V.—CRITICAL NOTICES.

An Essay on the Foundations of Geometry. By **BERTRAND A. W. RUSSELL**, M.A., Fellow of Trinity College, Cambridge. Cambridge: At the University Press, 1897. Pp. xvi., 201.

THIS book is in the main an attempt to determine the logical relations of the most elementary constituents of space. For this purpose Mr. Russell makes an extensive use of the modern mathematical developments of Geometry; and it is, therefore, very difficult to criticise his results without a considerable knowledge of Mathematics. This, I must confess, I do not possess; and, therefore, any statements I may make, involving mathematical technicalities, must be understood to be liable to correction. It seems impossible, however, to avoid such statements, since the arguments by which Mr. Russell endeavours to establish that certain axioms are *a priori* and others merely empirical always involve some reference to purely mathematical notions. And the chief value of the book undoubtedly lies in these discussions, since the purely philosophical questions as to how in general the *a priori* is to be distinguished from the empirical, and as to the relation of space to other objects of knowledge, are not treated at any great length; while the treatment of them, such as it is, seems liable to grave objections.

The book is divided into an introduction and four chapters.

The introduction (pp. 1-6) defines the problem with particular reference to Kant, attempting to distinguish more clearly than he had done between the *a priori* and the subjective, and announcing that the former notion alone will be used in the following discussion.

Chapter i. (pp. 7-53) is entitled 'A Short History of Metageometry'. In this Mr. Russell distinguishes three periods. In the first of these, which is very briefly handled, it was proved by Lobatchewsky and Bolyai that a system of geometry, in which the axiom of parallels was denied, could be consistently worked out. In the second period the chief names are Riemann and Helmholtz, whose object was largely philosophical. This period is marked by the use of analytical methods, particularly by the analytical conception of 'measure of curvature,' invented by Gauss, but first extended by Riemann and Helmholtz to a manifold of more than two dimensions, and by the discovery of a second system of Metageometry. The conception of measure of curvature as developed in this period is all-important for Meta-

geometry, since it makes it possible to define the quality which distinguishes non-Euclidean spaces from the Euclidean as an intrinsic property of the space in question. The third period finally is marked by the invention of Projective Geometry, a department of mathematics not hitherto discussed by philosophers, but which Mr. Russell considers to be of the utmost importance for the philosophy of space. He accordingly treats this period at greater length, endeavouring to show that Projective Geometry deals exclusively with the qualitative properties of space, and involves no reference to quantity. In this period the chief names are Cayley and Klein, the latter of whom invented a third system of Metageometry.

Chapter ii. (pp. 54-116) contains a critical discussion of 'some previous philosophical theories of Geometry'. In this chapter Mr. Russell contends that Kant's arguments suffice to prove the *apriority* of some form of externality, but are insufficient to determine which of the geometrical axioms are necessary to this ; and he points out that Lotze's attempted refutation of Metageometry is entirely based on mathematical mistakes. But by far the greater part of the chapter is occupied by a refutation of the views of Riemann, Helmholtz and Benno Erdmann, who wished to maintain that all the axioms peculiar to geometry, that is, all those that involve other conceptions than those of quantity, are empirical.

Chapter iii. (pp. 117-177) is divided into two sections, in the first of which Mr. Russell attempts to formulate the axioms of Projective Geometry, and to deduce their necessity from the conception of a form of externality. He finds three axioms which are thus sufficient and *a priori*. In the second section he does the same for Metrical Geometry, finding the three corresponding *a priori* axioms, but emphasising the fact that for Metrical Geometry a fourth axiom is needed, which, however, he maintains to be of an empirical nature. The *a priori* axioms, whether in their projective or their metrical form, are involved in Euclidean and non-Euclidean Geometries alike. The fourth purely metrical axiom serves to distinguish the one from the other.

Chapter iv. (pp. 178-201) contains a short survey of 'Philosophical Consequences'. The subjects discussed in it are two. The first of these is the general relation of a 'form of externality' to 'experience,' in connexion with which Mr. Russell attempts to show that time alone is not sufficient for experience. The second subject of discussion is the contradictions in space, of which Mr. Russell enumerates three, deciding that each of them may be overcome if we interpret space as consisting merely of relations between unextended atoms that are in no sense in space.

