'_B}{Z_B} = \frac{R_2 - \frac{1}{\cos^2 i}}{R_2} \quad \dots \quad (60)$$

By the aid of the preceding Table III., which gives the values of R for a spherical grating, we can readily obtain the values of R' for a parabolic grating, and the values of R'' giving the ratio of the aberrations of the two. These values of R' and R'' have been computed for the fields S , S' , and S'' for a grating $\beta_3 = .05$, and for fields S''' and S^{iv} for a grating $\beta_2 = .025$. They are tabulated in Table IV. Comparing the values of R' of this Table with those of R of the preceding Table, we see that for values of i less than 45° the *variation* in the amount of the aberration at different parts of the field is *greater* with the parabolic than with the spherical grating, but that the absolute amount, as shown by R'' , is everywhere *less*. Above 45° the reverse of both propositions is true. Hence, for large values of i the parabolic grating is inferior in definition to the spherical grating, although the field is *more uniform*; for small values of i the definition of the parabolic grating is much the *better* at the centre of the field, but the field is far *less* uniform.

A further inspection of the values of R'' for these small values of i shows that for values of κ greater than 1° the parabolic grating is not much superior to the spherical grating in definition. The smaller the value of β , the less pronounced is this difference. Hence, for points any distance from the axis of the grating, there is not much to choose between the two.

The maximum field of good definition that can be obtained with any given grating is found as before by putting

$$Z'_{B(\max)} = -\frac{\lambda}{4} \quad \dots \quad (61)$$

Since in this case the maximum aberration is always negative, this gives for the value of $\kappa_{\max}$,

$$\begin{aligned} \tan \kappa_{\pm m} = \pm \frac{1}{2a_1} & \left(\frac{\lambda}{W \sin^2 \beta} - \frac{a \sin \beta}{2} \tan^2 i \right) \left[1 - \frac{1}{8} \frac{ab \sin \beta}{a_1^2} \right. \\ & \left. \left(\frac{\lambda}{W \sin^2 \beta} - \frac{a \sin \beta}{2} \tan^2 i \right) \right] \quad \dots \quad (62) \end{aligned}$$

For $i=0$ (field S) this becomes simply

$$\tan \kappa = \pm \frac{\lambda}{4W \sin^2 \beta} \left[1 - \frac{9}{16} \frac{\lambda}{W \sin \beta} \right] \quad \dots \quad (63)$$

Aberration of the Concave Grating.

TABLE IV.

Grating.	$\beta_2 = 0.05.$											
	$S, (i=0^\circ).$						$S'', (i=15^\circ).$					
	$R_1', *$	$R_2', *$	$R''.$	$R_1', *$	$R_2', *$	$R''.$	$R_1', *$	$R_2', *$	$R''.$	$R_1', *$	$R_2', *$	$R''.$
0° 0' ...	+1.00	+1.00	± 0.000	+ 1.00	+ 1.00	- .008	+ 1.00	+ 1.00	-0.072	+ 1.00	+ 1.00	-0.323
5' ...	...	...	-0.101	+ 15.7	- 13.6	-0.108	+ 2.62	- 0.61	-0.168	+ 1.78	+ 0.22	-0.470
10' ...	...	...	-0.188	+ 31.5	- 29.4	-0.195	+ 4.35	- 2.34	-0.252	+ 2.61	- 0.61	-0.566
15' ...	...	...	-0.261	+ 47.1	- 45.1	-0.267	+ 6.07	- 4.07	-0.320	+ 3.44	- 1.44	-0.633
30' ...	$\pm \infty$	$\pm \infty$	-0.411	+ 92.5	- 90.1	-0.417	+ 11.1	- 9.03	-0.461	+ 5.82	- 3.82	-0.745
1° ...	...	...	-0.583	+ 184	- 181.0	-0.589	+ 21.1	- 19.0	-0.622	+ 10.66	- 8.64	-0.842
2° ...	...	...	-0.740	+ 368	- 364	-0.745	+ 41.4	- 39.1	-0.767	+ 20.4	- 18.3	-0.910
3° ...	...	...	-0.813	+ 553	- 546	-0.818	+ 61.8	- 59.2	-0.834	+ 30.1	- 28.0	-0.944
5° ...	...	...	-0.890	+ 927	- 909	-0.894	+ 103	- 99.1	-0.901	+ 49.7	- 47.3	-0.961
...	...	...	...	...	...	...	...	...	...	...	...	...

