one P along the axis, and the other R at right angles to the axis, in a plane through the axis and the point considered, while the magnetic force H, say, has a single component perpendicular to the plane. Thus two differential equations are got connecting P, R, and H, from which (P having first been found from the differential equation involving P alone) R and H are found at once in the forms $M\partial P/\partial r$, $N\partial P/\partial r$ where M and N are multipliers, and r is the radius drawn from the axis to the point considered.

It is to be noted also that the sign between the two groups of terms into which $K_0(x)$ is divided in (2), p. 263, should be the same as that before $\log x$ in the first group in brackets and that C should be taken with the same sign as $\log x$, and $\log 2$ with the opposite sign. This involves a correction likewise in the table of approximate values of the functions given lower down on the same page. Again, the same constant C, which has the value

$$\text{Lt}_{n \rightarrow \infty} \left(\sum_{n=1}^{\infty} \frac{1}{n} \log n \right)$$

is called Gauss's constant at p. 263, while the quantity $\gamma = e^{\gamma}$ is called Euler's constant at p. 430. The established usage seems to be to call C Euler's Constant from its discoverer, who gave its value (to sixteen places of decimals) in his *Institutiones Calculi Differentialis*.

The "throttling" of the current in wires subjected to rapidly alternating electromotive forces is fully considered for a cable with inner and outer coaxial conductors, and for two flat strips in parallel planes with a stratum of insulating material between them. In this connection the author first introduces Mr. Oliver Heaviside's word *impedance*. Writing E for the external electromotive force, I for the total current, and R and L for the effective resistance and self-inductance, we have (p. 272)

$$E = L \frac{\partial I}{\partial t} + RI.$$

R is called by Prof. Thomson the impedance. According to Heaviside's proposal it is $\sqrt{R^2 + n^2 L^2}$ that should be called the impedance, where $n = 2\pi/T$, T being the period of the alternation.

The manner in which the damping out of the vibration is taken account of by the complex analysis is well worth remarking. The eating up of the energy and consequent tapering off of the amplitude according to an exponential function of the distance from the starting end by the impinging of the oscillations in the dielectric on the conductors bounding it, and the lowering of speed of propagation of phase in the dielectric below the natural speed, that of light, all come out in the most beautiful manner.

Mr. Heaviside's careful synthetical explanations of such phenomena are well worth reading in this connection.

The author next passes to his own most valuable investigations regarding the effect of subdivision of iron on the dissipation of energy in the iron of a transformer, to electrical oscillations on cylinders and on spheres, and other problems of the greatest interest to all students of the later developments of Maxwell's great theory carried out by Hertz, now, alas, to be continued entirely by other hands.

The concluding portion of the book consists of a most valuable account of the work of Hertz, and forms the most appropriate supplement to Maxwell's great work that could have been written. The idea of Faraday tubes is well applied to picture the action of a Hertzian resonator in its different positions relatively to the vibrator in the experiments on direct radiation, and those on waves along wires. Not only is Hertz's own work fully described and explained, but the vast amount of fine work that has been done at Dublin, Liverpool, at Cambridge, and on the continent, is discussed, and much of it submitted to careful mathematical analysis.

Space does not permit of even a summary of the topics here treated, and we can only say that the reader who wishes to know these things well, and who shrinks from the labour of digging them out of *Proceedings*, *Annalen*, and *Berichte*, here, there, and everywhere, ought to read Prof. Thomson's work. Such a work is worthy not only of the author, but of the researches of the master and his great disciple who have passed away. A. GRAY.

GREENHILL'S ELLIPTIC FUNCTIONS.

The Applications of Elliptic Functions. By Alfred George Greenhill, F.R.S., Professor of Mathematics in the Artillery College, Woolwich. (London: Macmillan and Co., 1892.)

