SOME ILLUSTRATIONS OF MODES OF DECAY OF VIBRATORY MOTIONS

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1.

The cause of decay of vibratory motion which is to be considered here is the communication of a disturbance from the vibrator to the surrounding medium. The energy of the vibrator is gradually transmitted to the distant parts of the medium by wave-motion. When the vibrator is a solid elastic body and the medium is air there are customary certain processes for estimating the rate of decay of the vibrations. In one of these processes the vibrator is assumed to have its natural free period, the motion of the air is assumed to be that progressive wave-motion of simple harmonic type in the same period which would be forced if the free motion in question were maintained for an indefinitely long time in opposition to the reaction of the air, the rate of transmission of the energy across a surface surrounding the vibrator is calculated, and this is taken to be the rate at which the energy of the actual vibratory motion is diminishing owing to the presence of the air. In another process it is assumed that waves of simple harmonic type are propagated in air outwards from the vibrator, the reaction of the air on the vibrator is calculated in accordance with this assumption, the equation of motion of the vibrator subject to this reaction is formed, and, with a certain interpretation of symbols, it is found to be of the ordinary type of damped harmonic vibrations.* The rate of decay of the actual vibratory motion is taken to be that indicated by this equation. These two processes lead to the same result when the rate of decay is slow, and when this is the case there can be no doubt of their correctness as regards the motion of the vibrator; but it is manifest that they do not give a complete account of the motion of the medium. This motion cannot be a progressive wave-motion of simple harmonic type because in the neighbourhood of the vibrator its amplitude is continually diminishing. Further, the motion of the vibrator must have a beginning in time, and

* See Lord Rayleigh, *Theory of Sound*, Vol. II., § 302, and other Articles.

consequently the waves that are generated must have a front or boundary at any instant; beyond this moving boundary there is no disturbance. The existence of such a boundary implies conditions which the disturbance must satisfy, but the processes which are indicated above take no account of any such conditions. So far as I am aware, the only problems concerning such motions which have been solved completely relate to systems in one dimension,* and it will be found that three-dimensional systems present new features.

Even in their application to the motion of the vibrator it is clear that the success of the above described methods depends upon the possibility of free vibrations of the vibrator when isolated from the medium, and upon the smallness of the part played by the medium in the actual motion. In the case of a sounding body the density of the body must be much greater than that of the air. Otherwise the vibrations of the body in air will not be approximately the same as in a vacuum. This remark becomes of great importance when it is sought to extend the methods to electrical vibrations. In this case the essential phenomenon is the wave-motion excited in the æther, and there is in general no meaning in electrical vibrations independent of the surrounding medium. Exceptional cases are the vibrations of a condenser with or without a small aperture,† and vibrations within an insulating body of enormous specific inductive capacity.‡ These are examples of systems in which electrical vibrations that approximate to free vibrations are possible.§ The nearly dead-beat oscillations of a Hertzian vibrator differ essentially from those that occur in the above mentioned exceptional cases. The vibrator is not in any sense isolated from the medium; and the disturbance that takes place is much more accurately described as a change of state of the medium than as a change of state of the vibrator.

The fundamental tone of acoustical resonators is given out by a mode of vibration which depends essentially upon the neck making communication with the external medium. Air contained in a cavity within a rigid boundary having no aperture has definite modes of free vibration, but none of them is the same as the mode of vibration that is characteristic of the resonator made by producing an aperture in the boundary. The system acquires through the existence of the aperture a new mode of

* H. Lamb, *Proceedings*, Vol. XXXII., p. 208 (1900).

† J. Larmor, *Ibid.*, Vol. XXVI. (1895).

‡ H. Lamb, *Cambridge Phil. Soc. Trans.*, Vol. XVIII. (1900).

§ A conductor outside which the space is doubly or multiply connected—*e.g.*, an infinite cylinder or a ring—admits, when thin, of electrical oscillations which are very nearly free oscillations.

vibration, which decays through the transmission of energy to the external air. A system with a permanent vibration of nearly the same period and type, which shall differ from the system consisting of the air in the cavity only by having an additional degree of freedom, can be devised by imagining a piston in the aperture to be held nearly in position by a constant force equal to that exerted upon it by the pressure of the air inside when at rest, and neglecting the air outside. The mode of vibration characteristic of the resonator is that in which the piston oscillates to and fro within the walls of the aperture. When the external air is present and there is no piston a slightly damped harmonic vibration with nearly the same period is possible, and such vibrations are excited by any causes which vary the pressure over the aperture, just as the oscillation of the piston would be excited by varying the force applied to it. The electrical analogue of the air in the cavity would appear to be the dielectric plate of a condenser of which the conducting surfaces are closed. Making an aperture in the outer conductor appears in this case not to introduce any new electrical degrees of freedom, and the analogue of the aperture in the acoustical problem appears to be a wire joining the two conducting surfaces.* But the analogue is imperfect, inasmuch as opening a communication with the external medium is no longer the process by which the new degree of freedom is introduced.

In what follows there will be investigated some problems concerning the generation of sound waves in air and of electrical vibrations in free æther. It will be seen that the customary methods represent well the motion of a sounding body, but that the nature of the sound waves generated by the body is in general different from that assumed in these methods. For electrical vibrations the case chosen will be that of a spherical conductor over which a surface distribution of electricity variable from point to point is produced. For the sake of simplicity the sphere will be taken to be a perfect conductor.

In the ordinary method of treating this problem,† the disturbance is assumed to be of exponential type, and the possible exponents are determined by the condition that the electric force at the surface of the conductor is normal to the conductor. The exponents may be real and negative or complex with negative real parts. Thus the solutions that are found

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* H. M. Macdonald, *Electric Waves* (Cambridge, 1902), p. 57.

† J. J. Thomson, *Recent Researches*, pp. 361 et seq. The problem was treated by the same author in *Proceedings*, Vol. xv. (1884), and by H. Lamb in *Phil. Trans. Roy. Soc.*, Vol. CLXXIV. (1883).

contain factors of the forms

$$e^{-p(ct-r)} \quad \text{and} \quad e^{-p(ct-r)} \sin q\,(ct-r+\epsilon),$$

which tend to become infinite with r. These solutions cannot represent unlimited trains of waves propagated outwards. The waves that are actually propagated have a boundary which moves outwards with the velocity c. The effects due to the boundary of the waves are usually left out of account, and the disturbances of exponential type are also ignored. They will be found to represent an essential part of the disturbance. Whenever they can occur they are necessary to the continued advance of the wave-boundary.

In addition to problems of the decay of vibrations that are consequences of an initial state, some examples will be discussed of vibrations that are maintained for a time and are then left to decay when the cause that maintains them ceases to operate. These examples bring out the result that there is no essential difference in the modes of subsidence that are exhibited in the two cases.

2. *Introduction of Arbitrary Functions.*

In Prof. Lamb's paper* to which reference has been made there is given an illustration of the decay of vibratory motion by transmission of the energy to a distance. The system considered is a massive body attached to an infinitely long stretched string and capable of vibrating transversely under the action of a spring. The waves that are propagated along the string must be expressible by a function of the form $f(at-x)$. The initial state of the system being one of equilibrium, the body is struck transversely to the string, and the initial conditions together with the equation of motion of the body suffice to determine completely the value of the function f for all values of x and t. It appears that f is of the form $Ae^{-p(at-x)} \sin q(at-x)$ when $x < at$, but $f = 0$ when $x > at$. The step by which we can advance beyond the more customary and less satisfactory method of assuming that the waves in the string are of simple harmonic type is the substitution of an arbitrary function $f(at-x)$ for a function of the form $A \sin n(t-x/a)$. The solutions of problems connected with spherical boundaries which will be discussed below contain arbitrary functions of $t-r/a$, where r denotes distance from the centre of the sphere and a is the velocity of wave-propagation. In the case of sound waves the velocity potential ϕ satisfies the equation $\partial^2\phi/\partial t^2 = a^2\nabla^2\phi$,

* *Proceedings*, Vol. XXXII., p. 208.

and the most general solution that can express waves travelling outwards and be proportional to a spherical surface harmonic S_n is of the form

$$r^n S_n \left(\frac{1}{r}\frac{\partial}{\partial r}\right)^n \left\{\frac{1}{r}\chi(at-r)\right\}.$$