* The sign of Z_0' is negative. The signs of Z_B' are therefore opposite to those of R_1' .

Since the second term of the quantity within the brackets is always small compared to unity for any practicable values of β , it may in general be neglected. The only difference therefore between the expression (62) for a parabolic grating and the expression (40) for a spherical grating is in the presence of the factor $\tan^2 i$ as coefficient of the term in $\sin \beta$. This will increase the value of κ for values of i less than 45° , and decrease it for values above 45° . Thus, for the same grating before considered ($W \cong 73$ mm., $\beta = .011$), we find for κ_m

$$\left. \begin{aligned} \kappa_m &= \pm 54', & \text{for } i &= 0 \\ \kappa_m &= \pm 56', & \text{,, } i &= 15^\circ \\ \kappa_m &= \pm 2^\circ 10', & \text{, } i &= 60^\circ \end{aligned} \right\} \quad . \quad . \quad . \quad (64)$$

Comparing these results with those of (40), we see that the effect of parabolizing the surface of this given grating is to increase the field by only about 15 per cent. for small values of i , and decrease it about the same amount for large values of i .

For larger angular apertures the field of good definition is reduced, as we have already seen, in the inverse ratio of the square of the aperture. Thus for a grating of angular aperture $\beta_3 = .05$ the field of good definition is only a little over $5'$ ($2' 35''$ each side of the axis) for normal incidence, and it would become zero for a value of i a little less than 15° [see (55)].

For any given angular aperture β the value of κ_m varies also very nearly inversely as the linear aperture $2W$. Thus, if we decrease the size (aperture and focal length) of the grating one half, we shall double the field. Or, keeping the field the same, we may increase the maximum permissible value of $\beta_{\max}$. This will vary for any given value of i inversely as the cube root of the linear aperture [see (55)], the field being confined to the image on the axis of the grating. For any smaller values of i the field will be increased, and since κ_m decreases only as the square of the angular aperture, it follows that by decreasing the linear aperture we may simultaneously increase the value of $\beta_{\max}$ for large values of i , and the values of κ_m for small values of i . Thus, if we decrease the linear aperture of the grating considered in (54) and (64) to, say, 2 cm. ($W = 10$ mm.), we may increase permissible angular aperture 2β to nearly 0.10 ($\beta \cong .05$) for $i = 60^\circ$, and increase the angular field to about $0^\circ.6$ ($\kappa_m = 19'$ to $\kappa_m = 17'$) for $i = 0$ to $i = 30^\circ$.

It will also be instructive to compare the preceding results for the limiting values of κ_m and β_m from another point of

view, *i. e.*, from the standpoint of a given resolving-power. As the writer has previously shown*, the resolving-power of a grating of any form may be stated without reference to the ruling of the surface, in terms only of its aperture and total spectral deviation of the diffracted ray. So expressed, we have

$$r = \frac{2W}{\lambda} (\sin i - \sin \theta), \quad . \quad . \quad . \quad (65)$$

or for the A type of mounting above considered,

$$r = \frac{2W}{\lambda} \sin i. \quad . \quad . \quad . \quad (66)$$

Substituting this in (54) we get

$$\beta_{\max.} = \sqrt[3]{\frac{4 \sin i}{r a \tan^2 i}} = \sqrt[3]{\frac{1}{r} \frac{4}{(1 + \cos i) \sin i}}. \quad (67)$$

The general statement made in the previous paper (*Astro-physical Journal*, vol. iv. p. 61) is therefore rigorously correct only for certain values of *i*. It is, however, well within the truth for the maximum values of *i* and *r* there mentioned, viz., *i* = 10° and *r* = 30,000. For these values of *i* and *r* we get for $\beta_{\max.}$

$$\beta_{\max.} = \sqrt[3]{0.000387} \cong 0.073,$$

or for the angular aperture at the principal focus *o'*,

$$\beta_0 \cong 2\beta_{\max.}(1 + \cos i) \cong \frac{1}{3.5} \quad (\text{for } i < 15^\circ),$$

which is even larger than the limit $\beta_0 = \frac{1}{5}$ there imposed.