Mr. Russell's general philosophical position does not seem very much to affect the main value of his book. He does not indeed discuss it at any great length, referring us constantly for a fuller treatment of important points to the Logics of Bradley and Bosanquet. Only one philosophical distinction is absolutely

essential to his purpose—that between the *a priori* and empirical; the rest of his philosophical apparatus, in which a ‘form of externality’ is the central conception, may be regarded merely as establishing and explaining this distinction.

In fact, however, they seem rather to confuse the issue. Mr. Russell tells us, to begin with, that his test of apriority will be purely logical: ‘Would experience be impossible if a certain axiom or postulate were denied?’ (p. 3). This test, he considers, excludes ‘any psychological implication’ from the word *a priori*. But the meaning of the test obviously depends on what is to be meant by the possibility of experience; and Mr. Russell seems actually to use this conception in such a way as to incur both the objections which can be urged against Kant’s distinction between *a priori* and empirical, namely (1) that the distinction is not absolute, (2) that the *a priori* is confused with the object of psychological experience.

(1) That the distinction is not absolute would seem to follow from the twofold nature of the test which Mr. Russell gives us for the *a priori*. He tries to show not only that propositions which are presupposed in any particular science are *a priori*—a perfectly clear and simple test—but also to deduce the necessity of these from the necessity for experience of a ‘form of externality’. Necessity, as he holds to be ‘shown by modern logic,’ always involves a ground of necessity (p. 4). But this ground must itself either be simply categorical, or else it must itself be necessary and require a further ground. In the former case we are actually trying to deduce an *a priori* proposition from one that, as categorical, is merely empirical; in the latter, which Mr. Russell seems in the end inclined to accept, we must either allow an infinite regress of necessary propositions, and thus never reach the absolutely *a priori*, or else we must accept the view that knowledge is circular, and shall in the end return to the proposition from which we started as empirical, as being itself the ground of necessity of the *a priori*, and therefore itself as much *a priori* as the latter. Mr. Russell seems actually to accept this latter view (pp. 57-60)—a view which renders his logical criterion nugatory, since it asserts that that which is presupposed in the empirical equally and in the same sense presupposes the empirical. To put the matter shortly, to show that a ‘form of externality’ is necessary for the possibility of experience, can only mean to show that it is presupposed in our actual experience. And this can never prove that no experience would be possible without such a form, unless we assume that our actual experience is itself necessary, i.e., that no other experience is possible. (2) The only escape from this self-contradiction would seem to be by way of putting the ground of necessity in some psychological fact. And this is the subterfuge to which Mr. Russell seems actually to be driven when we find him defining his deduction from the possibility of experience as a proof that the falsehood of the judgment in

question would only be effected 'by a change which should render some branch of experience formally impossible, i.e., inaccessible to our methods of cognition' (p. 60). Similarly (p. 179) he asserts: 'The ground of necessity, we may safely say, arises from the mind'. Now that which is 'inaccessible to our methods of cognition' would seem only to mean that which we cannot know; it cannot imply that the judgments in question cannot be true. But apart from a proof that they cannot be true (in which case of course they cannot be *known* in the sense in which knowledge is distinguished from belief), it would seem that the only possible way of showing that they cannot be known must be simply a psychological inquiry into the conditions that are necessary for the production of actual beliefs. The same Kantian fallacy is betrayed again very clearly on page 135, where Mr. Russell asserts that 'what is purely intellectual cannot change'. Here by 'what is purely intellectual' must be meant either what is distinguished by certain psychological marks, or what is already known to be presupposed in experience. But in the former case the fact that it cannot change can be only known empirically, whereas in the latter 'intellectual' simply means what is *a priori*, as presupposed in experience, and therefore cannot be used to confirm this distinction.

Mr. Russell's professed *deduction* of his *a priori* axioms from a necessary 'form of externality' is therefore necessarily futile; since by the necessity of the form of externality can only be meant that it is presupposed in actual experience, which he has already shown his axioms to be. But there are two points in his actual argument for the necessity of such a form which seem to call for particular notice, as illustrating the psychological implications which must be introduced in order to give a meaning to such a deduction.