In the case of electric waves, we take c for the velocity of radiation and obtain solutions of equal generality* by assuming a vector (ξ, η, ζ) to be given by means of equations of the form

$$\xi = \left(y\frac{\partial \omega_n}{\partial z} - z\frac{\partial \omega_n}{\partial y}\right)\left(\frac{1}{r}\frac{\partial}{\partial r}\right)^n \frac{\chi(ct-r)}{r}, \tag{1}$$

in which ω_n is a spherical solid harmonic of order n, and η, ζ are obtained from ξ by cyclical interchanges of the letters x, y, z, while ω_n and χ remain unaltered. Then we may write at pleasure either

$$\left.\begin{aligned} (\alpha, \beta, \gamma) &= \frac{\partial}{\partial t}(\xi, \eta, \zeta) \\ (X, Y, Z) &= \mathrm{c}\,\mathrm{curl}(\xi, \eta, \zeta) \end{aligned}\right\} \tag{2}$$

or

$$\left.\begin{aligned} (X, Y, Z) &= \frac{\partial}{\partial t}(\xi, \eta, \zeta) \\ (\alpha, \beta, \gamma) &= -\mathrm{c}\,\mathrm{curl}(\xi, \eta, \zeta) \end{aligned}\right\}. \tag{3}$$

Here (X, Y, Z) represents electric force measured electrostatically, (α, β, γ) represents magnetic force measured electromagnetically, and the axes of (x, y, z) are a right-handed system. In the case expressed by (1) and (2) the normal component of (X, Y, Z) at the surface of a sphere of radius r can be shown to be

$$-\mathrm{c}\, n(n+1)\frac{\omega_n}{r}\left(\frac{1}{r}\frac{\partial}{\partial r}\right)^n \frac{\chi(ct-r)}{r}; \tag{4}$$

so that this form is suitable for representing oscillating electric charges on a sphere, the distribution of surface density being proportional to the surface harmonic contained in ω_n.

3. *Conditions to be satisfied at Wave-Fronts.*†

For the determination of the arbitrary functions χ which occur in such solutions as the above we may have, in addition to the conditions which

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* These solutions were given by the author in *Phil. Trans. Roy. Soc.*, Vol. CXCVII. (1901), as generalizations of the well-known forms in which χ is an exponential function of its argument.

† The results here stated for the case of waves advancing into a previously undisturbed portion of the medium have been given by the author in *Proceedings* (Ser. 2), Vol. 1, p. 37, and the extensions to cases in which the medium beyond the advancing wave-front is disturbed can be made without any difficulty.

must hold near to the vibrating nucleus, certain conditions which must be satisfied at the fronts of waves. In the case of a sound wave advancing into still air the wave-front advances with velocity a, and the velocity potential just behind the front must satisfy the condition

$$(\partial\phi/\partial t)+a(\partial\phi/\partial N) = 0,$$

where N denotes the direction of the normal to the front in the sense of advance of the front. If there is motion in the region into which the waves advance, we may denote by ϕ' the velocity potential just behind the wave-front and by ϕ'' that just ahead of the same surface, and then the condition to be satisfied is

$$(\partial\phi'/\partial t)+a(\partial\phi'/\partial N) = (\partial\phi''/\partial t)+a(\partial\phi''/\partial N). \tag{5}$$

The conditions $\phi'-\phi'' =$ const. in this case and $\phi =$ const. in the previous case also hold at the wave-front, but they will be found to be satisfied of themselves in the problems that we shall consider.

When a train of electric waves advances into a region in which the electric and magnetic forces are null the wave-front advances with the velocity c of radiation, and the electric force (X, Y, Z) and magnetic force (α, β, γ) just behind the advancing front are connected by the equations

$$\left.\begin{aligned} X &= \cos(z, N)\beta-\cos(y, N)\gamma \\ &\dots \quad \dots \quad \dots \quad \dots \\ -\alpha &= \cos(z, N)Y-\cos(y, N)Z \\ &\dots \quad \dots \quad \dots \quad \dots \end{aligned}\right\}, \tag{6}$$

where N is the normal to the wave-front drawn in that direction in which this front advances. If the magnetic and electric forces in the region into which the waves advance are not null, we may denote by (X', Y', Z') and $(\alpha', \beta', \gamma')$ the forces just behind the advancing front and by (X'', Y'', Z'') and $(\alpha'', \beta'', \gamma'')$ the forces just ahead of the same surface. Then the process by which equations (6) are established in the case where $X'', \dots, \alpha'', \dots$ are zero leads to two systems of equations, viz., three of the type

$$(X'-X'') = \cos(z, N)(\beta'-\beta'')-\cos(y, N)(\gamma'-\gamma'') \tag{7}$$

and three of the type

$$-(\alpha'-\alpha'') = \cos(z, N)(Y'-Y'')-\cos(y, N)(Z'-Z''). \tag{8}$$

The equations of types (7) and (8) are not independent. For example, the three equations of (8) show that

$$(\alpha'-\alpha'')\cos(x, N)+(\beta'-\beta'')\cos(y, N)+(\gamma'-\gamma'')\cos(z, N) = 0;$$

and, if this condition is satisfied, the three equations of (8) can be deduced from (7). The geometrical interpretation of the conditions is given in my paper already cited.

4. *Sphere Vibrating Radially in Air.*

The sphere will be treated as an elastic membrane of mass M and surface density σ, which is maintained nearly at a definite radius r_0 by springs. It will be supposed that, in the absence of the air, the frequency of vibration of the sphere would be $n/2\pi$. If ρ is the density of the air, r_0 the radius of the sphere when in equilibrium under the pressure of the air, $r_0+\xi$ its radius at time t, δp the excess of pressure above that in equilibrium, the equation of motion of the sphere is

$$M(\ddot{\xi}+n^2\xi) = -4\pi r_0^2\delta p,$$

where dots denote differentiation with respect to t.

With the ordinary approximations the velocity of the air and δp can be expressed in terms of the velocity potential ϕ. The above equation may be written

$$\ddot{\xi}+n^2\xi = \frac{\rho}{\sigma}\left(\frac{\partial\phi}{\partial t}\right)_{r=r_0}. \tag{9}$$

The velocity of sound in air being denoted by a, ϕ must satisfy the equation $\partial^2\phi/\partial t^2 = a^2\nabla^2\phi$ outside the sphere and the condition

$$\left(\frac{\partial\phi}{\partial r}\right)_{r=r_0} = \dot{\xi}. \tag{10}$$

The conditions of the problem being symmetrical about the centre of the sphere, ϕ must have the form $r^{-1}\chi(at-r)$, where χ is an unknown function. Equations (9) and (10) may be written

$$\ddot{\xi}+n^2\xi = \frac{a\rho}{\sigma r_0}\chi', \qquad \dot{\xi} = -\frac{1}{r_0^2}(\chi+r_0\chi'), \tag{11}$$

where accents denote differentiation of the function $\chi(at-r_0)$ with respect to its argument. The system of differential equations (11) is of the third order, and we may solve it by eliminating ξ and forming a differential equation for χ or *vice versa.* We should get a linear differential equation of the third order with constant coefficients. The three arbitrary constants of the solution of the equation for χ are definite multiples of the constants of the solution of the equation for ξ. Instead of proceeding in this way, we can obtain the complete primitive of the system of equations by assuming the forms

$$\chi(at-r) = Ae^{\lambda(at-r+r_0)}, \qquad \xi = Be^{\lambda at}, \tag{12}$$

the constant factor $e^{\lambda r_0}$ being inserted in the form of χ. Then

$$(n^2+\lambda^2a^2)B = \frac{a\rho}{\sigma r_0}\lambda A, \qquad \lambda a B = -\frac{1}{r_0^2}A(1+\lambda r_0). \tag{13}$$

It follows that λ satisfies the equation

$$(n^2+\lambda^2a^2)(1+\lambda r_0)+\frac{\rho r_0}{\sigma}\lambda^2a^2 = 0, \tag{14}$$

and that, if λ_1, λ_2, λ_3 are the roots of this equation, the complete primitive of the system of equations (11) leads to the following forms for ϕ and ξ:—

$$\phi = \frac{1}{r}\left[A_1e^{\lambda_1(at-r+r_0)}+A_2e^{\lambda_2(at-r+r_0)}+A_3e^{\lambda_3(at-r+r_0)}\right], \tag{15}$$

$$\xi = -\frac{1+\lambda_1r_0}{r_0^2a\lambda_1}A_1e^{\lambda_1at}-\frac{1+\lambda_2r_0}{r_0^2a\lambda_2}A_2e^{\lambda_2at}-\frac{1+\lambda_3r_0}{r_0^2a\lambda_3}A_3e^{\lambda_3at}. \tag{16}$$