The maximum linear aperture, however, must in this case not exceed 95 mm. ($W_{\max.} = 48$ mm.), if *r* is taken as 30,000 for $\lambda = 5500$ tenth-metres, as before.

Conversely, if we use a maximum angular aperture of 1/5 at the principal focus, $\beta \cong 0.05$, and we can, without injury to the definition, use a maximum resolving-power $r_{\max.}$ equal to

$$\frac{4}{\sin^3 \beta (1 + \cos i) \sin i} \quad . \quad . \quad . \quad (68)$$

$$= \frac{3200}{(1 + \cos i) \sin i}, \quad . \quad . \quad . \quad (68a)$$

or for *i* = 10°, as in the case just considered,

$$r_{\max.} \cong 93,000,$$

* *Phil. Mag.* vol. xliii. p. 319.

and the maximum linear aperture for wave-length $\lambda = 5500$ tenth-metres is correspondingly

$$2W_{\max.} = 294 \text{ mm.},$$

from which it appears that if we use angular apertures as large as $1/5$ only, we can use linear apertures as large as previously assumed, *i. e.* 30 cm., and resolving-powers at least three times larger (90,000 as against 30,000), without injury to definition, *provided*, first, we use only small angles of incidence, and provided, second, we confine ourselves to points *very close* to the axis of the grating. My preceding statement was incomplete, in not more carefully pointing out these restrictions.

It is the violation of the above conditions, particularly the second, that has caused trouble in the experimental work referred to at the beginning of this paper. If we express the limiting value of κ in terms of the resolving-power, we shall have from (62) and (66) (neglecting the small variation in r with κ),

$$\text{tang } \kappa_{\max.} = \frac{\sin i}{ra_1 \sin^2 \beta} - \frac{1}{4} a_0 \sin \beta. \quad . \quad . \quad (69)$$

For any given value of i the field of good definition will depend on the value of r . For $r = r_{\max.}$, as defined by (68), $\kappa = 0$; *i. e.*, the only well-defined line is that at the centre of the field. *To obtain field we must sacrifice resolving-power or reduce the angular aperture.*

If we make $r = 30,000$ and retain the same values of β and i as already used, we find for κ

$$\kappa_{\max.} = .0008 \simeq 3'.$$

In order to increase the total field of good definition to 1° , we must decrease r to about 4000 units, *i. e.* to the resolving-power of a single prism-spectroscope of about 4 cm. aperture.

If we put (66) in the form

$$\text{tang } \kappa_{\max.} = \text{tang } i \left(\frac{1}{r \sin^2 \beta (1 + \cos i)} - \frac{1}{4} \sin i \sin \beta \right) \quad (69a)$$

we see that for a given resolving-power the maximum field of good definition increases very nearly proportionally to $\text{tang } i$. Hence, if large fields are necessary or desirable, large angles of incidence are advantageous, a point that is also clearly brought out, although not in such definite proportional form, by the individual examples already considered in (40) and (64). These examples also show that in such cases spherical gratings are *better* than parabolic gratings. But if large angles of i are used, large values of β are not

permissible with either spherical or concave gratings [see (38), (55), (56), and (67)],—although in this case also the spherical grating has a decided advantage,—unless again we use very small resolving-powers.

