likewise for the second. Thus, for a regular polygon the value is $2\pi^2 \left(1 - \frac{1}{n}\right)$. For a finite straight line the value is π^2 , instead of $2\pi^2$; a result which may be verified, the tangents T, T' becoming here the vectors r, r' to the extremities of the line.

If, instead of the above system of tangents, altering by constant angles, we take an infinity of equidistant points ranged along the given contour, and draw tangents at these points, we shall find that the density of the intersections of these tangents is proportional to

$$\frac{\rho\rho'}{TT'} \sin \theta,$$

where ρ, ρ' are the radii of curvature of the contour at the points of contact of the tangents T, T' ; hence we obtain the integral

$$\iint \frac{\rho\rho'}{TT'} \sin \theta dS = \frac{1}{2}L^2,$$

extended over the whole outer surface; where L is the perimeter of the contour.

June 25th, 1868.

Prof. SYLVESTER, President, in the Chair.

Prince Camille de Polignac was elected a Member, and admitted into the Society by the President, as also were Messrs. W. J. C. Miller, J. M. Wilson, and W. S. B. Woolhouse.

A paper was read by Mr. J. M. Wilson on "Euclid as a Text-book of Elementary Geometry." Prof. Hirst, Messrs. De Morgan, Ellis, Hemming, and A. Smith took part in the discussion upon this paper, and Mr. Wilson replied. [The paper has since been printed *in extenso* in the *Educational Times* for September, 1868, and allusion is made to it in the *Athenæum*, July 18th, 1868, in a review of Mr. Wilson's Geometry.]

Prince Camille de Polignac then read the following paper:—

On a Problem in Combinations.

I. m letters $abc\dots a'b'c'\dots khl\dots$ being given, it is proposed to find the number of combinations which can be formed with these letters, n and n together, with the condition that a and a' must not enter the same combination, nor b and b' , c and c' &c.

The letters $khl\dots$, which are submitted to no restriction, we shall call *free* letters, the others *associative* letters; a and a' , b and b' , &c., will be called *associated* or *corresponding* letters.

The combinations themselves we shall call *restricted combinations*, and their number will be represented by the notation ${}^*K_{m,n}$ in which

m indicates the total number of letters, n how many enter each combination, and q the number of free letters. Thus $m-q$ indicates the number of *associative* letters, and it is plain that $m-q$ must be an even number.

To begin with the simplest case, let us first suppose $q = 0$, whence $m = 2p$. Supposing all the restricted combinations to have been written out, let us form in each all possible permutations of letters. We shall thus obtain the *restricted permutations* of $2p$ letters, n and n together, and if we designate their number by $M_{2p,n}$ we shall have

$$K_{2p,n} = \frac{M_{2p,n}}{\Pi_n},$$

where Π_n stands for the number of ordinary permutations of n letters, equal to $1.2.3\dots n$.

In order to define $M_{2p,n}$ let us throw together all the permutations *beginning* with the same letter. We shall thus form $2p$ groups, in which the first letters will be respectively, $a, b, c, \dots, a', b', c', \dots$, and it is obvious, from symmetry, that each group will be composed of the same number of permutations.

But if we strike off a from the first group, we shall obtain restricted permutations of $(n-2)$ letters, as a' does not enter that group. The number of permutations in the first group is therefore $M_{2p-2,n-1}$, and consequently

$$\begin{aligned} M_{2p,n} &= 2p M_{2p-2,n-1}; \\ \text{whence } M_{2p-2,n-1} &= (2p-2) M_{2p-4,n-2} \\ M_{2p-4,n-2} &= (2p-4) M_{2p-6,n-3} \\ &\vdots \\ M_{2p-2n+4,2} &= (2p-4) M_{2p-2n+2,1}. \end{aligned}$$

If we multiply together these equations, and observe that $M_{2p-2n+2} = 2p-2n+2$, we find

$$M_{2p,n} = 2p(2p-2)(2p-4)\dots(2p-2n+2);$$

$$\text{hence } K_{2p,n} = \frac{2p(2p-2)\dots(2p-2n+2)}{1.2.3\dots n} = 2^n C_{p,n},$$

$C_{p,n}$ indicating the number of ordinary combinations of n letters out of p .

$$\text{When } n = p, \quad K_{2p,p} = 2^p.$$

II. Let us now pass on to the general case in which there are q free letters and $m = 2p + q$.

Any specific combination out of the system whose number we have agreed to denote by ${}^qK_{2p+q,n}$ (and which, for brevity, we shall in future call a combination *appurtenant* to ${}^qK_{2p+q,n}$) will contain α free letters and β associative letters.