5. *Symmetrical Sound Wave produced by Initial Impulse.*

In the simplest case the system is set in motion by an impulse delivered at the instant $t = 0$. Then ϕ vanishes when t is negative, and ξ vanishes when $t = 0$, but $\dot{\xi}$ has a given value $\dot{\xi}_0$ when $t = 0$. The condition that ϕ vanishes for all negative values of t requires that $\chi(\zeta) = 0$ for all values of ζ which are less than $-r_0$. Hence the solution expressed by (15) holds only for values of r which are less than $at+r_0$. For greater values of r, $\chi(at-r) = 0$. Hence we have a wave with a boundary $r = at+r_0$ travelling outwards with velocity a. The values of $\partial\phi/\partial r$ and $\partial\phi/\partial t$ at the front of the wave must satisfy the condition expressed in § 3, viz.,

$$\frac{\partial\phi}{\partial t} = -a\frac{\partial\phi}{\partial r},$$

or $$\frac{A_1\lambda_1a}{r}+\frac{A_2\lambda_2a}{r}+\frac{A_3\lambda_3a}{r} = a\left[\frac{A_1\lambda_1}{r}+\frac{A_2\lambda_2}{r}+\frac{A_3\lambda_3}{r}+\frac{A_1+A_2+A_3}{r^2}\right],$$

or $$A_1+A_2+A_3 = 0.$$

The initial conditions in regard to ξ and $\dot{\xi}$ give the equations

$$\left(\frac{1}{\lambda_1}+r_0\right)A_1+\left(\frac{1}{\lambda_2}+r_0\right)A_2+\left(\frac{1}{\lambda_3}+r_0\right)A_3 = 0,$$

$$(1+\lambda_1r_0)A_1+(1+\lambda_2r_0)A_2+(1+\lambda_3r_0)A_3 = -r_0^2\dot{\xi}_0.$$

To determine the constants A_1, A_2, A_3 we have therefore the equations

$$\Sigma A = 0, \qquad \Sigma A/\lambda = 0, \qquad \Sigma A\lambda = -r_0\dot{\xi}_0, \tag{17}$$

and the complete solution of the problem is expressed by the equations

$$\xi = \frac{1}{ar_0}\dot{\xi}_0\left\{\frac{(1+\lambda_1 r_0)e^{\lambda_1 at}}{(\lambda_1-\lambda_2)(\lambda_1-\lambda_3)}+\frac{(1+\lambda_2 r_0)e^{\lambda_2 at}}{(\lambda_2-\lambda_3)(\lambda_2-\lambda_1)}+\frac{(1+\lambda_3 r_0)e^{\lambda_3 at}}{(\lambda_3-\lambda_1)(\lambda_3-\lambda_2)}\right\}, \tag{18}$$

$$\phi = -\frac{r_0}{r}\dot{\xi}_0\left\{\frac{\lambda_1 e^{\lambda_1(at-r+r_0)}}{(\lambda_1-\lambda_2)(\lambda_1-\lambda_3)}+\frac{\lambda_2 e^{\lambda_2(at-r+r_0)}}{(\lambda_2-\lambda_3)(\lambda_2-\lambda_1)}+\frac{\lambda_3 e^{\lambda_3(at-r+r_0)}}{(\lambda_3-\lambda_1)(\lambda_3-\lambda_2)}\right\}, \tag{19}$$

of which the latter holds when $at+r_0 > r > r_0$. When $r > at+r_0$ we must put $\phi = 0$. Here λ_1, λ_2, λ_3 are the roots of the equation (14).

The solution represents a composite system of waves. To interpret it we consider the case where the ratio $\rho r_0/\sigma$ is small. The roots of equation (14) are to a first approximation

$$\lambda_1 = -1/r_0, \qquad \lambda_2 = \iota n/a, \qquad \lambda_3 = -\iota n/a.$$

To this order of approximation the first term in ξ vanishes and ξ becomes $n^{-1}\dot{\xi}_0 \sin nt$. To the same order of approximation ϕ becomes

$$\frac{r_0^2 a^2}{a^2+n^2r_0^2}\dot{\xi}_0\frac{1}{r}\left[e^{-(at+r_0-r)/r_0}-\cos n\left(t-\frac{r-r_0}{a}\right)-\frac{nr_0}{a}\sin n\left(t-\frac{r-r_0}{a}\right)\right]. \tag{20}$$

Hence, to this order of approximation, the motion of the sphere is the same as it would be in the absence of the air, and the motion of the air consists of two wave-motions: one of simple harmonic type which would be forced by the maintenance for an indefinite time of this motion of the sphere, the other of exponential type. Near the sphere the latter is damped rapidly, but near the front of the wave it is of the same degree of importance as the simple harmonic wave. The wave of exponential type is practically confined to a small region near the front of the advancing wave, but, in this region, it is comparable with the simple harmonic wave and the coexistence of the two is required for the continued advance of the front.

When we proceed to a second approximation we find that, to the first order in $\lambda r_0/\sigma$,

$$\left.\begin{aligned}\lambda_1 &= -\frac{1}{r_0}-\frac{\rho}{\sigma}\frac{a^2}{a^2+n^2r_0^2}\\ \lambda_2 &= -\frac{\rho r_0}{2\sigma}\frac{n^2r_0}{a^2+n^2r_0^2}+\iota\left(\frac{n}{a}-\frac{\rho r_0}{2\sigma}\frac{na}{a^2+n^2r_0^2}\right)\end{aligned}\right\}, \tag{21}$$

and λ_3 is the imaginary conjugate to λ_2. It follows that the motion of the sphere consists of a motion of exponential type which decays very rapidly, compounded with a motion of the ordinary damped harmonic type. Since the coefficient of $e^{\lambda_1 at}$ in (18) is small of the order $\rho r_0/\sigma$, the former motion is small compared with the latter, and, since this coefficient is negative, the effect of this component of the motion is to make the maximum displacement of the surface slightly less than it would be in the absence of the air. The modulus of decay of the damped harmonic oscillations is $\frac{\rho r_0}{2\sigma}\frac{n^2 r_0 a}{a^2+n^2 r_0^2}$, which is the value that would be found by the customary methods. In the motion of the air the simple harmonic wave-trains obtained by the first approximation become damped harmonic wave-trains, so that the motion near the sphere subsides gradually in the same way as the motion of the sphere; but, since all the exponentials in (19) have the value unity at the wave-front, there is no damping at the front, and the motion at the front of the wave is subject to diminution through the law of spherical divergence only. As before, the co-existence of the exponential wave-train and the slightly damped harmonic wave-train is necessary to the continued advance of the wave-front.

When there is initial displacement as well as initial velocity the problem is but slightly more complicated. The second of equations (17) must then be replaced by $\Sigma A/\lambda = -r_0^2 a\xi_0$. *Keeping the first approximation only, we find that when t is positive and r is less than $at+r_0$ the forms for ξ and ϕ are

$$\xi = \xi_0 \cos nt + n^{-1}\dot{\xi}_0 \sin nt,$$

$$\phi = \frac{A}{r}\left[\sin\left\{n\left(t-\frac{r-r_0}{a}\right)+\alpha\right\} - \sin\alpha\, e^{-(at-r+r_0)/r_0}\right],$$

where A and α are given in terms of ξ_0 and $\dot{\xi}_0$ by the equations

$$A^2 = \frac{a^2 r_0^4(\dot{\xi}_0^2+n^2\xi_0^2)}{a^2+n^2 r_0^2}, \qquad \tan\alpha = \frac{a\dot{\xi}_0+n^2 r_0\xi_0}{n(r_0\dot{\xi}_0 - a\xi_0)}.$$

The form of $r\phi$, as the sum of a simple harmonic function of $n(t-r/a)$ and an exponential function of $(at-r)/r_0$, is determined by the conditions which hold at the surface of the vibrating sphere. The form of the ratio of the coefficients of these two terms, viz., $-\sin\alpha$, is determined by the conditions which hold at the front of the wave. The actual value of this

* A small addition (placed between asterisks) has here been made to the paper (April 17th, 1904). Prof. Larmor called my attention to the special case noted in equation (21a).

ratio is determined by the initial conditions. As in the case of initial velocity without initial displacement, the wave is in general composite. In one case it can be simple. This happens when ξ_0 and $\dot{\xi}_0$ are connected by the equation $\dot{\xi}_0 = -n^2 r_0 a^{-1} \xi_0$. In this case α vanishes and ϕ has the form

$$\phi = A r^{-1} \sin n \{t - (r - r_0)/a\}. \qquad (21a)^*$$

6. *Decay of Vibrations that have been maintained for a time.*

We may extend the method of Art. 5 to the case where the system is set in motion by forces which operate for a finite time. It will be sufficient to consider the motion due to periodic forces acting in the interval $t_1 > t > 0$, and to suppose that when $t < 0$ the sphere and the air surrounding it are at rest. Taking the force acting on the sphere to be proportional to $e^{\iota\kappa a t}$, equation (9) is replaced by an equation of the form

$$\ddot{\xi} + n^2 \xi = F e^{\lambda_0 a t} + \frac{\rho}{\sigma} \left(\frac{\partial \phi}{\partial t}\right)_{r=r_0}, \qquad (22)$$

where λ_0 is written for $\iota\kappa$. Equation (10) is unaltered, and the form of ϕ is the same as before, viz., $r^{-1}\chi(at-r)$. The system of equations (10) and (22) will possess a particular solution of the form

$$\phi = r^{-1} A_0 e^{\lambda_0 (at - r + r_0)}, \qquad \xi = B_0 e^{\lambda_0 a t}, \qquad (23)$$

where

$$\left.\begin{aligned} B_0 (n^2 + \lambda_0^2 a^2) - \frac{a\rho}{\sigma r_0} \lambda_0 A_0 &= F \\ B_0 a \lambda_0 + \frac{1}{r_0^2} (1 + \lambda_0 r_0) A_0 &= 0 \end{aligned}\right\}. \qquad (24)$$

Since equation (14) has not any pure imaginary roots, λ_0 cannot be a root of it, and the equations (24) determine A_0 and B_0 in terms of F. We shall therefore take A_0 and B_0 to be known. The complete expressions for ϕ and ξ are to be determined by adding to the right-hand members of (23) expressions of the forms given by (15) and (16), in which the constants A_1, A_2, A_3 are to be determined by the conditions that ξ and $\dot{\xi}$ vanish when $t = 0$ and that $(\partial\phi/\partial t) + a(\partial\phi/\partial r)$ vanishes at $r = at + r_0$. These conditions give

$$\sum_0^3 \left(\frac{1}{\lambda_s} + r_0\right) A_s = 0, \quad \sum_0^3 (1 + r_0 \lambda_s) A_s = 0, \quad \sum_0^3 A_s = 0, \qquad (25)$$

and these equations determine A_1, A_2, A_3 in terms of A_0. It follows that we may put

$$\left.\begin{aligned} \xi &= -\frac{1}{ar_0^2}\sum_0^3\left(\frac{1}{\lambda_s}+r_0\right)A_s e^{\lambda_s at} \\ \phi &= \frac{1}{r}\sum_0^3 A_s e^{\lambda_s(at-r+r_0)} \end{aligned}\right\}, \tag{26}$$

in which the A's are known in terms of F, λ_0 is $\iota\kappa$, and λ_1, λ_2, λ_3 are the roots of the equation (14). This solution holds for ξ when t is in the interval $t_1 > t > 0$, and it holds for ϕ when $at+r_0 > r > r_0$ and t is in the same interval. The motion of the sphere is compounded of three motions:—(1) a simple harmonic motion of the same period as the force and having a definite phase-relation to the force, (2) a motion of exponential type which is relatively very small when the sphere is massive, (3) a motion of slightly damped harmonic type. The second and third of these motions are of the same types as those which are consequent upon an initial disturbance. The motion of the air is compounded of three wave-motions of types corresponding exactly with the three motions of the sphere. When the force has been in action for a sufficiently long time the motion of the sphere is practically a simple harmonic motion, and the motion of the air near the sphere is practically a simple harmonic wave-train. These motions are represented by the particular solutions (23). But the motion of the air near the front of the waves never has this simple character. The co-existence of the three types of waves is necessary to the continued advance of the wave-front.

The mode of decay of the vibratory motion after the force has ceased to act will be determined by taking a new solution of the equations (9) and (10) in the forms

$$\left.\begin{aligned} \phi &= \frac{1}{r}\sum_1^3 A'_s e^{\lambda_s(at-at_1-r+r_0)} \\ \xi &= -\frac{1}{ar_0^2}\sum_1^3\left(\frac{1}{\lambda_s}+r_0\right)A'_s e^{\lambda_s(at-at_1)} \end{aligned}\right\}, \tag{27}$$

in which constant factors $e^{\lambda_s at_1}$ are absorbed in the constants A'_s. The constants A'_1, A'_2, A'_3 are determined by the conditions that ξ and $\dot{\xi}$ have given values when $t = t_1$, and that $(\partial\phi/\partial t)+a(\partial\phi/\partial r)$ is continuous at the surface $r = at-at_1+r_0$. The solution expressed by (27) will hold in the interval $t > t_1$ and in the region $r_0 < r < at-at_1+r_0$. The equations by which the constants A_s' are determined are accordingly

$$\left.\begin{array}{c}\sum_1^3 \left(\frac{1}{\lambda_s}+r_0\right) A_s' = \sum_0^3 \left(\frac{1}{\lambda_s}+r_0\right) A_s e^{\lambda_s a t_1} \\ \sum_1^3 (1+\lambda_s r_0) A_s' = \sum_0^3 (1+\lambda_s r_0) A_s e^{\lambda_s a t_1} \\ \sum_1^3 A_s' = \sum_0^3 A_s e^{\lambda_s a t_1}\end{array}\right\}. \tag{28}$$

The results show that the simple harmonic motion of the sphere with the period of the force ceases at once, and the subsequent motion of the sphere is of the same kind as the motion consequent upon given initial displacements and velocities. The motion of the air near the sphere is of the same kind as that determined by initial conditions. The two types of motion—exponential and slightly damped harmonic—must co-exist in order that the waves sent out in the subsequent motion may be continuous with the waves sent out by the maintained vibrations.

7. *Rigid Sphere vibrating in Air.*

As a second example, we may consider the vibrations of a rigid sphere of mass M controlled by a spring of such strength that in the absence of the air the frequency would be $n/2\pi$. The surface of the sphere at any time may be taken to be expressed by the equation $r = r_0 + \xi P_1$, where P_1, or more fully $P_1(\cos\theta)$, is the zonal surface harmonic of degree unity referred to the line of motion of the centre as axis. The motion of the air will be expressed by a velocity potential ϕ of the form given by the equation

$$\phi = rP_1 \left(\frac{1}{r}\frac{\partial}{\partial r}\right) \frac{\chi(at-r)}{r} = -P_1 r^{-2}(\chi + r\chi'). \tag{29}$$

The function χ is connected with the displacement ξ by two equations which hold at $r = r_0$. One of these is the condition of continuity of velocity normal to the surface, viz.,

$$\dot{\xi} = -\frac{\partial}{\partial r}\{r^{-2}\chi + r^{-1}\chi'\}_{r=r_0}, \tag{30}$$

and the other is the equation of motion of the sphere, viz.,

$$M(\ddot{\xi}+n^2\xi) = \int_0^\pi \left(-\rho\frac{\partial\phi}{\partial t}\right)_{r=r_0} (-\cos\theta)\, 2\pi r_0^2 \sin\theta \, d\theta, \tag{31}$$

where $\cos\theta$ is the argument of P_1. Taking σ for the density of the sphere, we may write this equation:—

$$\ddot{\xi}+n^2\xi = -\frac{\rho}{\sigma}\frac{a}{r_0^3}\{\chi'(at-r_0)+r_0\chi''(at-r_0)\}. \tag{32}$$

Equation (30) is

$$\dot{\xi} = r_0^{-3}\{2\chi(at-r_0)+2r_0\chi'(at-r_0)+r_0^2\chi''(at-r_0)\}. \tag{33}$$

To solve these equations we assume

$$\chi(at-r_0) = Ae^{\lambda at}, \qquad \xi = Be^{\lambda at}, \tag{34}$$

a factor $e^{-\lambda r_0}$ being absorbed in A. Then we have

$$\left.\begin{aligned} B(n^2+a^2\lambda^2) &= -\frac{\rho}{\sigma}\frac{a}{r_0^3}(\lambda+r_0\lambda^2)A \\ B\lambda a &= \frac{1}{r_0^3}(2+2r_0\lambda+r_0^2\lambda^2)A \end{aligned}\right\}, \tag{35}$$

so that λ must satisfy the equation

$$(n^2+a^2\lambda^2)(2+2r_0\lambda+r_0^2\lambda^2)+\frac{\rho}{\sigma}a^2\lambda^2(1+r_0\lambda) = 0. \tag{36}$$