We are thus confronted by two difficulties in the use of this form of objective spectroscope, and it seems impossible to overcome both simultaneously, except by sacrificing resolving-power*. This general conclusion is quite in harmony with the experimental results referred to at the beginning of the paper, with the single exception of those obtained by Mitchell. These require some further consideration. As already stated, Mitchell used a grating having a linear aperture of 146 mm. and a semi-angular aperture, β , of $\cdot 073$. He used the B and F types of mounting, and worked with plates about 127 mm. long. The value of i for the B mounting was about 15° , and the value of u [from (6a)] would therefore be about 508 mm. The plates (bent to a radius of about 254 mm.) would cover a field of over 14° , or 7° on each side of the axis of the grating. The semi-angular aperture β here exceeds the maximum limit imposed by the usual criterion of good definition for a grating of this size by more than 50 per cent. [see (56)]. Hence, judged by this standard, there would be no point in the field for which the definition would be good. Mitchell in his paper, however, states that the plates showed "excellent definition," presumably over the entire field. Either the preceding conclusions are radically wrong, or Mr. Mitchell's standard of "excellent definition" is decidedly different from that imposed by (37). As I shall attempt to show, the latter is not an impossible or even improper explanation of the discrepancy. Definition is, as has already been stated, a function of psychological and physiological phenomena, as well as of strictly physical conditions. As such it is strictly relative, and depends largely for its quantitative estimation on the basis of comparison employed. The latter again differs, or should differ, with the instrument used. If this basis of comparison is less exacting than it should be, then an aberration amounting to very much more than a quarter

* Certain special solutions of the general equations (42), (43), and (48) have been found which theoretically enable us to greatly reduce the value of Z' . No such solution has yet been found, however, that does not require us to satisfy certain mechanical and optical conditions of construction and mounting that are either impossible or highly impracticable. Some of these solutions, and some other points of interest that have been developed in connexion with them, will be given in another paper, "On the most Efficient Form of Surface for Concave Gratings and Concave Mirrors for Spectroscopes," now in course of preparation.

of a wave-period may pass unnoticed. On this point Lord Rayleigh* very pertinently remarks:—"The best results will be obtained with an aperture . . . giving an extreme aberration of from a quarter to half a period . . . with an increased aperture," (with proportionally increased aberration), "*aberration is not so much a direct cause of deterioration as an obstacle to the attainment of that improved definition which should accompany the increase of aperture.*" In other words, there will be no discrepancy between Mitchell's standard of "excellent definition" over a 7° field and the conditions of equation (37), if the resolving-power of his instrument did not exceed 210 units†. *On the same basis of comparison*, the definition would be equally "excellent" with an instrument of the resolving-power actually used (about 85,000), although the aberration would be enormously increased. In this connexion it should be noted that the character of the spectra photographed was such that it would be natural to apply a less rigorous standard of definition to the results than would be applied, for example, to solar-spectrum plates. As the writer has previously shown‡, a resolving-power of not more than 2000 is required to obtain as good definition on bright-line hydrogen-spectra as is obtained with a power of 400,000 on fine-line absorption-spectra from the sun.

Fortunately there is a much more definite and impersonal standard of performance of an optical instrument than that of definition. This is the standard of accuracy or metrological power, which has recently been investigated in some detail by the writer§. As has been there shown, the limiting metrological power ϵ of an instrument is about one-fifteenth as great as the limit of resolving-power α , or the limit of accuracy A is

$$A \simeq 15r.$$

For the grating used by Mitchell the value of A is about 1,270,000, and the limit of metrological power ϵ is therefore about 0.004 tenth-metre. This should be the mean or "brute" error (not the probable error) of an individual *setting* on a spectrum-line, with a grating of this resolving-power used under the best optical conditions.

* "Influence of Aberration," Phil. Mag. vol. viii. p. 403, Nov. 1879.

† Obtained by putting $\kappa=7^\circ$ and $\beta=.073$ in (69), and solving for r .

‡ Phil. Mag. vol. xliii. p. 336.

§ "On the Optical Conditions required to secure Maximum Accuracy of Measurement in the Use of the Telescope and Spectroscope," Astrophysical Journal, vol. xvi. p. 267; vol. xvii. pp. 1 and 100, Dec. 1902 and Jan.-Mar. 1903.