Mr. Russell recognises (p. 184 foll.) that it is essential for his purpose to contend that, if knowledge is to be possible, there must be a diversity 'not only of conceptual content but of existence'; since a form of externality means a form of relation between several existents, and not merely between diverse contents. He has therefore to argue that the mere diversity of content, which we may admit with him to be implied in any judgment, is itself impossible unless there be also a diversity of existents. His argument to prove this point seems to be purely psychological. For he assumes as a premiss that 'knowledge must *start from* perception' (p. 184). These words inevitably suggest that he is concerned only with the *genesis* of knowledge—a point of view which he has himself outruled as irrelevant to the discussion in hand. Unless he does mean this it would seem he must deny that any kind of cognition except perception is possible; since otherwise what is necessary for perception would not be necessary for that other kind. Now it is hardly possible he would refuse to distinguish between *e.g.* a mathematical cognition and

mere sense-perception—indeed he has already repeatedly done so. But in that case his whole argument vanishes; since it can at best only prove that a perception of different existents must precede, as psychological condition, any other kind of cognition.

Mr. Russell fails therefore to prove even that one form of externality, time, is necessary to the diversity implied in judgment; for he does not prove, what indeed he cannot prove, since he presupposes it, that there would be no diversity, unless we perceived diverse existents, but only that we could not come to know of it. But it is necessary for him to go farther, and to maintain that time alone is not sufficient. And here it seems possible to join issue with him on a mere question of fact. For he assumes that the complexity of an object presented in time alone must be purely adjectival, *i.e.* presumably containing a diversity of content only, not of existents. But, in fact, nothing seems more certain than that my mental state at any moment must be regarded as containing a diversity of existents, which are certainly not in space. The pleasure I have simultaneously with the cognition of a certain object is not merely diverse in content from that cognition; for this would be equally true of it when the cognition was repeated without the pleasure. There is therefore some other diversity between a cognition that is present and a pleasure that is present than the mere diversity between pleasure and cognition. And it seems impossible to distinguish this additional diversity from that diversity of existence which distinguishes different things in space, or which distinguishes my pleasure at one moment from my pleasure at another. It seems then certain that diverse mental things can both exist and be known to exist simultaneously, although they are not in space, and hence the argument that space is necessary for the experience of diversity is reduced to the psychological argument that a cognition of things in space is necessary either as cause or as simultaneous condition for the cognition of our own mental states.

There is one other point in Mr. Russell's philosophical conclusions about space which seems to rest on the same confusion about the relation of logical presupposition. This is his treatment of the antinomies. Mr. Russell thinks that these can all be avoided if 'space is regarded, so far as it is valid, as only spatial order' (p. 196) or 'as relations between unextended material atoms' (p. 198). 'Empty space, which arises by an inevitable illusion out of the spatial element in sense-perception, may be regarded, if we wish to retain it, as the bare principle of relativity, the bare logical possibility of relations between diverse things' (*ib.*). I think we might retort on Mr. Russell the dictum which he quotes from Lotze that 'contradictions in a necessary subjective intuition form as great a difficulty as in anything else' (p. 196); for it seems impossible to distinguish a 'necessary

subjective intuition' from his own '*unavoidable psychological illusion*': the only possible distinction lies in the word '*illusion*,' which, however, is not a psychological term, but is distinguished from '*intuition*,' only as implying that its object is false—the very thing which the '*intuition*' presupposed and was invented to explain. But we can not only say that such an explanation fails to avoid the difficulty; it seems that the reason why it does fail is that it involves a definite logical error. For it is only, if our intuition of space be true, that we have any reason to believe in these relations between unextended atoms; in other words, empty space and spatial order are equally presupposed in experience, and if we reject the one, we have no longer any reason for accepting the other. The same result seems to follow from a consideration of the other phrase by which Mr. Russell tries to explain away empty space. 'A mere name for the logical possibility of spatial relations' (p. 197) can surely mean nothing but 'that which renders spatial relations possible'; and hence if there be no empty space, it would seem to follow that there can be no spatial relations.