If λ_1, λ_2, λ_3, λ_4 are the roots of this equation, the forms for ξ and ϕ are

$$\xi = \sum_1^4 \frac{2+2r_0\lambda_s+r_0^2\lambda_s^2}{r_0^3 a\lambda_s}A_s e^{\lambda_s at}, \tag{37}$$

$$\phi = -\frac{P_1}{r^2}\sum_1^4 (1+r\lambda_s)A_s e^{\lambda_s(at-r+r_0)}. \tag{38}$$

This solution holds when $r < at+r_0$ and $t > 0$. It can be adapted, as before, to represent motions due to given initial values of ξ and $\dot{\xi}$. This adaptation yields two equations connecting the A's. The condition at the front of an advancing wave, viz., $(\partial\phi/\partial t)+a(\partial\phi/\partial r) = 0$, gives rise to the relation

$$\sum_1^4\left(\frac{2}{r^3}+\frac{\lambda_s}{r^2}\right)A_s = 0,$$

which must be satisfied when $r = at+r_0$. This condition, therefore, is equivalent to the two equations

$$\sum_1^4 A_s = 0, \qquad \sum_1^4 \lambda_s A_s = 0. \tag{39}$$

Hence all the constants are determinate when the initial state is given.

When ρ/σ is small two roots of the equation (36) are approximately

$$\lambda_1 = \iota n/a, \qquad \lambda_2 = -\iota n/a, \tag{40}$$

and the other two are approximately

$$\lambda_3 = \tfrac{1}{2}(-1+\iota)/r_0, \qquad \lambda_4 = \tfrac{1}{2}(-1-\iota)/r_0. \tag{41}$$

The coefficients of $e^{\lambda_3 at}$ and $e^{\lambda_4 at}$ in (37) are very small, and the motion of the sphere is very nearly a simple harmonic vibration of frequency $n/2\pi$. The motion of the air is compounded of two wave-motions: one wave-train is very nearly simple harmonic, with the same period as the motion of the sphere; and the other is very rapidly damped. Near the sphere the motion of the air is practically that belonging to the simple harmonic wave-train. Near the front of the wave the rapidly damped harmonic motion has the same degree of importance as the nearly simple harmonic motion, and the co-existence of the two is necessary to the continued advance of the front.

When we make a second approximation to the roots λ_1 and λ_2 we find

$$\lambda_1 = -\frac{\rho}{2\sigma}\,\frac{n^4 r_0^3}{4a^4+n^4 r_0^4} + \iota\frac{n}{a} - \iota\frac{\rho}{2\sigma}\,\frac{an(2a^2+n^2r_0^2)}{4a^4+n^4r_0^4}, \tag{42}$$

and λ_2 is the imaginary conjugate to λ_1. This approximation gives the same results as regards the effective inertia and the decay of the motion as are obtained by Lord Rayleigh (*Theory of Sound*, Vol. II., § 325).

Similar methods may be employed when the motion of the sphere is maintained periodic for a time and then allowed to decay, with results of the same kind as those for radial vibrations. Further, no essentially new feature is introduced when the normal displacement of the sphere depends upon a surface harmonic of order higher than unity.

8. *Electric Vibrations of Order Unity.*

The first case of electric vibrations to be discussed is that in which electrification is distributed over the surface of a conducting sphere with surface density proportional to the first zonal harmonic P_1. We shall suppose that before the instant $t = 0$ the electrostatic field of this electrification is established through all space outside the sphere $r = r_0$. The initial state of the medium outside this sphere is that expressed by the equations

$$\left.\begin{aligned} (X, Y, Z) &= E\left(\frac{3xz}{r^5}, \frac{3yz}{r^5}, -\frac{1}{r^3}+\frac{3z^2}{r^5}\right) \\ (\alpha, \beta, \gamma) &= 0 \end{aligned}\right\}, \tag{43}$$

in which E is a constant. The initial surface density on the sphere is then $EP_1/2\pi r_0^3$.

At the instant $t = 0$ the cause which previously maintained the field expressed by (43) is supposed to cease to operate. Thereafter the surface

$r = r_0$ is to be taken to be that of a perfect conductor. It is required to determine the subsequent state of the medium in accordance with the conditions: (i.) that the initial field is that expressed by (43), (ii.) that the tangential electric force vanishes at $r = r_0$.

It is clear that a new state of the medium arises, for the surface condition at $r = r_0$ is not satisfied by (43). It is clear also that the disturbed state of the medium cannot at any instant t have extended to the part of the medium beyond the sphere $r = ct + r_0$. Further, it is known that the problem can have only one solution.* The form of solution which suggests itself naturally involves the assumptions (i.) that the surface density on the sphere is always distributed so as to be proportional to P_1, (ii.) that the spherical surface $r = ct + r_0$ is the front of an advancing wave. We can show that the solution obtained by means of these assumptions satisfies all the conditions of the problem. In accordance with what has been said in § 2, we ought to take (X, Y, Z) and (α, β, γ) in the region $r < ct + r_0$ to be given by (1) and (2) with $n = 1$ and $\omega_n = z$. We take them, therefore, to be given by the equations

$$\left.\begin{aligned} (X, Y, Z) &= \mathrm{c}\,(xz,\ yz,\ -x^2-y^2)\left(\frac{1}{r}\frac{\partial}{\partial r}\right)^2 \frac{\chi(ct-r)}{r} \\ &\quad + \mathrm{c}\,(0,\ 0,\ -2)\left(\frac{1}{r}\frac{\partial}{\partial r}\right)\frac{\chi(ct-r)}{r} \\ (\alpha, \beta, \gamma) &= \mathrm{c}\,(y,\ -x,\ 0)\left(\frac{1}{r}\frac{\partial}{\partial r}\right)\frac{\chi'(ct-r)}{r} \end{aligned}\right\}, \tag{44}$$

which are the same as

$$(X, Y, Z) = \mathrm{c}\left(\frac{\partial^2}{\partial x\,\partial z},\ \frac{\partial^2}{\partial y\,\partial z},\ -\frac{\partial^2}{\partial x^2}-\frac{\partial^2}{\partial y^2}\right)\frac{\chi(ct-r)}{r},$$

$$(\alpha, \beta, \gamma) = \left(\frac{\partial^2}{\partial y\,\partial t},\ -\frac{\partial^2}{\partial x\,\partial t},\ 0\right)\frac{\chi(ct-r)}{r},$$

while (43) is the same as

$$(X, Y, Z) = \left(\frac{\partial^2}{\partial x\,\partial z},\ \frac{\partial^2}{\partial y\,\partial z},\ \frac{\partial^2}{\partial z^2}\right)\frac{E}{r}.$$

We then show that we can adjust the unknown function χ so as to satisfy the surface condition at the conductor $r = r_0$, and also to satisfy the conditions of the types (7) and (8) which must hold at the front of the advancing wave.

* For the proof of this theorem in the case where there is a moving surface of discontinuity, see my paper already cited in *Proceedings* (Ser. 2), Vol. 1.

To express the condition that the tangential component of (X, Y, Z) vanishes at $r = r_0$, denote by R the radial component of (X, Y, Z), so that

$$Rr = Xx + Yy + Zz.$$

Then, at this surface, R is the resultant of (X, Y, Z) and the direction of (X, Y, Z) is the same as that of r. Hence, at this surface, we have

$$\frac{X}{x} = \frac{Y}{y} = \frac{Z}{z} = \frac{R}{r}$$

or

$$X - Rx/r = 0, \quad \ldots .$$

I form therefore the vector $(X - Rx/r,\ Y - Ry/r,\ Z - Rz/r)$, and express the condition that it vanishes at $r = r_0$. I find

$$\left(X - R\frac{x}{r},\ Y - R\frac{y}{r},\ Z - R\frac{z}{r}\right)$$

$$= \mathrm{c}(xz,\ yz,\ -x^2-y^2)\left[\left(\frac{1}{r}\frac{\partial}{\partial r}\right)^2 \frac{\chi(\mathrm{c}t-r)}{r} + \frac{2}{r^3}\frac{\partial}{\partial r}\frac{\chi(\mathrm{c}t-r)}{r}\right], \quad (45)$$

and it vanishes at $r = r_0$, provided

$$\chi(\mathrm{c}t-r_0) + r_0\chi'(\mathrm{c}t-r_0) + r_0^2\chi''(\mathrm{c}t-r_0) = 0. \quad (46)$$

This holds for all positive values of t.