IT would be difficult to exaggerate the part which the study of elliptic functions has played in the pure mathematics of the present century. And this was to be expected; for whether we regard natural science as the application of common sense to the material needs of life, or as the outcome of the need for expansion in the mental world, and whether we consider mathematics as that exact basis without which progress was not permanently possible, or esteem it to be those higher Alps—

Where we can ever climb, and ever
To a finer air—

in either case we must see that a development of integral calculus—a development which was competent to fill so large a part of Legendre's life, which suggested such magnificent algebra as we find in Jacobi's *Fundamenta*, which promised, too, in Abel's hands such generalisations as are not even yet brought to perfection, such a theory, surely, was well worthy of persevering pursuit. And if we attribute the present extent of the theory of curves and of the theory of functions to the day when Riemann stood best man to the ideas of Cauchy and the suggestions of hydrodynamics, we must admit it was because his methods were employed upon the materials left by Abel that such results have come.

The importance of the present work lies in its recognition that the theory of elliptic functions arose as a development of integral calculus, and as such may be expected to supply a formulation of the solution of many problems of physics otherwise regarded as unfinished. Prof. Greenhill is well known to be a man who has not allowed his unwearied application to such problems to destroy his sympathy with pure mathematical speculation; on the contrary, he has sought, by every means in his power, to fill the difficult position of apostle to the Gentiles in this respect, by making as many of the results

of analysis as are susceptible of application to physics, easily intelligible to students of that subject. The present book, addressed, we are told, to the trained mathematical student, is stated to be primarily a collection of problems (mostly in dynamics and electric flow) whose solutions are expressible by elliptic functions; and it is intended that the properties of these functions should be suggested by, and developed simultaneously with, the problems in hand. Really, of course, it is much more. In fact, the student who works completely through the book will meet with a good many of the formulæ of common occurrence in the elementary part of the subject, and will, moreover, learn to manipulate them for himself; and whether he be interested most in the motion of tops, or the stability of ships, or the biquadratic form, he will probably be surprised at the amount of information condensed here.

The book opens with a consideration of the motion of the common pendulum. The fact that in this motion the angular displacement depends uniquely upon the time, suggests the inversion of the elliptic integral; the existence of a real period of the functions thus obtained, is suggested by the periodic motion of the pendulum. The functions are then immediately used to express the solution of Euler's equations for a body moving about a fixed point under no forces. Then follow seventy pages devoted to the expression of elliptic integrals in terms of the functions, in the course of which, beside a vast variety of examples collected from Legendre and elsewhere, are found a consideration of Watts' Governor, of the Elastica, of the Sumner lines on a Mercator chart, of the Catenoid, of quadrantal oscillations, and of other things—the notation being sometimes Jacobian and sometimes Weierstrassian. It is needless to say that here is a mine of wealth for the examiner. It is only in chapter iv., when we are a third of the way through the book, that the addition theorem of the functions becomes necessary. And while this is proved by a pendulum view of Jacobi's two-circle method, space is found for a thorough examination of Legendre's method and a detailed account of the porism of the in- and circum-scribed polygon for two circles, the diagrams being of the most painstaking character. Then follow sixty pages which will be perhaps the least interesting of the book—at least to the students for whom Prof. Greenhill writes—devoted to an algebraic exposition of the addition theorems for the three kinds of integrals. They contain an examination of the theorems of Fagnano and Graves for the ellipse and hyperbola. They are followed by an account of the tortuous elastica, succeeded by a resumption of the motion of a body about a fixed point under no forces, wherein the author introduces a very full account of the herpolhode. In the hundred pages remaining, the book may be said to be drawing to a conclusion, the double periodicity is considered, Cariesian ovals being introduced in connection with the expression of functions of a purely imaginary argument; a chapter is devoted to the factor expressions of the functions, here suggested by hydrodynamical considerations; and the last chapter is a summary of the earlier part of the theory of transformation, characterised, however, like the rest of the book, by the utmost particularity, numerical and otherwise.

This summary will show to some extent the scope of the work. It is essentially a student's book, written in a concise conversational style; but whether the student have more sympathy with physical or pure mathematics, he cannot fail to find much that is new to him, and be surprised at the detail with which it is given; and the air of practical reality which pervades every page, and the skill and originality with which the results are obtained, will atone for the tentative nature of many of the demonstrations.