This point seems to be closely connected with another philosophical error which constantly appears throughout Mr. Russell's mathematical discussions—an error which seems to arise from a certain ambiguity in the use of the term '*relativity*'. Thus Mr. Russell states on page 160 '*Position is not an intrinsic, but a purely relative, property of things in space*. If there could be such a thing as absolute position, in short, metrical Geometry would be impossible.' This statement Mr. Russell seems to regard as equivalent to the statement that all points are qualitatively similar; and he seems to base it upon the fact that 'so long as we leave matter out of account, one position is perfectly indistinguishable from another' (p. 77). A similar argument is used with regard to the relativity of magnitude. '*Magnitude*' he says 'is nothing apart from comparison' (p. 87); and again 'To speak of differences of magnitude in a case where comparison cannot reveal them, is absurd' (p. 154). But it would seem that this is a purely psychological argument: it assumes that nothing is distinct unless it has some property which will enable the mind to distinguish it. Mr. Russell himself seems sufficiently to expose the fallacy of this assumption in a note on page 165; where he says 'although measurement and the judgment of quantity express the result of comparison, yet the terms compared must exist before the comparison'. The principle implied here seems directly to contradict such a statement as we find only two pages earlier: '*Positions exist only by virtue of other positions*' (p. 163). The true view seems rather to be, that though positions have that qualitative similarity which is necessary for the axiom of free mobility, yet this in no way prevents them from having *also* a qualitative and intrinsic difference, though this perhaps we could not have discovered apart from the qualitative differences of

other things. As it is, we certainly have discovered it: we do distinguish different positions, as Mr. Russell says, in 'intuition'; and to assign a cause for the existence of this intuition does not impugn but rather presupposes its validity.

The assertion of absolute position in this sense obviously does not involve 'an action of mere space, *per se*, on things,' which Mr. Russell seems to regard as equivalent to it (p. 151). And thus the necessary connexion which he asserts between the axiom of free mobility and the relativity of position disappears. The assertion of this connexion seems, indeed, to be partly due to the way in which this first of Mr. Russell's *a priori* axioms is stated—a form which is indicated by the name. For Mr. Russell conceives this axiom, as distinguished from the corresponding projective axiom of the homogeneity of space, to introduce 'an entirely new idea,' 'namely, the idea of Motion'; for, he says, our results cannot be obtained 'without at least an ideal motion of our figures through space' (p. 149). It seems desirable to protest against this conception of the movement of spatial figures. It is implied in the view accepted by Mr. Russell, that superposition is the criterion of equality between spatial figures. But, in fact, 'the ideal motion' by which superposition is affected implies that we already know the figures in question to be equal. For to move a spatial figure can mean nothing more than to construct a figure equal to it in another part of space; and this can not be done, unless we know immediately when two figures in different parts of space are equal. This, therefore, 'that there are equal figures in all parts of space,' would seem to be the true form of the so-called 'Axiom of Free Mobility,' which is presupposed alike in the motion and in the measurement of things which occupy space. To conceive spatial figures as themselves moving, we must conceive them as things, whose identity in motion means no more than that they successively occupy equal spaces. For the equality of these spaces themselves there is then no possible criterion. Its necessity is proved by the fact that it is presupposed alike in the assumed equality or inequality of any things whatsoever that can occupy space.

The only other point in Mr. Russell's book on which I desire to touch is the possibility of non-Euclidean spaces. Mr. Russell agrees 'with Helmholtz in thinking the distinction between Euclidean and non-Euclidean spaces empirical' (p. 73). His arguments in favour of this contention seem to involve a considerable knowledge of mathematics, and yet his result is philosophical. Philosophers therefore would seem to be placed in an awkward dilemma: either they must accept the result merely on the author's authority—a proceeding not generally to be recommended in philosophy; or else they must be liable to be told that their criticisms are vitiated by mere mistakes in mathematics. In view of this dilemma, I shall only venture to mention some points which make Mr. Russell's arguments appear to me to be in-

conclusive. A fuller explanation of these points might, perhaps, make Mr. Russell's work more useful for philosophy. As it is it seems to be too mathematical for philosophers, even if it is not also too philosophical for mathematicians.

The first point, on which I think a fuller explanation would be desirable, may be raised in connexion with the argument on page 85. Mr. Russell here asserts that 'There is no qualitatively similar unit in the three kinds of space, by which quantitative comparison can be effected'. 'A debt of £300' he goes on 'may be represented as the asset of -£300, and the height of the Eiffel Tower is +300 metres; but it does not follow that the two are quantitatively comparable.' This analogy seems to me at least unfortunate. For though -£300 is not quantitatively comparable with +300 metres, yet, in order that -£300 may have any meaning at all (and Mr. Russell has admitted that one measure of curvature may be regarded as negative), there must be some conceivable positive quantity, £300 with which -£300 is quantitatively comparable, and what this is, in the case of a negative measure of curvature, Mr. Russell does not explain. His argument would seem rather to exclude the possibility of any such; and thus we should be left with a negative quantity which is a negative quantity of nothing. Moreover Mr. Russell repeatedly asserts that our actual space 'may have a very small space-constant' (p. 115), and in order that any meaning may be given to this assertion, it would seem that one space-constant must be quantitatively comparable with another. Unless one denotes a smaller, and the other a larger quantity of some common quality, none could be 'very small'. That two cannot be suffered 'to coexist in the same world' (p. 85) is surely utterly irrelevant, since we are quite well able to compare non-existent with existent magnitudes, provided only we know of what they are magnitudes.

My next point may perhaps involve nothing but a verbal error. It concerns the statement made on page 159, with reference to the geometry of non-congruent surfaces, that 'The fundamental formula, that for the length of an infinitesimal arc, is only obtained on the assumption that such an arc may be treated as a straight line, and that Euclidean Plane Geometry may be applied in the immediate neighbourhood of any point'. But on page 19, where Mr. Russell is explaining how we can find a 'sense of the measure of curvature in which it can be extended to space,' it appears that the formula given involves a reference to 'small arcs'. Now if by 'small' is here meant the same as by 'infinitesimal' in the former passage, it would seem either (1) that in any space which is defined by means of this formula 'Euclidean Plane Geometry may be applied in the immediate neighbourhood of any point,' or else (2) that on page 159 'the Plane Geometry of any congruent space' should be substituted for Euclidean Plane Geometry. The former alternative would seem to contradict Mr.

Russell's assertion of the qualitative dissimilarity of Euclidean and non-Euclidean spaces.

Again I find a difficulty in what seems to be implied on page 111, in the argument against M. Delboeuf, that in non-Euclidean spaces the space-constant may have a definite ratio to other magnitudes in the same space. If this is the case, it would seem it must be possible to measure the magnitude of the space-constant against other magnitudes in the same space; and measurement, as Mr. Russell often tells us, presupposes the straight line. But *the* straight line in any space means not that which is in common between Euclidean and non-Euclidean spaces, but *the* straight line of that space; and hence we should be measuring our space-constant by means of that which presupposes it. Or else, if by the straight line presupposed in measurement is meant the common quality of Euclidean and non-Euclidean straight lines it would seem that we could only use it in measurement if the things measured *are* nothing but a certain quantity of it; but in that case the space-constant of one space would differ only quantitatively from that of another, and there seems no room for the qualitative difference between spaces which Mr. Russell asserts to be the only one.

The conclusion which suggests itself to me from these considerations, is that all spatial magnitudes can only be regarded as magnitudes of some one quality. If we are to allow that the straight line has two qualities, one that is in common between Euclidean and non-Euclidean spaces, and another that is found in any given non-Euclidean space, then it seems that measurement must be impossible in such a space, since we have two *measures* of magnitude, which are consequently not comparable *inter se*. Measurement seems to presuppose only one measure of magnitude, and hence the truth of Euclid would be involved in the possibility of measurement, and so would be *a priori* in the same sense as the axiom of 'Free Mobility': since 'in Metageometry we have, while in Euclid we have not, a standard of comparison involved in the nature of our space as a whole' (p. 111). That the axiom of parallels is logically independent of the other axioms would seem to make nothing against this argument, since, if Mr. Russell's three *a priori* axioms differ from one another at all, they likewise must be logically independent of one another, in the sense that some part, at least, of one may be denied, without compelling us, on pain of contradiction, to deny the others also.

G. E. MOORE.