To deduce the form of χ let ζ stand for $\mathrm{c}t - r_0$. Then $\chi(\zeta)$ is a function of ζ which, for all values of ζ that are $> -r_0$, satisfies the equation

$$\chi''(\zeta) + r_0^{-1}\chi'(\zeta) + r_0^{-2}\chi(\zeta) = 0;$$

and therefore, for all such values, $\chi(\zeta)$ has the form

$$\chi(\zeta) = e^{-\frac{1}{2}\zeta/r_0}\left[A_1\cos\left(\tfrac{1}{2}\sqrt{3}\,\zeta/r_0\right) + B_1\sin\left(\tfrac{1}{2}\sqrt{3}\,\zeta/r_0\right)\right].$$

It follows that, for all values of r and t which satisfy the inequality $\mathrm{c}t > r - r_0$, $\chi(\mathrm{c}t - r)$ may be written in the form

$$\chi(\mathrm{c}t-r) = A\,e^{-\frac{1}{2}(\mathrm{c}t-r+r_0)/r_0}\sin\left\{\frac{\sqrt{3}}{2r_0}(\mathrm{c}t-r+r_0)+\epsilon\right\}, \quad (47)$$

where A and ϵ are arbitrary constants. It follows that a damped harmonic train of waves is propagated outwards; the period is $4\pi r_0/\mathrm{c}\sqrt{3}$, and the modulus of decay is $\mathrm{c}/2r_0$. The forms obtained by substituting this value of χ in (44) are those which are generally taken as the solution of the problem. This solution holds, however, only when $r < \mathrm{c}t + r_0$.

The conditions which have to be satisfied at the wave-front $r = \mathrm{c}t + r_0$ are three of the type

$$r(X - X_0) = z\beta - y\gamma \quad (48)$$

and three of the type $-r\alpha = z(Y-Y_0)-y(Z-Z_0)$, (49)

where X_0, Y_0, Z_0 are the X, Y, Z expressed in (43), and (X, Y, Z) and (α, β, γ) are given by (44). The first of equations (48) is

$$\frac{xz}{r^4}(3\chi+3r\chi'+r^2\chi'')-\frac{E}{c}\,\frac{3xz}{r^4}=\frac{xz}{r^3}(\chi'+r\chi''), \tag{50}$$

the second differs from this only by having yz in place of xz, and the third is

$$-\frac{x^2+y^2}{r^4}(3\chi+3r\chi'+r^2\chi'')+\frac{2}{r^3}(\chi+r\chi')-\frac{E}{c}\left\{\frac{2}{r^3}-\frac{3(x^2+y^2)}{r^4}\right\}$$
$$=-\frac{x^2+y^2}{r^3}(\chi'+r\chi''). \tag{51}$$

In these χ, χ', χ'' must have their values at the surface $r = ct+r_0$ and r must have this value. It follows that the value of χ' vanishes at this surface, and that the value of χ at this surface is E/c. When these conditions are satisfied, it appears that equations (49) are satisfied identically. Now we have

$$\chi'(ct-r) = -\tfrac{1}{2}Ar_0^{-1}e^{-\frac{1}{2}(ct-r+r_0)/r_0}\left[\sin\left\{\frac{\sqrt{3}}{2r_0}(ct-r+r_0)+\epsilon\right\}\right.$$
$$\left.-\sqrt{3}\cos\left\{\frac{\sqrt{3}}{2r_0}(ct-r+r_0)+\epsilon\right\}\right],$$

and this vanishes at the surface $r = ct+r_0$, if $\epsilon = \frac{1}{3}\pi$. Thus we have

$$\chi = \frac{2E}{c\sqrt{3}}e^{-\frac{1}{2}(ct-r+r_0)/r_0}\sin\left\{\frac{\sqrt{3}}{2r_0}(ct-r+r_0)+\frac{\pi}{3}\right\}, \tag{52}$$

and the constants A and ϵ are determined. With this form of χ we find

$$\left.\begin{aligned}\chi' &= -\frac{1}{r_0}\,\frac{2E}{c\sqrt{3}}e^{-\frac{1}{2}(ct-r+r_0)/r_0}\sin\left\{\frac{\sqrt{3}}{2r_0}(ct-r+r_0)\right\}\\ \chi'' &= \frac{1}{r_0^2}\,\frac{2E}{c\sqrt{3}}e^{-\frac{1}{2}(ct-r+r_0)/r_0}\sin\left\{\frac{\sqrt{3}}{2r_0}(ct-r+r_0)-\frac{\pi}{3}\right\}\end{aligned}\right\}. \tag{53}$$

It may be observed that with the above determination of A and ϵ the magnetic force along a circle of latitude is

$$-\frac{\sin\theta}{r}\,\frac{2E}{r_0^2\sqrt{3}}\left(1-\frac{r_0}{r}+\frac{r_0^2}{r^2}\right)^{\frac{1}{2}}e^{-\vartheta}\cos(\sqrt{3}\,\vartheta+\delta),$$

where $\vartheta = \frac{1}{2}(ct-r+r_0)/r_0$ and $\tan\delta = (r-2r_0)/r\sqrt{3}$;

also the radial electric force is

$$\frac{\cos\theta}{r^2}\,\frac{4E}{r_0\sqrt{3}}\left(1-\frac{r_0}{r}+\frac{r_0^2}{r^2}\right)^{\frac{1}{2}} e^{-\vartheta}\cos\left(\sqrt{3}\,\vartheta+\delta+\frac{\pi}{3}\right),$$

and the tangential electric force is

$$-\frac{\sin\theta}{r}\,\frac{2E}{r_0^2\sqrt{3}}\left(1-\frac{r_0}{r}\right)\left(1+\frac{r_0}{r}+\frac{r_0^2}{r^2}\right)^{\frac{1}{2}} e^{-\vartheta}\cos(\sqrt{3}\,\vartheta+\delta'),$$

where $\tan\delta' = (r-r_0)/(r+r_0)\sqrt{3}$. These results differ from those obtained by J. J. Thomson (*loc. cit.*) as regards phase. The phases given by him are not determined by the conditions which hold at the front of the wave-train.

It appears from the above analysis that the damped harmonic wave-train represented by the customary form of solution can advance into a region in which the electric field is expressed by (43). The same analysis can be applied at once to show that it cannot advance into a region free from electric and magnetic forces; it can also be applied to determine the mode of decay of the external field due to maintained electrical oscillations of the same surface harmonic type on the sphere. Exactly as in the problem of sound waves it appears that the forced wave must be accompanied by a wave of the type (47), and that the wave that is propagated outwards after the system is left to itself is also of the type expressed by (47); and the constants A and ϵ of these two waves can be adjusted so as to satisfy the conditions that hold at the front of the forced wave and at the common boundary of the two waves. The concurrent existence of a wave of type (47) along with the forced wave is necessary to the continued advance of the wave-front. Exactly as in the sound problems the damped harmonic wave-train is not damped at the front, but is subject only to the kind of diminution by spherical divergence that is appropriate to the spherical surface harmonic.