If from that combination we strike off the free letters, the remaining letters will give one restricted combination appurtenant to $K_{2p,\beta}$.

But if we strike off the associative letters, we shall obtain one ordinary combination appurtenant to $C_{q,\alpha}$.

Hence the number of restricted combinations in which occur α free, and β associative, letters is $C_{q,\alpha} \times K_{2p,\beta}$.

But we have obviously ${}^qK_{2p+q,n} = \Sigma C_{q,\alpha} \cdot K_{2p,\beta}$, where α and β can range from 0 to n , with the condition $\alpha + \beta = n$; hence

$${}^qK_{2p+q,n} = K_{2p,n} + C_{q,1} \cdot K_{2p,n-1} + C_{q,2} \cdot K_{2p,n-2} + \dots + C_{q,n-2} \cdot K_{2p,2} + C_{q,n-1} \cdot K_{2p,1} + C_{q,n},$$

and we know how to form $K_{2p,n}$, $K_{2p,n-2}$, &c.

III. On a determinant involving $K_{2p,n}$.

We shall now give a new expression for $K_{2p,n}$ based upon a method entirely different from the first, and which consists in striking off from the total number of ordinary combinations $C_{2p,n}$ such as contain one or more couples of corresponding letters.

Let Q_h represent the aggregate combinations which contain h letters and their associates; we shall have $K_{2p,n} = C_{2p,n} - \Sigma Q_h$, where h ranges from 1 to $\frac{n}{2}$ or $\frac{n-1}{2}$ according as n is even or odd.

But Q_h is composed of combinations such as $abc \dots a'b'c' \cdot P$, where P represents a product of $n-2h$ letters cleared of corresponding letters. P can therefore take $K_{2p-2h,n-2h}$ values.

On the other hand, the factor $abc \dots a'b'c' \dots$ can be written $\alpha\beta\gamma$, where every Greek letter stands for the product of a Roman letter and its associate; and, in this form, it is clear that $abc \dots a'b'c' \dots$ can take $C_{p,h}$ values; hence $Q_h = C_{p,h} \cdot K_{2p-2h,n-2h}$; whence finally

$$K_{2p,n} = C_{2p,n} - \Sigma C_{p,h} \cdot K_{2p-2h,n-2h}.$$

If n be even, replacing it by $2n$, we find

$$(1) \dots K_{2p,2n} = C_{2p,2n} - \{ C_{p,1} \cdot K_{2p-2,2n-2} + C_{p,2} \cdot K_{2p-4,2n-4} + \dots + C_{p,n-1} \cdot K_{2p-2n+2,2} + C_{p,n} \}.$$

Let us write, generally,*

$$X_n = K_{2p,2n}, \quad Q_n = C_{p,n} - C_{2p,2n}.$$

If in (1) we change successively p and n into $p-1$, $n-1$; $p-2$, $n-2$, &c., we shall form the n following equations.

$$\begin{aligned} C_{p,n-1} X_1 + C_{p,n-2} X_2 + \dots + C_{p,1} X_{n-1} + X_n + Q_n &= 0 \\ C_{p-1,n-2} X_1 + C_{p-1,n-3} X_2 + \dots + C_{p-1,1} X_{n-2} + X_{n-1} + Q_{n-1} &= 0 \\ \vdots & \\ C_{p-n+2,1} X_1 + X_2 + Q_2 &= 0 \\ X_1 + Q_1 &= 0 \end{aligned}$$

* When in these equations n is replaced by $n-h$, p must also be replaced by $p-h$. Thus $X_{n-1} = K_{2p-2,2n-2}$, $Q_{n-1} = C_{p-1,n-1} - C_{2p-2,2n-2}$, &c.

and the value of X_n will be obtained by equating to zero the determinant

$$\begin{vmatrix} C_{p,n-1}, & C_{p,n-2}, & \dots\dots\dots C_{p,2}, & C_{p,1}, & X_n + Q_n \\ C_{p-1,n-2}, & C_{p-1,n-3}, & \dots\dots\dots C_{p-1,1}, & 1, & Q_{n-1} \\ C_{p-2,n-3}, & C_{p-2,n-4}, & \dots\dots\dots 1, & 0, & Q_{n-2} \\ C_{p-3,n-4}, & C_{p-3,n-5}, & \dots\dots\dots 1, & 0, & Q_{n-3} \\ \vdots & & & & \\ C_{p-n+2,1}, & 1, & 0, \dots\dots\dots 0, & 0, & Q_2 \\ 1, & 0, & 0, \dots\dots\dots 0, & 0, & Q_1 \end{vmatrix}$$

and we learn that this determinant is equal to $X_n - 2^{2n} C_{p,2n}$. If we replace X_n by $2^{2n} C$, the determinant will vanish identically; if $2n=p$, it reduces to $X_n - 2^p$.

If now we suppose in the value $K_{2p,n}$ (p. 8), n to be an odd number, replacing it by $2n+1$, we find

$$(2) \dots K_{2p,2n+1} = C_{2p,2n+1} - \{C_{p,1} \cdot K_{2p-2,2n-1} + C_{p,2} \cdot K_{2p-4,2n-3} + \dots \\ \dots C_{p,n-1} \cdot K_{2p-2n+3,3} + C_{p,n} \cdot K_{2p-2n,1}\};$$

making $X_{n+1} = K_{2p,2n+1}$, the determinant will be

$$\begin{vmatrix} C_{p,n}, & C_{p,n-1}, & \dots\dots\dots C_{p,1}, & X_{n+1} - C_{2p,2n+1} \\ C_{p-1,n-1}, & C_{p-1,n-2}, & \dots\dots\dots 1, & -C_{2p-2,2n-1} \\ \vdots & & & \\ C_{p-n+1,1}, & 1, & 0, \dots\dots\dots 0, & -C_{2p-2n+3,3} \\ 1, & 0, & 0, \dots\dots\dots 0, & -C_{2p-2n,1} \end{vmatrix}$$

and its value

$$X_{n+1} - 2^{2p+1} C_{p,2p+1}.$$

We may obtain determinants involving ${}^q K_{2p+q,2n}$ and ${}^q K_{2p+q,2n+1}$ in precisely the same manner. The foregoing method applies identically to the general case, and it will be only necessary to write $2p+q$ for $2p$ in the above equations and determinants. Thus, placing

$$X'_n = {}^q K_{2p+2q,2n}, \quad Q'_n = C_{p,n} - C_{2p+q,2n},$$

we find

$$\begin{vmatrix} C_{p,n-1}, & C_{p,n-2}, & \dots\dots\dots C_{p,2}, & C_{p,1}, & X'_n + Q'_n \\ C_{p-1,n-2}, & C_{p-1,n-3}, & \dots\dots\dots C_{p-1,1}, & 1, & Q'_{n-1} \\ C_{p-2,n-3}, & & \dots\dots\dots 1, & 0, & Q'_{n-2} \\ \vdots & & & & \\ C_{p-n+2,2}, & 1, & 0, \dots\dots\dots 0, & 0, & Q'_2 \\ 1, & 0, & 0, \dots\dots\dots 0, & 0, & Q'_1 \end{vmatrix}$$

and we should in the same manner obtain the determinant involving ${}^q K_{2p+q,2n+1}$.

In order to illustrate the preceding theory, we here subjoin numerical examples.

Let $n=2$. If we use the first determinant, we shall have

$$\begin{vmatrix} C_{p,1}, & X_2 + Q_2 \\ 1, & Q_1 \end{vmatrix} \quad \text{and} \quad \begin{aligned} Q_2 &= C_{p,2} - C_{2p,4} \\ Q_1 &= C_{p-1,1} - C_{2p-2,2} \end{aligned}$$

and the determinant becomes

$$\begin{vmatrix} p, & X_2 + \frac{p(p-1)}{2} - \frac{2p(2p-1)(2p-2)(2p-3)}{1 \cdot 2 \cdot 3 \cdot 4} \\ 1, & p-1 - \frac{(2p-2)(2p-3)}{2} \end{vmatrix}$$

But $X_2 = K_{2p,4} = \frac{2p(2p-2)(2p-4)(2p-6)}{1 \cdot 2 \cdot 3 \cdot 4};$

hence the following determinant must vanish identically:—

$$\begin{vmatrix} p, & \frac{2p(2p-2)(2p-4)(2p-6)}{1 \cdot 2 \cdot 3 \cdot 4} + \frac{p(p-1)}{2} - \frac{2p(2p-1)(2p-2)(2p-3)}{1 \cdot 2 \cdot 3 \cdot 4} \\ 1, & p-1 - \frac{(2p-2)(2p-3)}{2} \end{vmatrix}$$