It is, in fact, in this regard that the reader may be most unfair to the author. It is no part of his plan to develop any demonstration beyond the nearest point at which it suggests the formula required, or to use any more general method of enquiry than is absolutely necessary, or to regard the subject in any other way than as a collection of formulæ. To forget this is to wish for many things to be differently stated—is, indeed, to wish for a quite different book. A few instances will suffice. The author frequently makes the remark that the present state of the theory is due to Abel's brilliant idea in inverting the elliptic integral of the first kind. One fears that the reader may enquire whether the inverse function is a one-valued function of its argument for all the values of the latter, or may forget that the expansion of p. 202 is not valid for all values of the quantities involved. He may even wish to invert the integral of the second kind, notwithstanding that it is here expressed in terms of the integral of the first kind. Or, again, the statement on p. 266, that ϕ and ψ "satisfy the conditions required of the potential and stream functions," may lead to misconception, for it is not sufficient that ϕ be infinite at A and C; it must be infinite in the neighbourhood of A like a multiple of the logarithm of the distance from A. And in the same way, on p. 281, in attempting to realise how a "uniform streaming motion parallel to the vector ma " is consistent with the motion in the strips which is represented by the other factors, we are liable to desire a proof that functions whose equality is not identical, but, as here, the result of proceeding to a limit, necessarily represent the same fluid motion. The fact is that the two functions considered here are not equal for $z = \infty$, where $\text{sine } z$ has a most essential singularity. Or, again, we may wish that the signs had received more attention; as, for instance, on p. 24, or throughout chapter ii., and in many other places. And this the more that Jacobi himself is known to have printed a mistaken sign (for $cn(K + iK')$). And this wish is not allayed by the fact that in the reservation of these difficulties, made on page 45, poles and branch points are mentioned together, as if similar singularities. While, lastly, if we forget the object of the book, we shall most devoutly wish a better recognition of the fact that the Jacobian functions and the Weierstrassian functions are not the fundamental fact of the theory. Underlying both is the same algebraic irrationality, now expressed by a binodal quartic, and now by a cubic curve—from either of which both these functions and many others can be constructed, the distinguishing mark being only the number and position of the poles. One does wish indeed that Prof. Greenhill had found occasion to state somewhere that the algebraic method he adopts throughout, fascinating as it certainly is, is also, in the strict sense employed by him, of only antiquarian interest, in view of the de-

scriptive methods that are available. It is, moreover, essentially incompetent, and therefore unsatisfactory.

But if we pass over such considerations as these, fundamental as they are from some points of view, recognising that in practical life we often count it a saving of time to exhaust the logical consequences of a belief, before painfully verifying the grounds of that belief and recognising that in a new subject it is always the most elementary method that furnishes the easiest introduction, we shall find very much for which to value the book before us—beside the excellent diagrams, index, printing, &c.

Note 1.—The result of carrying out in detail the work mentioned at the end of § 194 seems worth introducing into a new edition. The result is partly given in *Math. Tripos*. Part II. 1892. It seems a pity, too, that the expressions for the Jacobian $Z(u)$ in terms of the Weierstrassian $\zeta(u)$ are not given.

Note 2.—The example 15, p. 351 (though taken from *Math. Tripos*, Part II.), is wrong. The result should be

$$3 \frac{a - mc}{1 + m^3} = \frac{1}{m} (x_1 + x_2 + x_3) = \frac{x_1 - x_2}{y_1 - y_2} (x_1 + x_2 + x_3)$$

where $y = mx + c$ is the final position of the line.

Note 3.—The example 2 (i.), p. 140, is misprinted.

H. F. BAKER.

THE DISPERSAL OF SHELLS.