9. *Redistribution of the Energy.*

In the initial state the æther in the region between the spheres r and $r+dr$ possesses electric energy of amount $\frac{1}{8\pi}\,4\pi r^2 dr E^2\,\frac{2}{r^6}$ or $r^{-4}E^2 dr$, and the total energy of the field is $\frac{1}{3}E^2 r_0^{-3}$. In the subsequent state of wave-disturbance the same portion of the medium possesses magnetic energy of amount

$$\frac{1}{8\pi}\,4\pi r^2 dr\,\tfrac{2}{3}c^2 r^2\left\{\left(\frac{1}{r}\,\frac{\partial}{\partial r}\right)\frac{\chi'(ct-r)}{r}\right\}^2, \tag{54}$$

and it possesses electric energy of amount

$$\frac{1}{8\pi}\,4\pi r^2 dr\left[\tfrac{2}{3}c^2 r^4\left\{\left(\frac{1}{r}\frac{\partial}{\partial r}\right)^2\frac{\chi(ct-r)}{r}\right\}^2+4c^2\left\{\left(\frac{1}{r}\frac{\partial}{\partial r}\right)\frac{\chi(ct-r)}{r}\right\}^2\right.$$
$$\left.+\tfrac{8}{3}c^2 r^2\left\{\left(\frac{1}{r}\frac{\partial}{\partial r}\right)\frac{\chi(ct-r)}{r}\right\}\left\{\left(\frac{1}{r}\frac{\partial}{\partial r}\right)^2\frac{\chi(ct-r)}{r}\right\}\right], \quad (55)$$

where χ has the value already determined. The amount of the magnetic energy in the region is

$$\tfrac{4}{9}E^2\frac{dr}{r_0^4 r^2}\left\{r_0\sin\sqrt{3}\,\vartheta-r\sin\left(\sqrt{3}\,\vartheta-\frac{\pi}{3}\right)\right\}^2 e^{-2\vartheta}, \quad (56)$$

where ϑ is written for $\frac{1}{2}(ct-r+r_0)/r_0$. As soon as the wave-front has travelled to a distance from the conducting surface which is at all large compared with the radius of this surface the factor $e^{-2\vartheta}$ will be small except in the immediate neighbourhood of the wave-front, and thus we see that the energy of the wave-motion will be accumulated near the wave-front. Also when r is large compared with r_0 the above expression may approximately be replaced by

$$\tfrac{2}{9}\frac{E^2}{r_0^4}e^{-2\vartheta}\left\{1-\cos\left(2\sqrt{3}\,\vartheta-\frac{2\pi}{3}\right)\right\}dr. \quad (57)$$

We may calculate the energy between the wave-front and a spherical surface within it, and not far from it, by integrating this expression. Consider the case where the inner of these surfaces is at a distance of half a wave-length behind the front, *i.e.*, at a distance $2\pi r_0/\sqrt{3}$. The magnetic energy between the surfaces is approximately equal to

$$\tfrac{2}{9}\frac{E^2}{r_0^4}\int_0^{\pi/\sqrt{3}} e^{-2\vartheta}\left\{1+\tfrac{1}{2}\cos(2\sqrt{3}\,\vartheta)-\frac{\sqrt{3}}{2}\sin(2\sqrt{3}\,\vartheta)\right\}2r_0\,d\vartheta,$$

which is $$\tfrac{1}{6}E^2 r_0^{-3}(1-e^{-2\pi/\sqrt{3}}).$$

If we had taken the first wave-length of the advancing wave instead of the first half wave-length, we should have found $\frac{1}{6}E^2 r_0^{-3}(1-e^{-4\pi/\sqrt{3}})$ as the approximate value of the magnetic energy between the surfaces. If we calculate the electric energy in the same way and to the same order of approximation, we find the same values, so that the total energy between the two surfaces is approximately equal to $\frac{1}{3}E^2 r_0^{-3}(1-e^{-2\pi/\sqrt{3}})$ when the surfaces are half a wave-length apart, and to $\frac{1}{3}E^2 r_0^{-3}(1-e^{-4\pi/\sqrt{3}})$ when they are a wave-length apart. The terms omitted in the calculation are small compared with those retained in the order r_0/r and higher powers of r_0/r, r denoting the radius of the wave-front. It appears therefore that the energy of the initial electrostatic field is propagated outwards with the wave in such a way that the energy that was initially within a spherical surface surrounding the conductor is the energy of the wave-motion when that surface is the wave-front, and it is gathered up close behind the wave-front.* When the wave-front is at a great distance from the conductor the accumulation of energy at the front is so great that all but about $\frac{2}{75}$ of the total initial static energy of the field is gathered up in the first half-wave-length, and all but about $\frac{1}{1400}$ of it is gathered up in the first wave-length.

10. *Electrical Vibrations of Order* 2.

We suppose that the initial electrification has surface density proportional to the second zonal harmonic P_2, or to the solid harmonic $2z^2-x^2-y^2$, which is $2r_0^2P_2$, on the sphere of radius r_0, and we take the initial field to be given by the equation

$$(X, Y, Z) = E\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)\frac{x^2+y^2-2z^2}{r^5}, \tag{58}$$

which gives a surface density $(3/2\pi)Er_0^{-4}P_2$. The appropriate forms of (X, Y, Z) and (α, β, γ) in the ensuing wave-disturbance are expressed by (1) and (2), in which $n = 2$ and $\omega_n = 2z^2-x^2-y^2$, and we have

$$\left.\begin{aligned}
(\alpha, \beta, \gamma) &= 6c(yz, -zx, 0)\left(\frac{1}{r}\frac{\partial}{\partial r}\right)^2\frac{\chi'(ct-r)}{r}\\
X &= 6c\left\{x\left(\frac{1}{r}\frac{\partial}{\partial r}\right)^2\frac{\chi(ct-r)}{r}+z^2x\left(\frac{1}{r}\frac{\partial}{\partial r}\right)^3\frac{\chi(ct-r)}{r}\right\}\\
Y &= 6c\left\{y\left(\frac{1}{r}\frac{\partial}{\partial r}\right)^2\frac{\chi(ct-r)}{r}+z^2y\left(\frac{1}{r}\frac{\partial}{\partial r}\right)^3\frac{\chi(ct-r)}{r}\right\}\\
Z &= 6c\left\{-2z\left(\frac{1}{r}\frac{\partial}{\partial r}\right)^2\frac{\chi(ct-r)}{r}-(x^2+y^2)\,z\left(\frac{1}{r}\frac{\partial}{\partial r}\right)^3\frac{\chi(ct-r)}{r}\right\}
\end{aligned}\right\} \quad 59)$$

With these forms we find

$$X\frac{x}{r}+Y\frac{y}{r}+Z\frac{z}{r} = -6c\frac{2z^2-x^2-y^2}{r}\left(\frac{1}{r}\frac{\partial}{\partial r}\right)^2\frac{\chi(ct-r)}{r}, \tag{60}$$

and the (x, y, z)-components of a vector which has the same tangential

* The remark that it is the static electric energy of the field which is propagated with the waves was made by Prof. Larmor in a letter to the author before this paper began to be written.

components as (X, Y, Z), and no radial component can be written down in the forms

$$\left.\begin{aligned} X-\frac{x}{r}\left(X\frac{x}{r}+Y\frac{y}{r}+Z\frac{z}{r}\right) &= 6cxz^2\left\{\left(\frac{1}{r}\frac{\partial}{\partial r}\right)^3+\frac{3}{r^2}\left(\frac{1}{r}\frac{\partial}{\partial r}\right)^2\right\}\frac{\chi(ct-r)}{r} \\ Y-\frac{y}{r}\left(X\frac{x}{r}+Y\frac{y}{r}+Z\frac{z}{r}\right) &= 6cyz^2\left\{\left(\frac{1}{r}\frac{\partial}{\partial r}\right)^3+\frac{3}{r^2}\left(\frac{1}{r}\frac{\partial}{\partial r}\right)^2\right\}\frac{\chi(ct-r)}{r} \\ Z-\frac{z}{r}\left(X\frac{x}{r}+Y\frac{y}{r}+Z\frac{z}{r}\right) &= -6cz(x^2+y^2)\left\{\left(\frac{1}{r}\frac{\partial}{\partial r}\right)^3+\frac{3}{r^2}\left(\frac{1}{r}\frac{\partial}{\partial r}\right)^2\right\}\frac{\chi(ct-r)}{r} \end{aligned}\right\}. \quad (61)$$

Hence the condition that the surface $r = r_0$ may be that of a perfect conductor is

$$\chi'''(ct-r_0)+3r_0^{-1}\chi''(ct-r_0)+6r_0^{-2}\chi'(ct-r_0)+6r_0^{-3}\chi(ct-r_0) = 0, \quad (62)$$

and it follows that $\chi(ct-r)$ may be expressed in the form

$$\chi(ct-r) = A_1e^{\lambda_1(ct-r+r_0)}+A_2e^{\lambda_2(ct-r+r_0)}+A_3e^{\lambda_3(ct-r+r_0)}, \quad (63)$$

where λ_1, λ_2, λ_3 are the roots of the equation

$$r_0^3\lambda^3+3r_0^2\lambda^2+6r_0\lambda+6 = 0, \quad (64)$$

and A_1, A_2, A_3 are constants. The real root is approximately $-(1{\cdot}6)\,r_0^{-1}$, and the imaginary roots* are approximately $(-0{\cdot}7 \pm 1{\cdot}8\iota)r_0^{-1}$.