Reducing and suppressing common factors,

$$\begin{vmatrix} 1, & \frac{(2p-4)(2p-6) - (2p-1)(2p-3) + 3}{3 \cdot 4} \\ 1, & -(p-2) \end{vmatrix}$$

or, $\begin{vmatrix} 1, & -(p-2) \\ 1, & -(p-2) \end{vmatrix}$

If in the determinant

$$\begin{vmatrix} C_{p,1}, & X_2 + Q_2 \\ 1, & Q_1 \end{vmatrix}$$

we make $p=4$, we shall have $X_2 = K_{8,4}$, and we know that in this case $X_2 = 2^4$. Consequently the following determinant must vanish identically:—

$$\begin{vmatrix} 4, & 2^4 + \frac{4 \cdot 3}{2} - \frac{8 \cdot 7 \cdot 6 \cdot 5}{2 \cdot 3 \cdot 4} \\ 1, & 3 - \frac{6 \cdot 5}{2} \end{vmatrix}$$

or $\begin{vmatrix} 1, & 2^3 + \frac{3}{2} - \frac{7 \cdot 5}{2} \\ 1, & 3 - \frac{6 \cdot 5}{2} \end{vmatrix} = \begin{vmatrix} 1, & -12 \\ 1, & -12 \end{vmatrix}$

III. The well known equation $C_{m,n} = C_{m-1,n} + C_{m-1,n-1}$ can be translated into the language of restricted combinations, and will give us new expressions involving ${}^qK_{2p+q,n}$.

In the first place, if we arrange ${}^qK_{2p+q,n}$ in two groups with reference to any *free* letter, the first of which will *not* contain that letter, the second which *will* contain it, we shall find as above

$${}^qK_{2p+q,n} = {}^{q-1}K_{2p+q-1,n} + {}^{q-1}K_{2p+q-1,n-1}.$$

But if we form two similar groups with reference to any associative letter, we shall find

$${}^qK_{2p+q,n} = {}^{q+1}K_{2p+q-1,n} + {}^qK_{2p+q-2,n-1};$$

$$\text{hence } {}^{q-1}K_{2p+q-1,n} + {}^{q-1}K_{2p+q-1,n-1} = {}^{q+1}K_{2p+q-1,n} + {}^qK_{2p+q-2,n-1}.$$

We might also inquire what expression corresponds to the equation

$$C_{m,n} = C_{m,m-n},$$

and this will lead us to a new class of combinations; for if we divide the product $abc\dots a'b'c'\dots fgh\dots$ successively by all the combinations composing ${}^qK_{2p+q,n}$ we shall obtain combinations, *all* of which will contain *at least one* letter out of *every* pair of corresponding letters. Thus a new field for inquiry is afforded; and although no important consequences have been deduced from the above researches in this paper, yet as they have led us to a determinant, whose value it does not appear easy at first sight to obtain by actual expansion, I think the whole theory not unworthy of attention, both in itself and in its bearings upon the different branches of mathematics with which the theory of arrangements is connected.

NOTE.—Since this paper was read, Mr. M. Jenkins has pointed out to me the following generalisation of the problem:—

The actual combinations of a series of letters $a, a', a''\dots b, b'\dots c, c'\dots h, k, l$, taken n and n together, subject to the restriction that no two of the same group (e. g., a, a', a'' , &c., or $b, b', \dots$ &c.) shall enter into the same combination, is the coefficient of x^n in

$$\{1 + (a + a' + a'' + \dots)x\} \{1 + (b + b' + \dots)x\} \{ \quad \} (1 + hx)(1 + kx)(1 + lx).$$

Their number is obtained by putting

$$a = a' = a'' = \dots = b = b' = \dots h \dots = 1,$$

and is therefore the coefficient of x^n in $(1 + \mu_1 x)^p (1 + \mu_2 x)^q (1 + \mu_3 x)^r \dots$ if there be p groups of μ_1 letters, q of μ_2 , &c. ($\mu_1, \mu_2, \dots$ being all unequal).

The result obtained in p. 70, Art. I., or $2^n C_{p,n}$, is therefore the coefficient of x^n in $(1 + 2x)^p$; and the result in p. 71, Art. II., is the coefficient of x^n in $(1 + 2x)^p (1 + x)^q$.

The President next explained one of his Anallagmatic Squares, formed in connexion with his Theory of Sequences.

Mr. Woolhouse gave an abstract of the following paper:—