The Dispersal of Shells: an Inquiry into the Means of Dispersal possessed by Fresh-water and Land Mollusca. By Harry Wallis Kew, F.Z.S. With a Preface by Alfred Russel Wallace, LL.D., F.R.S. With Illustrations. Pp. xi. 291. International Scientific Series, Vol. lxxv. 8vo. (London: Kegan Paul, Trench, Trübner and Co., 1893.)

IT is strange that we have had to wait so long for a manual on dispersal. Many books have been written on the geographical distribution of animals and plants; and islands and even continents have been raised or lowered to account for the strange anomalies. Yet comparatively little attention has been paid to a study that must be undertaken before we are qualified to express an opinion on geographical distribution. Darwin and Lyell, however, thoroughly recognised the importance of the subject, and the former made many experiments on the vitality of seeds under trying circumstances—such as being immersed in sea-water, or eaten by birds. Direct observation of the species in transit, under natural conditions, has been less attended to, except in the case of flying animals and of certain plants. The cause of this neglect is easy to understand: dispersal, in the groups that are not specially modified to assist the process, is mainly the result of the accumulation of rare accidents, such as would only occasionally be noticed by some naturalist engaged in quite different observations. It is useless to go into the field on purpose to watch the dispersal of snails; the entomologist, ornithologist, fisherman, or sportsman may once in a season obtain a direct observation, and it is to such observers that we must principally trust.

In certain respects the land and fresh-water mollusca are peculiarly valuable for the study of geographical distribution; they are essentially sedentary animals; some of them can float, but scarcely any except

Dreissena have an active free-swimming larval stage. Few of the species are specialised for dispersal; though we do not think that there is such a complete absence of specialisation as would at first appear. The study of the dispersal of the mollusca becomes, under these circumstances, of great importance to the naturalist; for if snails or their eggs can cross rivers and straits, it is probable that other sedentary groups can do so also.

The system on which Mr. Kew has worked is to collect all the facts relating to the dispersal of land and fresh-water mollusca, giving the authority for each statement. He has thus gathered into one small volume an enormous amount of information, much of which will be quite new to naturalists. Beginning with the fresh-water shells, he treats first of the anomalies in their local distribution, such as their occurrence in perfectly isolated ponds. Then follow chapters dealing with the means of dispersal; and it is surprising how varied these are. Not only are the animals transported down stream on various floating objects, but the author can quote an actual instance in which a number of fresh-water mollusca (*Anodon*) were carried by a whirlwind and fell with the rain. Another interesting case of transportation over dry land is that mentioned by Canon Tristram, who found the eggs of some mollusc, probably *Succinea* attached to the foot of a passing mallard shot by him in the Sahara, a hundred miles from water. A few instances are noted in which birds on the wing have been shot with bivalves adhering to their toes; but there seems to be no recorded case of the occurrence of molluscs or their eggs in the bits of water-weed that so often catch on the feet of aquatic birds. It is probable that this means of transport is common; but being less striking than the other modes, it has not yet been observed. Insects also lend their aid, and a water-beetle (*Dytiscus marginalis*) has twice been captured on the wing with *Spharium* attached to its legs; another specimen was caught with *Ancylus* attached to its wing-case. Various other aquatic insects have often been found with mollusca attached to them, though they were not actually caught on the wing.

As regards the land-shells, there is a singular dearth of direct evidence: Mr. Kew is able to mention various ways in which they may have been transported; but the only cases in which the process has actually been observed were some live *Helix caperata* found in a wood-pigeon three days after it had been shot, and an operculated land snail which had caught the foot of a bumble-bee, and was being dragged along. We cannot help thinking, however, that the dispersal of land-shells is a much rarer process than the carrying of fresh-water species. An isolated dew-pond after an existence of ten years will generally yield several species of fresh-water mollusca, and a mediæval fish-pond has quite a large fauna. A church or castle built of limestone, but surrounded by non-calcareous desert, is, for a large group of land snails, the equivalent of an isolated pond; but it is only on very old buildings that one finds colonies of the special limestone species. We have never come across an isolated colony of this sort on a building less than a hundred years old, and have never noticed more than two or three species on a single ruin under such circumstances.