To determine the constants A_1, A_2, A_3 we have the conditions at the front of the wave, *i.e.*, at the surface $r = r_0+ct$. We use the same equations (48) and (49) as in the previous problem, but now X_0, Y_0, Z_0 are given by the equations

$$\left.\begin{aligned} X_0 &= -E\frac{3x}{r^5}\left(1-\frac{5z^2}{r^2}\right) \\ Y_0 &= -E\frac{3y}{r^5}\left(1-\frac{5z^2}{r^2}\right) \\ Z_0 &= -E\frac{3z}{r^5}\left(3-\frac{5z^2}{r^2}\right) \end{aligned}\right\}, \quad (65)$$

* J. J. Thomson, *Recent Researches*, p. 371, gives the imaginary roots.

and (X, Y, Z) and (α, β, γ) are given by (59). Now we have

$$\left.\begin{aligned} rX-z\beta+y\gamma &= 6c\left[\frac{3x}{r^4}\left\{\left(1-\frac{5z^2}{r^2}\right)\chi+\left(1-\frac{4z^2}{r^2}\right)r\chi'\right\}+\frac{x}{r^2}\left(1-\frac{3z^2}{r^2}\right)\chi''\right] \\ rY-x\gamma+z\alpha &= 6c\left[\frac{3y}{r^4}\left\{\left(1-\frac{5z^2}{r^2}\right)\chi+\left(1-\frac{4z^2}{r^2}\right)r\chi'\right\}+\frac{y}{r^2}\left(1-\frac{3z^2}{r^2}\right)\chi''\right] \\ rZ-y\alpha+x\beta &= 6c\left[\frac{3z}{r^4}\left\{\left(3-\frac{5z^2}{r^2}\right)\chi+2\left(1-\frac{2z^2}{r^2}\right)r\chi'\right\}+\frac{z}{r^2}\left(1-\frac{3z^2}{r^2}\right)\chi''\right] \end{aligned}\right\}, \tag{66}$$

so that equations (48) require that at the surface $r = ct+r_0$ we should have

$$\chi' = 0, \qquad \chi'' = 0, \qquad \chi = -E/6c,$$

and it will be found that when these equations are satisfied equations (49) are satisfied identically. Hence the constants A_1, A_2, A_3 are determined by the equations

$$\sum_1^3 A_s = -E/6c, \qquad \sum_1^3 A_s\lambda_s = 0, \qquad \sum_1^3 A_s\lambda_s^2 = 0. \tag{67}$$

The important result is that the A's are definite multiples of E/c, and, in particular, that the A that corresponds with the real value of λ is not small in comparison with the other A's.

Hence in this case the wave-motion that ensues when the initial statical field is left to subside is compounded of two wave-motions—one of exponential type determined by the real value of λ and the corresponding value of A, and the other of damped harmonic type determined by the conjugate complex values of λ and the conjugate complex values of A that belong to them. Neither of these waves can be propagated except in company with the other, for the co-existence of the two is requisite to the continued advance of the wave-front. Near the conductor the field subsides very rapidly, but near the common front of the waves the fields that belong to them are not subject to damping, but merely diminish according to the law of spherical divergence which is appropriate to the spherical surface harmonic concerned.

11. *Generalization of the Results for "Electrical Vibrations on a Spherical Conductor."*

We may proceed to discuss waves that correspond with surface harmonics of higher orders. In any case we have to form the condition which is to be satisfied at the surface of the conductor. Taking the forms (1) and (2), we may show that

$$X - \frac{x}{r}\left(X\frac{x}{r} + Y\frac{y}{r} + Z\frac{z}{r}\right)$$
$$= -\frac{c}{2n+1}\left\{(n+1)\frac{\partial\omega_n}{\partial x} - nr^{2n+1}\frac{\partial}{\partial x}\left(\frac{\omega_n}{r^{2n+1}}\right)\right\}$$
$$\times\left\{(n+1)\left(\frac{1}{r}\frac{\partial}{\partial r}\right)^n\frac{\chi}{r} + r^2\left(\frac{1}{r}\frac{\partial}{\partial r}\right)^{n+1}\frac{\chi}{r}\right\}, \quad (69)$$

and so on. Thus the condition is expressed by the equation

$$\frac{\partial}{\partial r}\left\{r^{n+1}\left(\frac{1}{r}\frac{\partial}{\partial r}\right)^n\frac{\chi(ct-r)}{r}\right\} = 0, \quad (70)$$

which must hold when $r = r_0$. This is a linear differential equation of the $(n+1)$-th order with constant coefficients satisfied by $\chi(ct-r_0)$, and it serves for the determination of $\chi(ct-r)$ in the form

$$\chi(ct-r) = \sum_1^{n+1} A_s e^{\lambda_s(ct-r+r_0)}, \quad (71)$$

where the values of λr_0 are roots of an equation of the $(n+1)$-th degree with determinate numerical coefficients. When n is even one root is real, and we may expect it to be negative; we may also expect the remaining roots to be complex with negative real parts, and when n is odd we may expect all the roots to have this character.* The constants A will be determined by the conditions which hold at the wave-front $r = ct+r_0$. In general these conditions can be shown to lead to the equations

$$\chi' = 0, \quad \chi'' = 0, \quad \ldots, \quad \chi^{(n)} = 0,$$

and $\chi =$ a given constant. These hold at $r = ct+r_0$, and they suffice to determine the constants $A_1, A_2, \ldots, A_{n+1}$. It follows that, in general, with an initial distribution of surface density proportional to a definite surface harmonic of order $2m$ a wave of exponential type and m waves of damped harmonic type are propagated together with a common front, and that when the harmonic is of order $2m+1$ the waves propagated are all of damped harmonic type and are in number $m+1$. In both cases the field near the sphere subsides rapidly, nearly all its energy being transferred to a relatively thin spherical shell near the wave-front. The field of each of the waves near their common front diminishes only through the kind of

* J. J. Thomson (*loc. cit.*) gives the roots in case $n = 3$.

spherical divergence that is appropriate to the spherical harmonic concerned. In all cases the co-existence of the various waves of the system is requisite for the continued advance of the wave-front.

12. *General Conclusion.*

In our problems of sound waves we found that, besides the slightly damped harmonic wave-trains which have periods nearly equal to the natural periods of the vibrator, there must be others of exponential or rapidly damped harmonic type, which accompany the former as they travel outwards and serve to establish continually the front of the advancing waves. These subsidiary wave-trains have little influence on the motion of the vibrator, but they play a large part in the motion of the medium, especially in the region near the wave-front. The number of them increases with the complexity of the mode of vibration (expressed in the case of a sphere by the order of a surface harmonic), and all those that correspond with a particular mode of vibration must co-exist along with the slightly damped harmonic wave-train that is characteristic of the mode. They cannot exist independently. The motion of the medium that belongs to any particular vibration of the nucleus may be analysed as above into a system of damped harmonic and exponential wave-trains, but the analysis is entirely mathematical and does not correspond with a possible physical analysis into motions that can be executed independently of each other.

When a distribution of charge, which would not be possible for a free charge, is maintained over a conductor, and suddenly released, electric waves travel out into the medium. The waves may be expected to fall into classes determined by the modes of distribution of the charge, and the number of waves in a class may be expected to be definite. The different waves in a class are of exponential or damped harmonic types, and they are distinguished from each other by the exponents and periods.* They can have no physical existence independently of each other; all the waves in a class must co-exist.† The composite system of electric waves which consists of all the waves in a class advances into the statical field due to the initial distribution of the charge, and the co-existence of the

* It is possible that the types may involve a dependence on time of a more complicated character than exponential or damped harmonic when the conductor is not spherical.

† Mr. Macdonald has called my attention to the fact that a similar result was found by Heaviside, *Electrical Papers*, Vol. II., p. 408.—*April* 17*th*, 1904.

several waves is necessary to the continued advance of the front. As the wave advances, it transforms into electromagnetic energy the excess of the statical energy of the initial field over that of a free charge of the same total amount on the same conductor; and this electromagnetic energy is transferred continually towards the front of the advancing wave, in such a way that at a distance from the conductor the wave practically passes as a pulse. The electric waves that are thus generated appear to have little analogy to the sound waves sent out from a vibrator, and having nearly the natural period of the vibrator, but to be analogous rather to the subsidiary sound waves which accompany these and serve to establish the advancing wave-fronts without having a sensible influence upon the vibrator. This theory should be applicable to all cases in which the space outside the conductor is simply-connected; there may be exceptions when the space is multiply-connected or when this condition is nearly realized—for example, when a gap is made in a conducting ring.