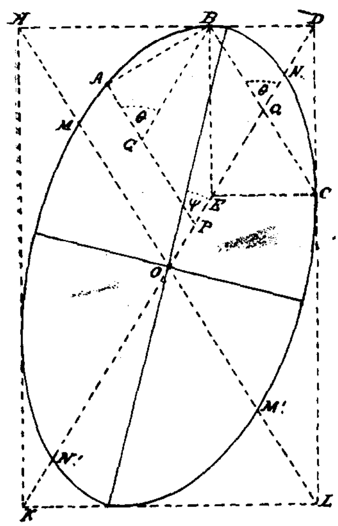
A NEWLY-DISCOVERED PROPERTY OF THE ELLIPSE.

BY L. D'AURIA.

In studying certain relations between physical quantities relatively to tidal rivers, the following problem of the ellipse occurred to the writer:

Given, of an ellipse, two tangents, $D\ B, D\ C$, at right angle with each other, the distances of their points of tangency, B, C, from their point of intersection D, and a third point A, determine the ellipse.



Being unable to find the solution of this problem in books of analytical geometry, the writer proceeded to solve it by the following method:

Let $A, B, C \ldots$ represent the desired ellipse, and upon the lines $B\ D$, $C\ D$ construct the rectangle $B\ D\ C\ E$. The diagonal $D\ E$ of such rectangle will determine the position of the transversal axis $N\ N'$ of the ellipse, and the other diagonal $B\ C$ will determine the inclination of the conjugate axis $M\ M'$, since $B\ C$ is bisected by $N\ N'$.

Now denote by (x_1, y_1) and (x_2, y_2) the co-ordinates of the points A and B, respectively; and by α, β the conjugate semi-axes $O\ N$, $O\ M$ of the ellipse. We have the following equations:

$$\alpha^2 y_1^2 + \beta^2 x_1^2 = \alpha^2 \beta^2; \quad \alpha^2 y_2^2 + \beta^2 x_2^2 = \alpha^2 \beta^2;$$

and can deduce

$$x_1^2 = x_2^2 - \frac{\alpha^2}{\beta^2}(y_1^2 - y_2^2). \tag{1}$$

Since the angular coefficients of the tangents BD, CD are in the present case equal to *one*, because $BQ = QC = QD$; and since the analytical expression of such coefficient for the tangent DB is

$$\frac{\beta^2 x_2}{\alpha^2 y_2},$$

we obtain the relation

$$\frac{\alpha^2}{\beta^2} = \frac{x_2}{y_2};$$

and substituting in equation (1), we have

$$x_1^2 = x_2^2 - \frac{x_2}{y_2}(y_1^2 - y_2^2). \tag{2}$$

Put $AB = l$; $AG = h$; $PQ = k$, will be

$$l^2 = h^2 + k(x_2 - x_1) - 2hk \cos\theta;$$

which gives

$$x_1 = x_2 + \frac{h^2 - l^2 - 2hk\cos\theta}{k}. \tag{3}$$

Put for brevity

$$\frac{h^2 - l^2 - 2hk\cos\theta}{k} = -m;$$

will be

$$x_1^2 = x_2^2 - 2mx_2 + m^2; \tag{4}$$

and comparing with equation (2), we have

$$2my_2x_2 - m^2y_2 = y_1^2x_2 - x_2y_2^2,$$

or

$$x_2 = \frac{m^2 y_2}{2my_2 + y_2^2 - y_1^2}, \tag{5}$$

which determines the centre O of the ellipse.