An examination of the two tables of wave-length determinations made by Mitchell with his instrument shows that the mean error of measurement was very much greater than this. In the case of the wave-length of the series of hydrogen-lines in the spectrum of Rigel, the mean error of *three* determinations on each of the twenty-two lines (H_{β} to head of series) as compared with the theoretical wave-lengths calculated from Balmer's law is about 0.120 tenth-metre. No data are given from which the individual errors of setting on each line can be determined, but they must necessarily have been *greater* than this. In the case of the wave-length measurements of the lines in the nebula of Orion the mean error as determined by a comparison of the two series of measurements is about 0.480 tenth-metre. Here also each determination tabulated is presumably (although it is not so stated) the mean of several settings, and if so, the mean error of individual settings would again be much greater than that given. Even without any allowance for such an increase, the errors of setting as they appear are respectively about thirty times and one hundred and twenty times what they would be if the full optical power of the grating were being realized.

In the case of Table II. of Mitchell's paper, there is further evidence that the grating was not giving the results which one of this aperture should attain under proper optical conditions. There is a group of three well-separated lines in the nebula of Orion which, according to Wright, have wave-lengths $\lambda = 3970.24$; 3967.69 ; and 3965.1 ; and intensities of 5, 3, and 1 respectively. These lines were clearly resolved and accurately measured to about 0.01 tenth-metre by Wright with the Mills spectrograph, which has a theoretical resolving-power in this region of the spectrum practically equal to that of Mitchell's instrument. On Mitchell's list only one of these lines—the brighter one—appears at all, and the error of measurement on this amounts to between 0.6 and 0.7 tenth-metre.

These comparisons of Mr. Mitchell's work are instituted, not because of any wish to criticize the work itself, *per se*, but only to show that the general conclusions reached in this paper by the mathematical analysis are not incompatible with his experimental results. There now seems to be no question but that both Dr. Poor and Mr. Mitchell, in common with Mohler and Daniel and Jewell, have considerably overrated the advantages of the direct concave-grating spectroscope, or, as the writer prefers to call it, the concave objective-grating spectroscope. I am not myself in a position to

criticize these views very strongly, for the reason that, as pointed out at the beginning of the paper, I appear to be the first to have published the general theory of the instrument and to have advocated its use. I can only say that since then my own views have been decidedly modified by a further experimental and mathematical investigation of the capabilities of the instrument. After the publication of Poor and Mitchell's first paper, I pointed out some of the objections to the instrument as used by them in photographing stellar spectra, on the score of its low light efficiency as compared with other forms of stellar spectrographs, particularly the transmission (wire) objective grating*. It was not until later, in the use of the instrument in solar eclipse work, that its limitations in the directions of resolving-power and photographic field were clearly recognized. This led me, as already stated, to the present investigation. It now appears from this that perhaps the most serious objection of all to the instrument is in its lack of accuracy. This point has been very briefly touched upon in this paper and in the one on "Conditions of Accuracy, &c." † already referred to, but will be more fully developed in a paper dealing with a discussion of Frost's and Humphrey's results on the flash-spectra ‡.

In conclusion we may quote with a considerable degree of pertinency to the present question the remark of Schuster with reference to the prism-spectroscope. He says ('Enc. Brit.' vol. xxii. p. 373):—"The dispersion and magnifying-power of a prism can be considerably altered by a change in its position, and a knowledge of this fact is of great value to an experienced observer. The use of a prism in a position different from that of minimum deviation is, however, *a luxury which only those acquainted with the laws of optics can indulge in with safety.*" Change "prism" to grating, and "minimum deviation" to Rowland's mounting, and the applicability of Schuster's statement to the case we have been discussing is very evident.

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1901-1902.

* See "Note on the Use of the Grating in Stellar Spectroscopic Work," *Astrophysical Journal*, vol. vii. p. 198; also "The Objective Spectroscope," *ibid.* vol. iv. p. 75 *et seq.*

† *Astrophysical Journal*, vol. xvi. pp. 286-288. On p. 288 there is a typographical error which is corrected in the March number of the *Journal*. The values of $\theta_{\max}$, as given in the December number are ten times too large, *i. e.*, in place of $2^{\circ} 20'$ and 6° the values of $\theta_{\max}$, should be $0^{\circ} 22$ and $0^{\circ} 6$.

‡ "On Measurements of Wave-Length with the Concave Grating Objective Spectroscope," *Astrophysical Journal*, vol. xviii. p. 1, July 1903.