Solving the equations

$$\alpha^2 y_2^2 + \beta^2 x_2^2 = \alpha^2 \beta^2, \quad \frac{\alpha^2}{\beta^2} = \frac{x_2}{y_2},$$

for α^2 and β^2 will be found

$$\alpha^2 = x_2 (x_2 + y_2): \tag{6}$$

$$\beta^2 = y_2 (y_2 + x_2). \tag{7}$$

Since the value of x_2 is given by equation (5), the values of α and β can be determined by equations (6) and (7); so that denoting by a, b, the principal semi-axes of the ellipse, the well known equations

$$\left.\begin{aligned}(a+b)^2 &= \alpha^2 + \beta^2 + 2\,\alpha\,\beta \sin\theta\,; \\ (a-b)^2 &= \alpha^2 + \beta^2 - 2\,\alpha\,\beta \sin\theta\,;\end{aligned}\right\} \tag{8}$$

will afford the determination of a and b since the angle θ is known. In fact, solving equations (9) for a and b will be found

$$a = \tfrac{1}{2}\left\{ \sqrt{\alpha^2 + \beta^2 + 2\,\alpha\,\beta \sin\theta} + \sqrt{\alpha^2 + \beta^2 - 2\,\alpha\,\beta \sin\theta} \right\}$$

$$b = \tfrac{1}{2}\left\{ \sqrt{\alpha^2 + \beta^2 + 2\,\alpha\,\beta \sin\theta} - \sqrt{\alpha^2 + \beta^2 - 2\,\alpha\,\beta \sin\theta} \right\}$$

Now let ψ represent the angle formed by $O\,D$ with the principal semi-axes a of the ellipse, will be

$$\alpha^2 = \frac{a^2 b^2}{a^2 \sin^2\psi + b^2 \cos^2\psi} = \frac{a^2 b^2}{\sin^2\psi\,(a^2 - b^2) + b^2},$$

hence

$$\sin\psi = \frac{b\sqrt{a^2 - \alpha^2}}{\alpha\sqrt{a^2 - b^2}}. \tag{9}$$

Having determined the position of the centre of the ellipse by equation (5), the angle ψ will enable to describe the ellipse upon its principal axes by the well-known methods given in books of analytical geometry.

In the solution of the above problem, the relations (6) and (7) appeared to the writer to be of some importance. After little examination it became apparent that by summing them up, the following remarkable relation would result, viz.:

$$\alpha^2 + \beta^2 = (x_2 + y_2)^2,$$

and observing that

$$x_2 = O\,Q,\ y_2 = Q\,B = Q\,D,\ x_2 + y_2 = O\,D,$$

it follows

$$\sqrt{\overline{O\,M}^2 + \overline{O\,N}^2} = O\,D \tag{10}$$

This expresses a new property of the ellipse, namely:

If on the continuation of one of the conjugate semi-axes of an ellipse a point is taken whose distance from the centre is equal to the square root of the sum of the squares of such semi-axes, the tangents thrown to the ellipse from such point will be at right angle with each other.

The FUTURE WATER-SUPPLY OF PHILADELPHIA.

[*Abstract of the Report of* RUDOLPH HERING, *Engineer in charge of Surveys for the Future Water-Supply of the City of Philadelphia.*]

The final report of Rudolph Hering, engineer in charge of surveys for the future water-supply of the city of Philadelphia, was lately presented to the Water Committee of the City Councils. This report closes the three years' investigations, by the city, of all the available sources for the future water-supply of Philadelphia. It is a very voluminous document, and embodies what is regarded by scientific engineers as the most complete of any similar known work. The survey cost the city the aggregate sum of $80,000. Nearly 100 persons were engaged in the survey and sanitary investigation of the water-shed from which Philadelphia's future supply must come. Accompanying the report is a large number of plates, charts, and tables, showing the population of the surveyed territory, the topography of the country, the rain-fall of the various districts within a radius of 150 miles of Philadelphia, the location of proposed reservoirs, and a variety of other valuable detail information on the subject. The Councils had deferred consideration of certain propositions and of the subject of the water supply in general, during the summer months, awaiting the completion of this survey, and, with the information contained in the report now before them, it is expected that the subject will be taken up, and decided at an early date.

THE REPORT.

Mr. Hering says that the office corps has been engaged in computing the stream-flows of the Perkiomen, Tohickon and Neshaminy Creeks, in ascertaining the available storage in each of the respective valleys, in estimating its cost, and in arranging and compiling the tables, maps, and charts for the final report, and then continues:

In approaching the solution of the question as to where the city should go for better water when the Schuylkill River is no longer a fit source of supply, the definite conclusions arrived at in the previous reports were substantially as follows: