This is a contradiction, and therefore there can only be a finite number of the intervals D_i which contain points of the given set. Let these be $D_1, D_2, \dots, D_k$. In any one of these, say D_i , the points of the closed set form a closed set lying entirely internal to D_i . Hence we can assign two points P and Q internal to D_i such that the points of the closed set which lie in D_i form a closed set lying between P and Q (not inclusive).

Now between P and Q , (both inclusive), there are only internal points of the given set of intervals (§ 5); hence, by the Heine-Borel theorem, we can assign a *finite* number of the given intervals so as to entirely cover up the segment (P, Q) . In this finite number of the given intervals every point of the closed set of points in D_i must lie. Performing this for all values of i from 1 to k , we got a finite number of the original intervals having the required property.

The Continuations of certain Fundamental Power Series. By M. J. M. HILL, M.A., Sc.D., F.R.S., Astor Professor of Mathematics at University College, London. Received, in revised form, February 6th, 1903.*

1. The theory of the continuation of power series has not, so far as I can ascertain, been hitherto illustrated by applications to simple cases where the work is unartificial.

By means of some well known formulæ, proved rigidly for the first time by Abel, in his famous memoir on *The Binomial Series*, I have succeeded in working out the continuations of the binomial series, the logarithmic series, and the series for $\arctan x$ along arbitrary circuits. Using conjugate functions I have also succeeded in finding the continuation of the series for $\arcsin x$.

* The paper has been condensed from two papers communicated at the meetings of November 13th, 1902, and December 11th, 1902; the latter dealt exclusively with the case of $\arcsin x$.

The continuation of the logarithmic series is obtained by Harkness and Morley in their treatise on *The Theory of Functions*, pp. 123-4, by considering a circuit in the form of a regular polygon containing an infinite number of sides.

The observation that the result must be the same for an arbitrary circuit led me to the consideration of a circuit of that kind, and finally to the results in this paper.

The continuation of the binomial series will be found at the end of Art. 3 and Art. 4, of the logarithmic series in Art. 7, of the series for $\arctan x$ in Art. 8, of the series for $\arcsin x$ at the end of Art. 17.

2. First Illustration.—Continuation of the Binomial Series.

The work depends on a formula which can be deduced from those proved in Abel's memoir on *The Binomial Theorem*.

If $|x| < 1$, and if k_n be an abbreviation for $k(k-1) \dots (k-n+1)$, then

$$1 + k_1 x + \frac{k_2 x^2}{2!} + \dots + \frac{k_n x^n}{n!} + \dots$$

$$= |1+x|^k [\cos k \operatorname{amp}(1+x) + i \sin k \operatorname{amp}(1+x)],$$

where $\operatorname{amp}(1+x)$ is the principal amplitude of $1+x$. This principal amplitude lies between $-\frac{\pi}{2}$ and $+\frac{\pi}{2}$, because $|x| < 1$.

$$3. \text{ Let } P(x) = 1 + k_1 x + k_2 \frac{x^2}{2!} + \dots + k_n \frac{x^n}{n!} + \dots, \quad (\text{I.})$$

where $|x| < 1$.

$$\text{Put } x = x_1 + (x - x_1).$$

$$\text{Then } P(x|x_1) = P(x_1) + (x-x_1)P'(x_1) + \frac{(x-x_1)^2}{2!}P''(x_1) + \dots$$

$$\dots + \frac{(x-x_1)^n}{n!}P^n(x_1) + \dots \quad (\text{II.})$$

The condition for convergency will presently be shown to be

$$|x-x_1| < |1+x_1|.$$

In order to calculate the right-hand side of (II.) observe that

$$P(cx) = 1 + k_1(cx) + \frac{k_2(cx)^2}{2!} + \dots + \frac{k_n(cx)^n}{n!} + \dots,$$

where

$$|cx| < 1. \quad (\text{III.})$$

Then by actual differentiation, and substitution in the equation

$$(1+cx) \frac{d^r P(cx)}{dx^r} = c(k-r+1) \frac{d^{r-1} P(cx)}{dx^{r-1}}, \quad (\text{IV.})$$

the validity of (IV.) can be established. From this it follows that

$$\frac{d^r P(cx)}{dx^r} = k_r \left(\frac{c}{1+cx} \right)^r P(cx). \quad (\text{V.})$$

Using the formula (V.) in (II.) it follows that

$$P(x|x_1) = P(x_1) \left[1 + k_1 \frac{x-x_1}{1+x_1} + \frac{k_2}{2!} \left(\frac{x-x_1}{1+x_1} \right)^2 + \dots + \frac{k_n}{n!} \left(\frac{x-x_1}{1+x_1} \right)^n + \dots \right];$$

therefore
$$P(x|x_1) = P(x_1) P\left(\frac{x-x_1}{1+x_1}\right), \quad (\text{VI.})$$

provided that $|x-x_1| < |1+x_1|$.

Put next $x-x_1 = (x_2-x_1) + (x-x_2)$.

$$\begin{aligned} \text{Then } P\left(\frac{x-x_1}{1+x_1}\right) &= P\left(\frac{x_2-x_1}{1+x_1}\right) + (x-x_2) \left[\frac{d}{dx} P\left(\frac{x-x_1}{1+x_1}\right) \right]_{x=x_2} \\ &\quad + \frac{(x-x_2)^2}{2!} \left[\frac{d^2}{dx^2} P\left(\frac{x-x_1}{1+x_1}\right) \right]_{x=x_2} + \dots \\ &\quad + \frac{(x-x_2)^n}{n!} \left[\frac{d^n}{dx^n} P\left(\frac{x-x_1}{1+x_1}\right) \right]_{x=x_2} + \dots \end{aligned}$$

Now, by (V.),

$$\frac{d^n P\left(\frac{x-x_1}{1+x_1}\right)}{dx^n} = k_n \left\{ \frac{\left(\frac{1}{1+x_1}\right)}{1+\frac{x-x_1}{1+x_1}} \right\}^n P\left(\frac{x-x_1}{1+x_1}\right);$$

therefore
$$\left[\frac{d^n P\left(\frac{x-x_1}{1+x_1}\right)}{dx^n} \right]_{x=x_2} = \frac{k_n}{(1+x_2)^n} P\left(\frac{x_2-x_1}{1+x_1}\right);$$

therefore

$$\begin{aligned} P\left(\frac{x-x_1}{1+x_1}\right) &= P\left(\frac{x_2-x_1}{1+x_1}\right) \left[1 + k_1 \frac{x-x_2}{1+x_2} + \frac{k_2}{2!} \left(\frac{x-x_2}{1+x_2} \right)^2 + \dots \right. \\ &\quad \left. \dots + \frac{k_n}{n!} \left(\frac{x-x_2}{1+x_2} \right)^n + \dots \right] \\ &= P\left(\frac{x_2-x_1}{1+x_1}\right) P\left(\frac{x-x_2}{1+x_2}\right), \end{aligned}$$

provided $|x-x_2| < |1+x_2|$.

Hence (VI.) becomes

$$P(x | x_1 | x_2) = P(x_1) P\left(\frac{x_2 - x_1}{1 + x_1}\right) P\left(\frac{x - x_2}{1 + x_2}\right), \quad (\text{VII.})$$

and so on;

$$P(x | x_1 | x_2 | \dots | x_s)$$

$$= P(x_1) P\left(\frac{x_2 - x_1}{1 + x_1}\right) P\left(\frac{x_3 - x_2}{1 + x_2}\right) \dots P\left(\frac{x_s - x_{s-1}}{1 + x_{s-1}}\right) P\left(\frac{x - x_s}{1 + x_s}\right), \quad (\text{VIII.})$$

provided $|x - x_s| < |1 + x_s|$.

4. Now let a series of points be taken which form a complete circuit. We can suppose x_s to be the same point as x_1 .

Put $x_s = x_1$ in (VIII.), and then using (VI.) we have

$$\frac{P(x | x_1 | x_2 | \dots | x_{s-1} | x_1)}{P(x | x_1)} = P\left(\frac{x_2 - x_1}{1 + x_1}\right) P\left(\frac{x_3 - x_2}{1 + x_2}\right) \dots P\left(\frac{x_1 - x_{s-1}}{1 + x_{s-1}}\right). \quad (\text{IX.})$$

To calculate the value of the expression last written, consider the expression

$$P\left(\frac{x_{r+1} - x_r}{1 + x_r}\right).$$

The point x_{r+1} is inside the circle of convergence whose centre is x_r ; therefore

$$\left| \frac{x_{r+1} - x_r}{1 + x_r} \right| < 1;$$

therefore the amplitude of $1 + \frac{x_{r+1} - x_r}{1 + x_r}$ lies between $-\frac{\pi}{2}$ and $+\frac{\pi}{2}$

This amplitude is the amplitude of $\frac{1 + x_{r+1}}{1 + x_r}$, and it is therefore the acute angle, positive or negative, through which the vector $1 + x_r$ must be turned to fall on the vector $1 + x_{r+1}$.

Hence by the result due to Abel quoted above

$$\begin{aligned} P\left(\frac{x_{r+1} - x_r}{1 + x_r}\right) \\ = \left| \frac{1 + x_{r+1}}{1 + x_r} \right|^k \left[\cos k \text{ amp} \left(\frac{1 + x_{r+1}}{1 + x_r} \right) + i \sin k \text{ amp} \left(\frac{1 + x_{r+1}}{1 + x_r} \right) \right]; \end{aligned}$$

therefore

$$\begin{aligned} P\left(\frac{x_2 - x_1}{1 + x_1}\right) P\left(\frac{x_3 - x_2}{1 + x_2}\right) \dots P\left(\frac{x_1 - x_{s-1}}{1 + x_{s-1}}\right) \\ = \left\{ \begin{aligned} &\cos k \left[\text{amp} \left(\frac{1 + x_2}{1 + x_1} \right) + \text{amp} \left(\frac{1 + x_3}{1 + x_2} \right) + \dots + \text{amp} \left(\frac{1 + x_1}{1 + x_{s-1}} \right) \right] \\ &+ i \sin k \left[\text{amp} \left(\frac{1 + x_2}{1 + x_1} \right) + \text{amp} \left(\frac{1 + x_3}{1 + x_2} \right) + \dots + \text{amp} \left(\frac{1 + x_1}{1 + x_{s-1}} \right) \right] \end{aligned} \right\}. \end{aligned}$$

$$\text{Now} \quad \left[\text{amp} \left(\frac{1+x_2}{1+x_1} \right) + \text{amp} \left(\frac{1+x_3}{1+x_2} \right) + \dots + \text{amp} \left(\frac{1+x_s}{1+x_{s-1}} \right) \right]$$

is the angle through which the radius vector joining x to the point (-1) turns round as x describes the circuit $x_1 x_2 \dots x_{s-1} x_1$.

If the circuit do not include the point (-1) , then the angle is zero, and

$$P(x | x_1 | \dots x_{s-1} | x_1) = P(x | x_1).$$

If the circuit go once round the point (-1) in the positive direction, then the angle is 2π , and

$$P(x | x_1 | \dots x_{s-1} | x_1) = P(x | x_1) [\cos 2k\pi + i \sin 2k\pi],$$

so that, if k be a positive or negative integer, then

$$P(x | x_1 | \dots x_{s-1} | x_1) = P(x | x_1).$$

But, if k be equal to $\frac{1}{n}$, where n is a positive integer,

$$P(x | x_1 | \dots x_{s-1} | x_1) = P(x | x_1) \left(\cos \frac{2\pi}{n} + i \sin \frac{2\pi}{n} \right).$$

Hence the point x must go n times round the circuit including the singular point in the positive direction before the continuation can return to the original value $P(x | x_1)$.

5. A particular case of the preceding results will be required for the continuation of the series for $\arcsin x$.

$$\text{Let} \quad Q(x) = \sum_{n=0}^{\infty} \frac{1.3 \dots (2n-1)}{2.4 \dots (2n)} x^n,$$

which is the power series for $(1-x)^{-\frac{1}{2}}$. Let

$$R(x) = \sum_{n=0}^{\infty} (-1)^n \frac{1.3 \dots (2n-1)}{2.4 \dots (2n)} x^n,$$

which is the power series for $(1+x)^{-\frac{1}{2}}$. Let

$$P(x) = \sum_{n=0}^{\infty} \frac{1.3 \dots (2n-1)}{2.4 \dots (2n)} x^{2n},$$

which is the power series for $(1-x^2)^{-\frac{1}{2}}$. These are all convergent, and

$$P(x) = Q(x) R(x),$$

if $|x| < 1$. Also

$$Q(x | x_1 | \dots x_{s-1} | x_s) = Q(x_1) Q\left(\frac{x_2-x_1}{1-x_1}\right) \dots Q\left(\frac{x_s-x_{s-1}}{1-x_{s-1}}\right) Q\left(\frac{x-x_s}{1-x_s}\right)$$

and

$$R(x | x_1 | \dots x_{s-1} | x_s) = R(x_1) R\left(\frac{x_2-x_1}{1+x_1}\right) \dots R\left(\frac{x_s-x_{s-1}}{1+x_{s-1}}\right) R\left(\frac{x-x_s}{1+x_s}\right).$$

Consequently $P(x | x_1 | \dots x_{s-1} | x_s)$ is equal to

$$Q(x_1) Q\left(\frac{x_2 - x_1}{1 - x_1}\right) \dots Q\left(\frac{x_s - x_{s-1}}{1 - x_{s-1}}\right) \quad (\text{A})$$

multiplied by $R(x_1) R\left(\frac{x_2 - x_1}{1 + x_1}\right) \dots R\left(\frac{x_s - x_{s-1}}{1 + x_{s-1}}\right) \quad (\text{B})$

multiplied by $Q\left(\frac{x - x_s}{1 - x_s}\right) R\left(\frac{x - x_s}{1 + x_s}\right). \quad (\text{C})$

The product of the two power series in the line marked (C) is a convergent power series in $(x - x_s)$ if $|x - x_s|$ is less than the smaller of the two quantities $|1 - x_s|$ and $|1 + x_s|$.

If now $x_1 x_2 \dots x_{s-1} x_s$ form a circuit including $x = 1$, but not including $x = -1$, and x_s be the same point as x_1 , then, using the results of Art. 4, and putting therein $k = -\frac{1}{2}$, we have

$$Q\left(\frac{x_2 - x_1}{1 - x_1}\right) \dots Q\left(\frac{x_1 - x_{s-1}}{1 - x_{s-1}}\right) = \cos\left(-\frac{1}{2}\right) 2\pi + i \sin\left(-\frac{1}{2}\right) 2\pi = -1,$$

$$R\left(\frac{x_2 - x_1}{1 + x_1}\right) \dots R\left(\frac{x_1 - x_{s-1}}{1 + x_{s-1}}\right) = \cos\left(-\frac{1}{2}\right) 0 + i \sin\left(-\frac{1}{2}\right) 0 = +1$$

therefore

$$\begin{aligned} P(x | x_1 | \dots x_{s-1} | x_1) &= -Q(x_1) R(x_1) Q\left(\frac{x - x_1}{1 - x_1}\right) R\left(\frac{x - x_1}{1 + x_1}\right) \\ &= -P(x | x_1). \end{aligned}$$

In like manner, if the circuit include $x = -1$, but does not include $x = +1$, the same result will be obtained. But a circuit round both $x = +1$ and $x = -1$ brings the function back to its original value.

6. To apply the results to the series for $\arcsin x$, it is necessary to transform the preceding results by the aid of conjugate functions.

If $x = \sin y,$

and $x = \xi + i\eta, \quad y = u + iv,$

then $\xi = \sin u \cosh v, \quad (\text{X.})$

$$\eta = \cos u \sinh v; \quad (\text{XI.})$$

therefore $(\xi \operatorname{cosec} u)^2 - (\eta \sec u)^2 = 1, \quad (\text{XII.})$

$$(\xi \operatorname{sech} v)^2 + (\eta \operatorname{cosech} v)^2 = 1. \quad (\text{XIII.})$$

The equations (XII.) and (XIII.) are not equivalent to (X.) and

(XI.), because it has been necessary to square (X.) and (XI.) to find (XII.) and (XIII.).

It is convenient, however, to have (XII.) and (XIII.), and we will call (XII.) the equation of the hyperbola u , and (XIII.) the equation of the ellipse v .

Let the Riemann surface for the x -plane be constructed. Let the sheet corresponding to the principal branch be placed immediately below that corresponding to the $(+1)$ -th branch. Then, if a point in the sheet corresponding to the principal branch be denoted by any letter K , the point immediately above it in the sheet corresponding to the $(+1)$ -th branch will be denoted by K_1 .

The x -plane will be slit along the ξ -axis from $+1$ to $+\infty$, and from -1 to $-\infty$; and there will be a junction from each edge of the slit $(+1$ to $+\infty)$ for points to pass from the principal branch to the $(+1)$ -th branch, and conversely.

Let corresponding points in the x and y planes be denoted by the same letter, and let us consider the circuit in the x -plane which corresponds to the following circuit in the y -plane. Let the circuit commence at the point $M(u = a, v = b)$, and pass along the line $u = a$ to the point $N(u = a, v = -b)$, then pass along the line $v = -b$ to the point $M_1(u = \pi - a, v = -b)$, then pass along the line $u = \pi - a$ to the point $N_1(u = \pi - a, v = +b)$, then pass along the line $v = +b$ to the starting point $M(u = a, v = b)$.

FIG. 1.

THE x -PLANE.

$$x = \xi + i\eta.$$

FIG. 2.

FIG. 3.

The point M , $u = a$, $v = b$, in the y -plane corresponds to

$$\xi = \sin a \cosh b, \quad \eta = \cos a \sinh b,$$

in the x -plane.

To the path MFN along $u = a$ in the y -plane corresponds the path MFN along the hyperbola $u = a$.

To the point N , $u = a$, $v = -b$ corresponds

$$\xi = \sin a \cosh b, \quad \eta = -\cos a \sinh b.$$

To the path NP along $v = -b$ corresponds the path NP along the ellipse $v = -b$.

At the point P the path crosses out of the sheet of the Riemann's surface representing the principal branch into the sheet representing the $(+1)$ -th branch. It still remains, however, on the ellipse $v = -b$. When $u = \pi - a$, $v = -b$,

$$\xi = \sin a \cosh b, \quad \eta = \cos a \sinh b;$$

so that the point corresponding to M_1 is in the $(+1)$ -th branch vertically above the point corresponding to M .

Hence in the x -plane a complete circuit has been made, corresponding to the path MNM_1 in the y -plane.

The portion $MFNP$ of the path is in the sheet corresponding to the principal branch.

The portion Q_1M_1 of the path, where M_1 is exactly above M , is in the sheet corresponding to the $(+1)$ -th branch.

It remains to consider the path in the x -plane, which corresponds to $M_1F_1N_1P_1QM$ in the y -plane. $M_1F_1N_1$ is an arc of the hyperbola $u = \pi - a$.

This is in the sheet corresponding to the $(+1)$ -th branch, and is immediately above the hyperbola $u = a$, which is in the sheet corresponding to the principal branch.

From N_1 to P_1 the point is moving along the ellipse $v = +b$.

At P_1 it crosses from the sheet representing the $(+1)$ -th branch into the sheet representing the principal branch, and goes from Q to M , the original starting point along the ellipse $v = +b$. So that a second circuit in the x -plane has now been completed.

It is to be noticed that the ellipse $v = +b$ has the* upper half in the sheet corresponding to the principal branch, and the lower half in the sheet corresponding to the $(+1)$ -th branch; whilst the ellipse $v = -b$ has the upper half in the sheet corresponding to the $(+1)$ -th branch, and the lower half in the sheet corresponding to the principal branch.

Since $x = \sin u \cosh v + i \cos u \sinh v$,

therefore $x_s = \sin u_s \cosh v_s + i \cos u_s \sinh v_s$;

therefore

$$Q(x_1) Q\left(\frac{x_2 - x_1}{1 - x_1}\right) \dots Q\left(\frac{x_s - x_{s-1}}{1 - x_{s-1}}\right) \\ = |1 - x_s|^{-\frac{1}{2}} \left[\cos\left(-\frac{1}{2}\right) \sigma_s + i \sin\left(-\frac{1}{2}\right) \sigma_s \right],$$

where σ_s is the angle through which the vector $1 - x$, starting from a position in the positive direction of the real axis, must turn to fall on the vector $1 - x_s$ when the point x goes round the circuit. Similarly,

$$R(x_1) R\left(\frac{x_2 - x_1}{1 + x_1}\right) \dots R\left(\frac{x_s - x_{s-1}}{1 + x_{s-1}}\right) \\ = |1 + x_s|^{-\frac{1}{2}} \left[\cos\left(-\frac{1}{2}\right) \tau_s + i \sin\left(-\frac{1}{2}\right) \tau_s \right],$$

* For the upper half $\eta > 0$, for the lower half $\eta < 0$.

where τ , is the angle through which the vector $1+x$, starting from a position in the positive direction of the real axis, must turn to fall on the vector $1+x$, when the point x goes round the circuit.

We have also the following expressions:—

$$|1-x_s| = \cosh v_s - \sin u_s,$$

$$\cos \text{amp}(1-x_s) = \frac{1 - \sin u_s \cosh v_s}{\cosh v_s - \sin u_s},$$

$$\sin \text{amp}(1-x_s) = \frac{-\cos u_s \sinh v_s}{\cosh v_s - \sin u_s},$$

$$\tan \frac{1}{2} \text{amp}(1-x_s) = -\tanh \frac{1}{2} v_s \cot \left(\frac{\pi}{4} - \frac{1}{2} u_s \right),$$

$$\cos \frac{1}{2} \text{amp}(1-x_s) = \pm \frac{\cosh \frac{1}{2} v_s \sin \left(\frac{\pi}{4} - \frac{1}{2} u_s \right)}{\sqrt{\sin^2 \left(\frac{\pi}{4} - \frac{1}{2} u_s \right) + \sinh^2 \frac{1}{2} v_s}},$$

$$\sin \frac{1}{2} \text{amp}(1-x_s) = \mp \frac{\sinh \frac{1}{2} v_s \cos \left(\frac{\pi}{4} - \frac{1}{2} u_s \right)}{\sqrt{\sin^2 \left(\frac{\pi}{4} - \frac{1}{2} u_s \right) + \sinh^2 \frac{1}{2} v_s}};$$

$$|1+x_s| = \cosh v_s + \sin u_s,$$

$$\cos \text{amp}(1+x_s) = \frac{1 + \sin u_s \cosh v_s}{\cosh v_s + \sin u_s},$$

$$\sin \text{amp}(1+x_s) = \frac{\cos u_s \sinh v_s}{\cosh v_s + \sin u_s},$$

$$\tan \frac{1}{2} \text{amp}(1+x_s) = \tanh \frac{1}{2} v_s \tan \left(\frac{\pi}{4} - \frac{1}{2} u_s \right),$$

$$\cos \frac{1}{2} \text{amp}(1+x_s) = \pm \frac{\cosh \frac{1}{2} v_s \cos \left(\frac{\pi}{4} - \frac{1}{2} u_s \right)}{\sqrt{\cos^2 \left(\frac{\pi}{4} - \frac{1}{2} u_s \right) + \sinh^2 \frac{1}{2} v_s}},$$

$$\sin \frac{1}{2} \text{amp}(1+x_s) = \pm \frac{\sinh \frac{1}{2} v_s \sin \left(\frac{\pi}{4} - \frac{1}{2} u_s \right)}{\sqrt{\cos^2 \left(\frac{\pi}{4} - \frac{1}{2} u_s \right) + \sinh^2 \frac{1}{2} v_s}}.$$

The ambiguous signs will now be determined. To do this consider a

portion $x, x_2, x_3 \dots x_4$ of a circuit round $x = 1$, but not round $x = -1$, as in the figure.

FIG. 4.

In this figure $\text{amp}(1-x_2) = \angle \xi l A$,

and also $\sigma_2 = \angle \xi l A$.

We have also $0 < u_2 < \frac{1}{2}\pi$;

therefore $\cos u_2 > 0$.

If, as in this figure, $\sigma_2 < \pi$, $\cos \frac{1}{2}\sigma_2 > 0$,

and $\sigma_2 = \text{amp}(1-x_2)$,

therefore

$$\cos \frac{1}{2}\sigma_2 = \frac{\cosh \frac{1}{2}v_2 \left| \sin \left(\frac{\pi}{4} - \frac{1}{2}u_2 \right) \right|}{\sqrt{\sin^2 \left(\frac{\pi}{4} - \frac{1}{2}u_2 \right) + \sinh^2 \frac{1}{2}v_2}},$$

$$\sin \frac{1}{2}\sigma_2 = \frac{-\sinh \frac{1}{2}v_2 \cot \left(\frac{\pi}{4} - \frac{1}{2}u_2 \right) \left| \sin \left(\frac{\pi}{4} - \frac{1}{2}u_2 \right) \right|}{\sqrt{\sin^2 \left(\frac{\pi}{4} - \frac{1}{2}u_2 \right) + \sinh^2 \frac{1}{2}v_2}}.$$

Also in this figure $-\frac{\pi}{2} < \text{amp}(1+x_2) < +\frac{\pi}{2}$,

and $\tau_2 = \text{amp}(1+x_2)$;

therefore $\cos \frac{1}{2}\tau_2 > 0$;

therefore

$$\cos \frac{1}{2} \tau = \frac{\cosh \frac{1}{2} v \left| \cos \left(\frac{\pi}{4} - \frac{1}{2} u \right) \right|}{\sqrt{\cos^2 \left(\frac{\pi}{4} - \frac{1}{2} u \right) + \sinh^2 \frac{1}{2} v}},$$

$$\sin \frac{1}{2} \tau = \frac{\sinh \frac{1}{2} v \tan \left(\frac{\pi}{4} - \frac{1}{2} u \right) \left| \cos \left(\frac{\pi}{4} - \frac{1}{2} u \right) \right|}{\sqrt{\cos^2 \left(\frac{\pi}{4} - \frac{1}{2} u \right) + \sinh^2 \frac{1}{2} v}};$$

therefore $\cos \frac{1}{2} (\sigma + \tau) = \frac{|\cos u| \cosh v}{\sqrt{\cos^2 u + \sinh^2 v}},$

$$\sin \frac{1}{2} (\sigma + \tau) = \frac{-\sinh v \sin u \frac{|\cos u|}{\cos u}}{\sqrt{\cos^2 u + \sinh^2 v}};$$

therefore

$$\begin{aligned} \cos \frac{1}{2} (\sigma + \tau) + i \sin \frac{1}{2} (\sigma + \tau) &= \frac{|\cos u|}{\cos u} \frac{\cos u \cosh v - i \sin u \sinh v}{\sqrt{\cos^2 u + \sinh^2 v}} \\ &= \frac{\cos u \cosh v - i \sin u \sinh v}{\sqrt{\cos^2 u + \sinh^2 v}}; \end{aligned}$$

because $\cos u > 0$. Also

$$|1 - x_i^2| = \cos^2 u + \sinh^2 v;$$

therefore

$$\begin{aligned} |1 - x_i^2|^{-\frac{1}{2}} \left[\cos \left(-\frac{1}{2} \right) (\sigma + \tau) + i \sin \left(-\frac{1}{2} \right) (\sigma + \tau) \right] \\ = \frac{1}{\cos u \cosh v - i \sin u \sinh v} = \sec(u + iv). \end{aligned}$$

The next case to be considered is that in which x_i is on the real axis between $+1$ and $+\infty$.

FIG. 5.

Here

$$\text{amp}(1-x_1) = \sigma_1 = \pi,$$

$$u_1 = \frac{\pi}{2},$$

$$\tau_1 = 0,$$

$$|1+x_1| = \cosh v_1 + 1,$$

$$|1-x_1| = \cosh v_1 - 1,$$

$$|1-x_1^2| = \sinh^2 v_1,$$

$$|1-x_1^2|^{-1} = -\text{cosech } v_1,$$

the negative sign being taken, because here v_1 is negative. [Compare this with Fig. 2. For the part of the path NP , v is negative.]

$$\text{Therefore } \cos \frac{1}{2}(\sigma_1 + \tau_1) = 0, \quad \sin \frac{1}{2}(\sigma_1 + \tau_1) = 1;$$

$$\text{therefore } |1-x_1^2|^{-1} \left[\cos(-\tfrac{1}{2})(\sigma_1 + \tau_1) + i \sin(-\tfrac{1}{2})(\sigma_1 + \tau_1) \right] \\ = -\text{cosech } v_1 (-i) = i \text{ cosech } v_1.$$

[This is included in the previous result, as is seen by putting $u_1 = \frac{\pi}{2}$ in it.]

There remains only the case in which $\pi < \sigma_1 < 2\pi$.

FIG. 6.

Here

$$\sigma_1 = \pi + OIA,$$

$$\text{amp}(1-x_1) = -(\pi - OIA);$$

therefore

$$\sigma_1 = \text{amp}(1-x_1) + 2\pi;$$

$$\tfrac{1}{2}\sigma_1 = \pi + \tfrac{1}{2} \text{amp}(1-x_1).$$

Hence the expressions for $\cos \frac{1}{2}\sigma_1$ and $\sin \frac{1}{2}\sigma_1$ have the opposite signs to those obtained in the first case.

But $\cos \frac{1}{2}\tau_1$ and $\sin \frac{1}{2}\tau_1$ are unchanged.

Hence $\cos \frac{1}{2}(\sigma_1 + \tau_1)$ and $\sin \frac{1}{2}(\sigma_1 + \tau_1)$ have the opposite signs to those obtained in the first case.

Therefore

$$\begin{aligned} \cos \frac{1}{2}(\sigma_s + \tau_s) + i \sin \frac{1}{2}(\sigma_s + \tau_s) \\ = - \frac{|\cos u_s|}{\cos u_s} \frac{\cos u_s \cosh v_s - i \sin u_s \sinh v_s}{\sqrt{\cos^2 u_s + \sinh^2 v_s}}. \end{aligned}$$

Now, in this case,

$$\frac{\pi}{2} < u_s < \pi, \quad \cos u_s < 0;$$

therefore

$$\cos \frac{1}{2}(\sigma_s + \tau_s) + i \sin \frac{1}{2}(\sigma_s + \tau_s) = \frac{\cos u_s \cosh v_s - i \sin u_s \sinh v_s}{\sqrt{\cos^2 u_s + \sinh^2 v_s}},$$

as in the first case.

Further, because $|1 - x_s^2| = \cos^2 u_s + \sinh^2 v_s$,

therefore

$$\begin{aligned} |1 - x_s^2|^{-\frac{1}{2}} \left[\cos \left(-\frac{1}{2}\right)(\sigma_s + \tau_s) + i \sin \left(-\frac{1}{2}\right)(\sigma_s + \tau_s) \right] \\ = \frac{1}{\cos u_s \cosh v_s - i \sin u_s \sinh v_s} = \sec(u_s + iv_s), \end{aligned}$$

as in the first case.

Hence it is proved in all cases that

$$\begin{aligned} Q(x_1) Q\left(\frac{x_2 - x_1}{1 - x_1}\right) \dots Q\left(\frac{x_s - x_{s-1}}{1 - x_{s-1}}\right) \times R(x_1) \left(\frac{x_2 - x_1}{1 + x_1}\right) \dots R\left(\frac{x_s - x_{s-1}}{1 + x_{s-1}}\right) \\ = \sec(u_s + iv_s). \end{aligned}$$

Hence the continuation of the power series for $(1 - x^2)^{-\frac{1}{2}}$, when $x = \sin(u + iv)$, has everywhere the value $\sec(u + iv)$. In this form it is easy to trace the changes in the value of the power series.

When $u = a$, $v = b$,

$$\sec(u + iv) = \sec(a + ib).$$

When $u = a$, $v = -b$,

$$\sec(u + iv) = \sec(a - ib).$$

When $u = \pi - a$, $v = -b$,

$$\sec(u + iv) = -\sec(a + ib).$$

Hence $u + iv$ has changed from $a + ib$ to $\pi - (a + ib)$, and the value of the power series has merely changed its sign.

It is possible to deduce from the above the corresponding results for the power series for $(1 + x^2)^{-\frac{1}{2}}$ [which is not in reality a distinct series from the preceding].

The result is that the power series for $(1 + x^2)^{-\frac{1}{2}}$, where

$x = \sinh(u + iv)$, and its continuations are everywhere represented by the expression $\operatorname{sech}(u + iv)$.

7. *Second Illustration.—The Continuation of the Logarithmic Series.*

The results only are given.

The work depends on the theorem due to Abel,

$$x - \frac{x^2}{2} + \dots + (-1)^{n-1} \frac{x^n}{n} + \dots = \log |1+x| + i \operatorname{amp}(1+x),$$

where $|x| < 1$, and $\operatorname{amp}(1+x)$ is the principal amplitude of $(1+x)$.

Since $|x| < 1$, this principal amplitude lies between $-\frac{\pi}{2}$ and $+\frac{\pi}{2}$.

$$\text{Let} \quad P(x) = x - \frac{x^2}{2} + \dots + (-1)^{r-1} \frac{x^r}{r} + \dots \quad (|x| < 1).$$

$$\text{Then} \quad P(x|x_1) = P(x_1) + P\left(\frac{x-x_1}{1+x_1}\right),$$

$$\begin{aligned} P(x|x_1|\dots|x_{s-1}|x_s) \\ = P(x_1) + P\left(\frac{x_2-x_1}{1+x_1}\right) + \dots + P\left(\frac{x_s-x_{s-1}}{1+x_{s-1}}\right) + P\left(\frac{x-x_s}{1+x_s}\right) \end{aligned}$$

If therefore x_s be the same point as x_1 ,

$$P(x|x_1|\dots|x_{s-1}|x_1) - P(x|x_1) = P\left(\frac{x_2-x_1}{1+x_1}\right) + \dots + P\left(\frac{x_1-x_{s-1}}{1+x_{s-1}}\right).$$

$$\begin{aligned} \text{Now} \quad P\left(\frac{x_{r+1}-x_r}{1+x_r}\right) &= \log \left| 1 + \frac{x_{r+1}-x_r}{1+x_r} \right| + i \operatorname{amp} \left(1 + \frac{x_{r+1}-x_r}{1+x_r} \right) \\ &= \log \left| \frac{1+x_{r+1}}{1+x_r} \right| + i \operatorname{amp} \left(\frac{1+x_{r+1}}{1+x_r} \right); \end{aligned}$$

$$\begin{aligned} \text{therefore} \quad P(x|x_1|\dots|x_{s-1}|x_1) - P(x|x_1) \\ = \log \left\{ \left| \frac{1+x_2}{1+x_1} \right| \left| \frac{1+x_3}{1+x_2} \right| \dots \left| \frac{1+x_1}{1+x_{s-1}} \right| \right\} \\ + i \left[\operatorname{amp} \frac{1+x_2}{1+x_1} + \operatorname{amp} \frac{1+x_3}{1+x_2} + \dots + \operatorname{amp} \frac{1+x_1}{1+x_{s-1}} \right]. \end{aligned}$$

The logarithm vanishes.

In consequence of the hypothesis regarding the vertices of the polygon, viz., that each is in the circle of convergence whose centre is the preceding vertex, the value of the coefficient of i is equal to the angle which the radius vector joining x to the point (-1) describes as x describes the polygon. It is therefore zero if the polygon do not contain the point (-1) , but equal to 2π if the point (-1) lie in the interior of the polygon, and the describing point has

gone once round the origin in the positive direction. Therefore

$$[P(x|x_1|x_2\ldots|x_{s-1}|x_1)] - P(x|x_1) = 2i\pi$$

if the polygon contain the point (-1) . Also

$$[P(x|x_1|x_2\ldots|x_{s-1}|x_1)] = P(x|x_1)$$

if the polygon do not contain the point (-1) .

8. *Third Illustration.*—Continuation of the Series for $\arctan x$.

The results only are given.

$$\text{Let} \quad P(x) = x - \frac{x^3}{3} + \frac{x^5}{5} - \dots,$$

where

$$|x| < 1.$$

$$\text{Let} \quad Q(x) = x - \frac{1}{2}x^3 + \dots + \frac{(-1)^{n-1}}{n}x^n + \dots$$

$$\text{Then} \quad P(x|x_1) = P(x_1) + \frac{1}{2i} \left[Q\left(\frac{x-x_1}{x_1-i}\right) - Q\left(\frac{x-x_1}{x_1+i}\right) \right];$$

therefore

$$\begin{aligned} P(x|x_1\ldots|x_{s-1}|x_s) \\ = P(x_1) + \frac{1}{2i} \left\{ \begin{aligned} &Q\left(\frac{x_2-x_1}{x_1-i}\right) + \dots + Q\left(\frac{x_s-x_{s-1}}{x_{s-1}-i}\right) + Q\left(\frac{x-x_s}{x_s-i}\right) \\ &- Q\left(\frac{x_2-x_1}{x_1+i}\right) - \dots - Q\left(\frac{x_s-x_{s-1}}{x_{s-1}+i}\right) - Q\left(\frac{x-x_s}{x_s+i}\right) \end{aligned} \right\}. \end{aligned}$$

If x_s be the same point as x_1 , then

$$\begin{aligned} P(x|x_1\ldots|x_{s-1}|x_1) - P(x|x_1) \\ = \frac{1}{2i} \left\{ \begin{aligned} &Q\left(\frac{x_2-x_1}{x_1-i}\right) + \dots + Q\left(\frac{x_1-x_{s-1}}{x_{s-1}-i}\right) \\ &- Q\left(\frac{x_2-x_1}{x_1+i}\right) - \dots - Q\left(\frac{x_1-x_{s-1}}{x_{s-1}+i}\right) \end{aligned} \right\} \\ = \frac{1}{2} \left\{ \begin{aligned} &\text{amp}\left(\frac{x_2-i}{x_1-i}\right) + \text{amp}\left(\frac{x_3-i}{x_2-i}\right) + \dots + \text{amp}\left(\frac{x_1-i}{x_{s-1}-i}\right) \\ &- \text{amp}\left(\frac{x_2+i}{x_1+i}\right) - \text{amp}\left(\frac{x_3+i}{x_2+i}\right) - \dots - \text{amp}\left(\frac{x_1+i}{x_{s-1}+i}\right) \end{aligned} \right\} \\ = \pi \quad \text{for a circuit round } x=i \text{ in the positive direction,} \\ \text{but equal to } -\pi \quad \text{,, ,, } x=-i \quad \text{,, ,, } \\ \text{and equal to 0 for a circuit including neither } x=i \text{ nor } x=-i \text{ or for} \\ \text{a circuit including both } x=i \text{ and } x=-i. \end{aligned}$$

9. *Fourth Illustration.—Continuation of the Power Series for arc sin x.**General Formulæ for the Continuation of Power Series.—*

If $f(x)$ be a power series, then

$$f(x|x_1) = f(x_1) + \sum_{n=1}^{\infty} \frac{(x-x_1)^n}{n!} f^n(x_1);$$

$$\text{therefore} \quad [f(x|x_1)]_{x=x_2} = f(x_1) + \sum_{n=1}^{\infty} \frac{(x_2-x_1)^n}{n!} f^n(x_1).$$

$$\text{Putting} \quad x-x_1 = (x_2-x_1) + (x-x_2),$$

it follows that

$$\begin{aligned} f(x|x_1|x_2) &= \left[f(x_1) + \sum_{n=1}^{\infty} \frac{(x_2-x_1)^n}{n!} f^n(x_1) \right] + (x-x_2) \left[f'(x_1) + \sum_{n=1}^{\infty} \frac{(x_2-x_1)^n}{n!} f^{n+1}(x_1) \right] \\ &\quad + \frac{(x-x_2)^2}{2!} \left[f''(x_1) + \sum_{n=1}^{\infty} \frac{(x_2-x_1)^n}{n!} f^{n+2}(x_1) \right] \\ &\quad + \dots \\ &= [f(x|x_1)]_{x=x_2} + (x-x_2) \left[\frac{df(x|x_1)}{dx} \right]_{x=x_2} + \frac{(x-x_2)^2}{2!} \left[\frac{d^2f(x|x_1)}{dx^2} \right]_{x=x_2} + \dots \end{aligned}$$

$$\text{Putting} \quad x-x_2 = (x_3-x_2) + (x-x_3),$$

it follows in like manner that

$$\begin{aligned} f(x|x_1|x_2|x_3) &= [f(x|x_1|x_2)]_{x=x_3} + (x-x_3) \left[\frac{df(x|x_1|x_2)}{dx} \right]_{x=x_3} \\ &\quad + \frac{(x-x_3)^2}{2!} \left[\frac{d^2f(x|x_1|x_2)}{dx^2} \right]_{x=x_3} + \dots, \end{aligned}$$

and so on; therefore

$$\begin{aligned} f(x|x_1|x_2|x_3) &= f(x_1) + \left\{ (x_2-x_1) \left[\frac{df(x)}{dx} \right]_{x=x_1} + \frac{(x_2-x_1)^2}{2!} \left[\frac{d^2f(x)}{dx^2} \right]_{x=x_1} + \dots \right\} \\ &\quad + \left\{ (x_3-x_2) \left[\frac{df(x|x_1)}{dx} \right]_{x=x_2} + \frac{(x_3-x_2)^2}{2!} \left[\frac{d^2f(x|x_1)}{dx^2} \right]_{x=x_2} + \dots \right\} \\ &\quad + \left\{ (x-x_3) \left[\frac{df(x|x_1|x_2)}{dx} \right]_{x=x_3} + \frac{(x-x_3)^2}{2!} \left[\frac{d^2f(x|x_1|x_2)}{dx^2} \right]_{x=x_3} + \dots \right\}, \end{aligned}$$

and so on.

Denoting therefore the series

$$\sum_{n=0}^{\infty} \frac{(x-x_r)^{n+1}}{(n+1)!} \left[\frac{d^{n+1}f(x|x_1|\dots|x_{r-1})}{dx^{n+1}} \right]_{x=x_r},$$

by $\chi(x, x_r)$, it follows that

$$f(x|x_1|\dots|x_r) = f(x_1) + \chi(x_2, x_1) + \chi(x_3, x_2) + \dots + \chi(x_{r-1}, x_r) + \chi(x, x_r).$$

10. The quantity χ satisfies the following relations :—

$$\begin{aligned} \frac{\partial \chi(x, x_r)}{\partial x} &= \sum_{n=0}^{\infty} \frac{(x-x_r)^n}{n!} \left[\frac{d^{n+1}f(x|x_1|\dots|x_{r-1})}{dx^{n+1}} \right]_{x=x_r} \\ &= \frac{df(x|x_1|\dots|x_{r-1}|x_r)}{dx}, \\ \frac{\partial \chi(x, x_r)}{\partial x_r} &= - \left[\frac{df(x|x_1|\dots|x_{r-1})}{dx} \right]_{x=x_r}. \end{aligned}$$

The quantity $\chi(x, x_r)$ does not involve $x_1, x_2, \dots, x_{r-1}$, except in so far as the choice of these points affects the branch of the function $f(x)$ which is represented by $f(x|x_1|\dots|x_r)$.

Its value will be determined later on in the case where $f(x)$ is the power series for arc sin x .

11. Calculation of the Differential Coefficients of the Continuations of the Power Series for arc sin x .

We will now suppose that $f(x)$ is the power series

$$\sum_{n=0}^{\infty} \frac{1.3 \dots (2n-1)}{2.4 \dots (2n)} \frac{x^{2n+1}}{2n+1},$$

which is convergent if $|x| \leq 1$. And we choose x_1 so that $|x_1| < 1$.

$$\frac{df(x)}{dx} = \sum_{n=0}^{\infty} \frac{1.3 \dots (2n-1)}{2.4 \dots (2n)} x^{2n} = \text{power series for } \frac{1}{\sqrt{1-x^2}};$$

therefore, using the notation of Art. 5,

$$\begin{aligned} \frac{df(x|x_1|\dots|x_r)}{dx} &= P(x|x_1|\dots|x_r) \\ &= Q(x_1) Q\left(\frac{x_2-x_1}{1-x_1}\right) \dots Q\left(\frac{x_r-x_{r-1}}{1-x_{r-1}}\right) Q\left(\frac{x-x_r}{1-x_r}\right) \\ &\quad \times R(x_1) R\left(\frac{x_2-x_1}{1+x_1}\right) \dots R\left(\frac{x_r-x_{r-1}}{1+x_{r-1}}\right) R\left(\frac{x-x_r}{1+x_r}\right); \end{aligned}$$

therefore

$$\begin{aligned} \frac{d^{n+1}f(x|x_1|\dots|x_r)}{dx^{n+1}} &= Q(x_1) Q\left(\frac{x_2-x_1}{1-x_1}\right) \dots Q\left(\frac{x_r-x_{r-1}}{1-x_{r-1}}\right) \\ &\times R(x_1) R\left(\frac{x_2-x_1}{1+x_1}\right) \dots R\left(\frac{x_r-x_{r-1}}{1+x_{r-1}}\right) \\ &\times \frac{d^n}{dx^n} \left[Q\left(\frac{x-x_r}{1-x_r}\right) R\left(\frac{x-x_r}{1+x_r}\right) \right]. \end{aligned}$$

$$\begin{aligned} \text{Now } \frac{d^n}{dx^n} \left[Q\left(\frac{x-x_r}{1-x_r}\right) R\left(\frac{x-x_r}{1+x_r}\right) \right] \\ = \sum_{s=0}^{\infty} {}_n C_s \left[\frac{d^{n-s}}{dx^{n-s}} Q\left(\frac{x-x_r}{1-x_r}\right) \right] \left[\frac{d^s}{dx^s} R\left(\frac{x-x_r}{1+x_r}\right) \right]. \end{aligned}$$

$$\text{Now } Q\left(\frac{x-\xi}{1-\xi}\right) = \sum_{t=0}^{\infty} \frac{1.3 \dots (2t-1)}{2.4 \dots (2t)} \left(\frac{x-\xi}{1-\xi}\right)^t;$$

therefore

$$\begin{aligned} \frac{d^v}{dx^v} Q\left(\frac{x-\xi}{1-\xi}\right) &= \sum_{t=0}^{\infty} \frac{1.3 \dots (2t-1)}{2.4 \dots (2t)} \frac{t(t-1) \dots (t-v+1)}{(1-\xi)^v} \left(\frac{x-\xi}{1-\xi}\right)^{t-v} \\ &= \sum_{w=0}^{\infty} \frac{1.3 \dots (2v+2w-1)}{2.4 \dots (2v+2w)} \frac{(v+w)(v+w-1) \dots (w+1)}{(1-\xi)^v} \left(\frac{x-\xi}{1-\xi}\right)^w \\ &= \sum_{w=0}^{\infty} \frac{1.3 \dots (2v+2w-1)}{2^{v+w} w!} \frac{1}{(1-\xi)^v} \left(\frac{x-\xi}{1-\xi}\right)^w. \end{aligned}$$

I will prove that this is equal to

$$\left(-\frac{1}{2}\right)_v \frac{Q\left(\frac{x-\xi}{1-\xi}\right)}{(x-1)^v}$$

$$\text{if } \left| \frac{x-\xi}{1-\xi} \right| < 1.$$

$$\text{Now } \left(-\frac{1}{2}\right)_v \frac{Q\left(\frac{x-\xi}{1-\xi}\right)}{(x-1)^v} = \left(-\frac{1}{2}\right)_v \left(-\frac{1}{1-\xi}\right)^v \frac{Q\left(\frac{x-\xi}{1-\xi}\right)}{\left[1-\frac{x-\xi}{1-\xi}\right]^v},$$

i.e., to

$$\begin{aligned} \left(-\frac{1}{2}\right)_v \left(-\frac{1}{1-\xi}\right)^v \sum_{w=0}^{\infty} \frac{1.3 \dots (2w-1)}{2.4 \dots (2w)} \left(\frac{x-\xi}{1-\xi}\right)^w \\ \times \left[1 + v \left(\frac{x-\xi}{1-\xi}\right) + \frac{v(v+1)}{2!} \left(\frac{x-\xi}{1-\xi}\right)^2 + \dots \right. \\ \left. \dots + \frac{v(v+1) \dots (v+w-1)}{w!} \left(\frac{x-\xi}{1-\xi}\right)^w + \dots \right]. \end{aligned}$$

The coefficient of $\left(\frac{x-\xi}{1-\xi}\right)^w$ in this product is

$$\begin{aligned} & \left(-\frac{1}{2}\right)_v \left(-\frac{1}{1-\xi}\right)^v \\ & \times \left[\frac{v(v+1) \dots (v+w-1)}{w!} \right. \\ & \quad + \frac{1}{2} \frac{v(v+1) \dots (v+w-2)}{(w-1)!} + \frac{1 \cdot 3}{2 \cdot 4} \frac{v(v+1) \dots (v+w-3)}{(w-2)!} + \dots \\ & \quad \left. + \frac{1 \cdot 3 \dots (2w-3)}{2 \cdot 4 \dots (2w-2)} v + \frac{1 \cdot 3 \dots (2w-1)}{2 \cdot 4 \dots (2w)} \right]. \end{aligned}$$

The bracket is the coefficient of z^w in the product of the power series for $(1-z)^{-v}$ multiplied by the power series for $(1-z)^{-1}$.

Hence it is the coefficient of z^w in $(1-z)^{-v-1}$. It is therefore

$$\frac{(v+\frac{1}{2})(v+\frac{3}{2}) \dots \left(v+\frac{2w-1}{2}\right)}{w!}.$$

The coefficient of $\left(\frac{x-\xi}{1-\xi}\right)^w$ is therefore

$$\begin{aligned} & \left(\frac{1}{1-\xi}\right)^v \frac{\left(\frac{1}{2} \cdot \frac{3}{2} \dots \frac{2v-1}{2}\right) \left(\frac{2v+1}{2}\right) \left(\frac{2v+3}{2}\right) \dots \left(\frac{2v+2w-1}{2}\right)}{w!} \\ & = \frac{1}{(1-\xi)^v} \frac{1 \cdot 3 \dots (2v+2w-1)}{2^{v+w} w!}; \end{aligned}$$

therefore
$$\frac{d^v}{dx^v} Q\left(\frac{x-\xi}{1-\xi}\right) = \left(-\frac{1}{2}\right)_v \frac{Q\left(\frac{x-\xi}{1-\xi}\right)}{(x-1)^v},$$

provided $|x-\xi| < |1-\xi|$. Similarly,

$$\frac{d^v}{dx^v} R\left(\frac{x-\xi}{1+\xi}\right) = \left(-\frac{1}{2}\right)_v \frac{R\left(\frac{x-\xi}{1+\xi}\right)}{(x+1)^v};$$

therefore

$$\begin{aligned} & \frac{d^n}{dx^n} \left[Q\left(\frac{x-x_r}{1-x_r}\right) R\left(\frac{x-x_r}{1+x_r}\right) \right] \\ & = \sum_{s=0}^{n-n} \left[{}_n C_s \left(-\frac{1}{2}\right)_{n-s} \left(-\frac{1}{2}\right)_s Q\left(\frac{x-x_r}{1-x_r}\right) R\left(\frac{x-x_r}{1+x_r}\right) \frac{1}{(x-1)^{n-s}} \frac{1}{(x+1)^s} \right] \\ & = Q\left(\frac{x-x_r}{1-x_r}\right) R\left(\frac{x-x_r}{1+x_r}\right) \sum_{s=0}^{n-n} \left[{}_n C_s \left(-\frac{1}{2}\right)_{n-s} \left(-\frac{1}{2}\right)_s \frac{1}{(x-1)^{n-s}} \frac{1}{(x+1)^s} \right]; \end{aligned}$$

therefore

$$\begin{aligned} \frac{1}{(n+1)!} \frac{d^{n+1}f(x|x_1|\dots|x_r)}{dx^{n+1}} \\ = Q(x_1) Q\left(\frac{x_2-x_1}{1-x_1}\right) \dots Q\left(\frac{x_r-x_{r-1}}{1-x_{r-1}}\right) Q\left(\frac{x-x_r}{1-x_r}\right) \\ \times R(x_1) R\left(\frac{x_2-x_1}{1+x_1}\right) \dots R\left(\frac{x_r-x_{r-1}}{1+x_{r-1}}\right) R\left(\frac{x-x_r}{1+x_r}\right) \\ \times \frac{1}{n+1} \sum_{s=0}^{s=n} \left[\frac{\left(-\frac{1}{2}\right)_{n-s}}{(n-s)!} \frac{\left(-\frac{1}{2}\right)_s}{s!} \frac{1}{(x-1)^{n-s}} \frac{1}{(x+1)^s} \right]. \end{aligned}$$

$$\begin{aligned} \text{Hence } \frac{(x-x_r)^{n+1}}{(n+1)!} \left[\frac{d^{n+1}f(x|x_1|\dots|x_{r-1})}{dx^{n+1}} \right]_{x=x_r} \\ = Q(x_1) Q\left(\frac{x_2-x_1}{1-x_1}\right) \dots Q\left(\frac{x_r-x_{r-1}}{1-x_{r-1}}\right) \\ \times R(x_1) R\left(\frac{x_2-x_1}{1+x_1}\right) \dots R\left(\frac{x_r-x_{r-1}}{1+x_{r-1}}\right) \\ \times \frac{(x-x_r)^{n+1}}{n+1} \sum_{s=0}^{s=n} \left[\frac{\left(-\frac{1}{2}\right)_{n-s}}{(n-s)!} \frac{\left(-\frac{1}{2}\right)_s}{s!} \frac{1}{(x_r-1)^{n-s}} \frac{1}{(x_r+1)^s} \right]. \end{aligned}$$

12: Determination of the Quantities $\chi(x, x_r)$, $\chi(x_{r+1}, x_r)$.

Hence

$$\begin{aligned} \chi(x, x_r) = Q(x_1) Q\left(\frac{x_2-x_1}{1-x_1}\right) \dots Q\left(\frac{x_r-x_{r-1}}{1-x_{r-1}}\right) \\ \times R(x_1) R\left(\frac{x_2-x_1}{1+x_1}\right) \dots R\left(\frac{x_r-x_{r-1}}{1+x_{r-1}}\right) \\ \times \sum_{n=0}^{\infty} \left\{ \frac{(x-x_r)^{n+1}}{n+1} \sum_{s=0}^{s=n} \left[\frac{\left(-\frac{1}{2}\right)_{n-s}}{(n-s)!} \frac{\left(-\frac{1}{2}\right)_s}{s!} \frac{1}{(x_r-1)^{n-s}} \frac{1}{(x_r+1)^s} \right] \right\}. \end{aligned}$$

Now take a circuit round $x=1$, such that $|x_r-1|$ is always less than $|x_r+1|$; then it will be proved that the above series is convergent if

$$|x-x_r| < |x_r-1|.$$

Since $\left| \frac{\left(-\frac{1}{2}\right)_{n-s}}{(n-s)!} \right|$ and $\left| \frac{\left(-\frac{1}{2}\right)_s}{s!} \right|$ are each less than unity, therefore the modulus of the general term of the series is less than

$$\frac{|x-x_r|^{n+1}}{n+1} \sum_{s=0}^{s=n} \frac{1}{|x_r-1|^{n-s}} \frac{1}{|x_r+1|^s} < \frac{|x-x_r|^{n+1}}{|x_r-1|^n} \frac{1}{n+1} \sum_{s=0}^{s=n} \left| \frac{x_r-1}{x_r+1} \right|^s;$$

and, since

$$\left| \frac{x_r - 1}{x_r + 1} \right| < 1,$$

therefore

$$\frac{1}{n+1} \sum_{s=0}^{\infty} \left| \frac{x_r - 1}{x_r + 1} \right|^s < 1.$$

Hence the modulus of the general term of the series is less than

$$|x - x_r| \left| \frac{x - x_r}{x_r - 1} \right|^n$$

Hence the series is convergent if

$$|x - x_r| < |x_r - 1|.$$

In like manner, if a circuit round $x = -1$ be taken, the series will be convergent if

$$|x - x_r| < |x_r + 1|.$$

Hence the series will be convergent if x be so taken that $|x - x_r|$ is always less than the smaller of the two moduli $|x - x_r|$ and $|x_r + 1|$.

Hence $\chi(x, x_r)$ and $\chi(x_{s+1}, x_s)$ will be convergent if the vertices $x_1, x_2, \dots, x_r$ are all chosen so that each of them is in the circle of convergence whose centre is at the preceding vertex.

In order to evaluate $\chi(x_{s+1}, x_s)$ we will suppose that x_s and x_{s+1} both lie either on one of the hyperbolas for which $u = \text{constant}$, or on one of the ellipses for which $v = \text{constant}$. This is not in reality a restriction, for, if x_s and x_{s+1} are not so placed, it is possible to draw the ellipse (or hyperbola) through x_s , and find where it meets the hyperbola (or ellipse) through x_{s+1} and take their point of intersection as a vertex of the polygon intermediate between x_s and x_{s+1} .

Now it was proved in Art. 10 that

$$\frac{\partial \chi(x, x_r)}{\partial x} = \frac{df(x | x_1 | \dots | x_r)}{dx}$$

and

$$\frac{\partial \chi(x, x_r)}{\partial x_r} = - \left[\frac{df(x | x_1 | \dots | x_{r-1})}{dx} \right]_{x=x_r};$$

therefore

$$\frac{\partial \chi(x_{s+1}, x_s)}{\partial x_{s+1}} = \left[\frac{df(x | x_1 | \dots | x_s)}{dx} \right]_{x=x_{s+1}},$$

$$\frac{\partial \chi(x_{s+1}, x_s)}{\partial x_s} = - \left[\frac{df(x | x_1 | \dots | x_{s-1})}{dx} \right]_{x=x_s}.$$

Therefore, by Art. 11,

$$\begin{aligned} \frac{\partial \chi(x_{s+1}, x_s)}{\partial x_{s+1}} &= Q(x_1) Q\left(\frac{x_2-x_1}{1-x_1}\right) \dots Q\left(\frac{x_{s+1}-x_s}{1-x_s}\right) \\ &\quad \times R(x_1) R\left(\frac{x_2-x_1}{1+x_1}\right) \dots R\left(\frac{x_{s+1}-x_s}{1+x_s}\right) \\ &= \frac{1}{\cos u_{s+1} \cosh v_{s+1} - i \sin u_{s+1} \sinh v_{s+1}} \quad (\text{by Art. 6}), \\ \frac{\partial \chi(x_{s+1}, x_s)}{\partial x_s} &= -Q(x_1) Q\left(\frac{x_2-x_1}{1-x_1}\right) \dots Q\left(\frac{x_s-x_{s-1}}{1-x_{s-1}}\right) \\ &\quad \times R(x_1) R\left(\frac{x_2-x_1}{1+x_1}\right) \dots R\left(\frac{x_s-x_{s-1}}{1+x_{s-1}}\right) \\ &= \frac{-1}{\cos u_s \cosh v_s - i \sin u_s \sinh v_s} \quad (\text{by Art. 6}). \end{aligned}$$

Now suppose that x_s and x_{s+1} both lie on the ellipse for which $v = b$. Then

$$x_s = \sin u_s \cosh b + i \cos u_s \sinh b,$$

$$x_{s+1} = \sin u_{s+1} \cosh b + i \cos u_{s+1} \sinh b;$$

therefore

$$\begin{aligned} \frac{D\chi(x_{s+1}, x_s)}{Db} &= \frac{\partial \chi(x_{s+1}, x_s)}{\partial x_s} \frac{Dx_s}{Db} + \frac{\partial \chi(x_{s+1}, x_s)}{\partial x_{s+1}} \frac{Dx_{s+1}}{Db} \\ &= \frac{-1}{\cos u_s \cosh b - i \sin u_s \sinh b} (\sin u_s \sinh b + i \cos u_s \cosh b) \\ &\quad + \frac{1}{\cos u_{s+1} \cosh b - i \sin u_{s+1} \sinh b} (\sin u_{s+1} \sinh b + i \cos u_{s+1} \cosh b) \\ &= -i + i = 0. \end{aligned}$$

In this case then we may find the value of $\chi(x_{s+1}, x_s)$ by putting $b = 0$; then x_s becomes $\sin u_s$ and x_{s+1} becomes $\sin u_{s+1}$. Here u_s and u_{s+1} are the parameters* of the hyperbolas through the vertices x_s, x_{s+1} of the polygon.

Again, if x_s and x_{s+1} both lie on the hyperbola for which $v = a$, then

$$x_s = \sin a \cosh v_s + i \cos a \sinh v_s,$$

$$x_{s+1} = \sin a \cosh v_{s+1} + i \cos a \sinh v_{s+1},$$

* The values of the parameters depend on the path of the variable x .

$$\begin{aligned}
& \frac{D\chi(x_{s+1}, x_s)}{Da} \\
&= \frac{\partial\chi(x_{s+1}, x_s)}{\partial x_s} \frac{Dx_s}{Da} + \frac{\partial\chi(x_{s+1}, x_s)}{\partial x_{s+1}} \frac{Dx_{s+1}}{Da} \\
&= \frac{-1}{\cos a \cosh v_s - i \sin a \sinh v_s} (\cos a \cosh v_s - i \sin a \sinh v_s) \\
&\quad + \frac{1}{\cos a \cosh v_{s+1} - i \sin a \sinh v_{s+1}} (\cos a \cosh v_{s+1} - i \sin a \sinh v_{s+1}) \\
&= -1 + 1 = 0.
\end{aligned}$$

In this case we may find the value of $\chi(x_{s+1}, x_s)$ by putting $a = 0$; then x_s becomes $i \sinh v_s$, x_{s+1} becomes $i \sinh v_{s+1}$. Here v_s and v_{s+1} are the parameters* of the ellipses which pass through the vertices x_s and x_{s+1} of the polygon.

13. Now consider $\chi(x_{s+1}, x_s)$ in the case in which x_s and x_{s+1} both lie on the ellipse whose parameter = b . Its value is, by Art. 12,

$$\begin{aligned}
& Q(x_1) Q\left(\frac{x_2 - x_1}{1 - x_1}\right) \dots Q\left(\frac{x_s - x_{s-1}}{1 - x_{s-1}}\right) \\
& \times R(x_1) R\left(\frac{x_2 - x_1}{1 + x_1}\right) \dots R\left(\frac{x_s - x_{s-1}}{1 + x_{s-1}}\right) \\
& \times \sum_{n=0}^{\infty} \left[\frac{(x_{s+1} - x_s)^{n+1}}{(n+1)} \sum_{t=0}^{t=n} \frac{(-\frac{1}{2})_{n-t}}{(n-t)!} \frac{(-\frac{1}{2})_t}{t!} \frac{1}{(x_s - 1)^{n-t}} \frac{1}{(x_s + 1)^t} \right] \\
&= \frac{1}{\cos u_s \cosh b - i \sin u_s \sinh b} \\
& \times \sum_{n=0}^{\infty} \left[\frac{(x_{s+1} - x_s)^{n+1}}{(n+1)} \sum_{t=0}^{t=n} \frac{(-\frac{1}{2})_{n-t}}{(n-t)!} \frac{(-\frac{1}{2})_t}{t!} \frac{1}{(x_s - 1)^{n-t}} \frac{1}{(x_s + 1)^t} \right].
\end{aligned}$$

Now we have seen that the value of this does not depend on the value of b ; therefore, putting $b = 0$, and consequently $x_s = \sin u_s$, $x_{s+1} = \sin u_{s+1}$, $\chi(x_{s+1}, x_s)$ becomes

$$\begin{aligned}
& \frac{1}{\cos u_s} \sum_{n=0}^{\infty} \left[\frac{(\sin u_{s+1} - \sin u_s)^{n+1}}{(n+1)} \sum_{t=0}^{t=n} \frac{(-\frac{1}{2})_{n-t}}{(n-t)!} \frac{(-\frac{1}{2})_t}{t!} \right. \\
& \quad \left. \times \frac{1}{(\sin u_s - 1)^{n-t}} \frac{1}{(\sin u_s + 1)^t} \right].
\end{aligned}$$

* The values of the parameters depend on the path of the variable x .

Now, by Taylor's theorem,

the acute angle whose sine is x_1 = the acute angle whose sine is x_2

$$+ \frac{1}{\sqrt{1-x_1^2}} \sum_{n=0}^{\infty} \left[\frac{(x_2-x_1)^{n+1}}{(n+1)} \sum_{t=0}^{n+1} \frac{(-\frac{1}{2})_{n-t}}{(n-t)!} \frac{(-\frac{1}{2})_t}{t!} \frac{1}{(x_1-1)^{n-t}} \frac{1}{(x_1+1)^t} \right],$$

where the square root is to be taken with the positive sign.

Now, if u , and u_{s+1} are both acute angles, we can put

$$x_1 = \sin u, \quad x_2 = \sin u_{s+1}, \\ \sqrt{1-x_1^2} = \cos u.$$

Also the acute angles whose sines are x_1 and x_2 are u , u_{s+1} respectively.

Hence

$$u_{s+1} = u + \chi(x_{s+1}, x_s);$$

therefore

$$\chi(x_{s+1}, x_s) = u_{s+1} - u.$$

The next case that comes up for consideration in forming a circuit is that in which u , is an acute angle, but u_{s+1} an obtuse angle. There is in this case a difficulty. As b approaches zero the ellipse shrinks up into the line joining the points -1 and $+1$. But it always has two sides, and we have to work out the continuation of the series from the point $\sin u$, on one side up to the point $+1$, and round on the other side to the point $\sin u_{s+1}$. It is therefore better to regard it as consisting of two parts, viz., of the part from $\sin u$, to $\sin \frac{\pi}{2}$, which belongs to the preceding case; and of the part from $\sin \frac{\pi}{2}$ to $\sin u_{s+1}$, which belongs to the following case.

Consider therefore the case where both u , and u_{s+1} are obtuse.

We put here, in applying Taylor's theorem,

$$x_1 = \sin(\pi - u), \quad x_2 = \sin(\pi - u_{s+1}), \\ \sqrt{1-x_1^2} = \cos(\pi - u).$$

In this case the acute angles whose sines are x_1 and x_2 are $\pi - u$, and $\pi - u_{s+1}$ respectively; therefore

$$\pi - u_{s+1} = \pi - u + \frac{1}{\cos(\pi - u)} \sum_{n=0}^{\infty} \left[\frac{(\sin u_{s+1} - \sin u)^{n+1}}{n+1} \right. \\ \left. \times \sum_{t=0}^{n+1} \frac{(-\frac{1}{2})_{n-t}}{(n-t)!} \frac{(-\frac{1}{2})_t}{t!} \frac{1}{(\sin u - 1)^{n-t}} \frac{1}{(\sin u + 1)^t} \right];$$

therefore

$$\pi - u_{s+1} = \pi - u - \chi(x_{s+1}, x_s);$$

therefore

$$\chi(x_{s+1}, x_s) = u_{s+1} - u_s,$$

as before.

We are now in a position to take the case where u_s is an acute angle and u_{s+1} is obtuse. Let us call the point $+1$, x_r . We take first u from u_s to $\frac{\pi}{2}$, which gives by the first case

$$\chi(x_r, x_s) = \frac{\pi}{2} - u_s.$$

Then we take u from $\frac{\pi}{2}$ to u_{s+1} , which gives by the last case

$$\chi(x_{s+1}, x_r) = u_{s+1} - \frac{\pi}{2},$$

Hence

$$\chi(x_r, x_s) + \chi(x_{s+1}, x_r) = u_{s+1} - u_s.$$

Hence in all cases we have

$$\chi(x_{s+1}, x_s) = u_{s+1} - u_s.$$

14. We have now to proceed to the case where x_{s+1} and x_s both lie on the hyperbola for which the parameter is equal to a . Here

$$\begin{aligned} \chi(x_{s+1}, x_s) &= \frac{1}{\cos a \cosh v_s - i \sin a \sinh v_s} \\ &\times \sum_{n=0}^{\infty} \left[\frac{(x_{s+1} - x_s)^{n+1}}{(n+1)} \sum_{t=0}^{n+1} \frac{(-\frac{1}{2})_{n-t}}{(n-t)!} \frac{(-\frac{1}{2})_t}{t!} \frac{1}{(x_s - 1)^{n-t}} \frac{1}{(x_s + 1)^t} \right]. \end{aligned}$$

The value of this is independent of the value of a . So we can put $a = 0$. Then x_s, x_{s+1} become $i \sinh v_s, i \sinh v_{s+1}$ respectively. And $\chi(x_{s+1}, x_s)$ becomes

$$\begin{aligned} &\frac{1}{\cosh v_s} \left\{ \sum_{n=0}^{\infty} \frac{(i \sinh v_{s+1} - i \sinh v_s)^{n+1}}{n+1} \right. \\ &\quad \times \left[\sum_{t=0}^{n+1} \frac{(-\frac{1}{2})_{n-t}}{(n-t)!} \frac{(-\frac{1}{2})_t}{t!} \frac{1}{(i \sinh v_s - 1)^{n-t}} \frac{1}{(i \sinh v_s + 1)^t} \right] \Big\} \\ &= \frac{i}{\cosh v_s} \left\{ \sum_{n=0}^{\infty} \frac{(\sinh v_{s+1} - \sinh v_s)^{n+1}}{n+1} \right. \\ &\quad \times \left[\sum_{t=0}^{n+1} \frac{(-\frac{1}{2})_{n-t}}{(n-t)!} \frac{(-\frac{1}{2})_t}{t!} \frac{1}{(\sinh v_s + i)^{n-t}} \frac{1}{(\sinh v_s - i)^t} \right] \Big\}. \end{aligned}$$

To evaluate this we consider the real logarithm of $z + \sqrt{1+z^2}$ where z is real. Call this real logarithm

$$\log_e(z + \sqrt{1+z^2}).$$

Then
$$\frac{d \log_e(z + \sqrt{1+z^2})}{dz} = \frac{1}{\sqrt{1+z^2}} = (z+i)^{-1}(z-i)^{-1};$$

therefore

$$\frac{d^{n+1} \log_e(z + \sqrt{1+z^2})}{dz^{n+1}} = \sum_{t=0}^{n+1} C_t \left(-\frac{1}{2}\right)_{n-t} (z+i)^{-n+t-1} \left(-\frac{1}{2}\right)_t (z-i)^{-t-1};$$

therefore

$$\begin{aligned} \frac{1}{(n+1)!} \frac{d^{n+1} \log_e(z + \sqrt{1+z^2})}{dz^{n+1}} \\ = \frac{1}{n+1} \frac{1}{\sqrt{1+z^2}} \sum_{t=0}^{n+1} \frac{\left(-\frac{1}{2}\right)_{n-t}}{(n-t)!} \frac{\left(-\frac{1}{2}\right)_t}{t!} \frac{1}{(z+i)^{n-t}} \frac{1}{(z-i)^t}, \end{aligned}$$

where the square root is to be taken with the positive sign.

Put now
$$z = \sinh v_s;$$

therefore

$$\begin{aligned} \left[\frac{1}{(n+1)!} \frac{d^{n+1} \log_e(z + \sqrt{1+z^2})}{dz^{n+1}} \right]_{z = \sinh v_s} \\ = \frac{1}{n+1} \frac{1}{\cosh v_s} \sum_{t=0}^{n+1} \frac{\left(-\frac{1}{2}\right)_{n-t}}{(n-t)!} \frac{\left(-\frac{1}{2}\right)_t}{t!} \frac{1}{(\sinh v_s + i)^{n-t}} \frac{1}{(\sinh v_s - i)^t}. \end{aligned}$$

Hence the value of $\chi(x_{s+1}, x_s)$ to be obtained is

$$\begin{aligned} &= i \sum_{n=0}^{\infty} \frac{(\sinh v_{s+1} - \sinh v_s)^{n+1}}{(n+1)!} \left[\frac{d^{n+1} \log_e(z + \sqrt{1+z^2})}{dz^{n+1}} \right]_{z = \sinh v_s} \\ &= i \left[\log_e(z + \sqrt{1+z^2})_{z = \sinh v_{s+1}} - \log_e(z + \sqrt{1+z^2})_{z = \sinh v_s} \right] \\ &= i(v_{s+1} - v_s). \end{aligned}$$

The series is convergent if $|\sinh v_{s+1} - \sinh v_s|$ is less than the smaller of the two moduli $|\sinh v_s - i|$ and $|\sinh v_s + i|$.

15. Hence, by Art. 13, if the vertices x, x_{s+1} be taken along an arc of one of the ellipses $v = \text{constant}$,

$$\begin{aligned}\chi(x_{s+1}, x_s) &= \text{value of } u \text{ at } x_{s+1} - \text{value of } u \text{ at } x_s \\ &= \{ \text{value of } (u+iv) \text{ at } x_{s+1} - \text{value of } (u+iv) \text{ at } x_s \},\end{aligned}$$

since the values of v at x_{s+1} and x_s are the same.

If, however, x, x_{s+1} be taken along an arc of one of the hyperbolas $u = \text{constant}$, then, by Art. 14,

$$\begin{aligned}\chi(x_{s+1}, x_s) &= i(\text{value of } v \text{ at } x_{s+1} - \text{value of } v \text{ at } x_s) \\ &= \text{value of } (u+iv) \text{ at } x_{s+1} - \text{value of } (u+iv) \text{ at } x_s,\end{aligned}$$

since the values of u at x_{s+1}, x_s are the same. Putting the two results together, when x_s and x_{s+1} are not situated as above, we may, by interposing one or more intermediary points, prove that

$$\chi(x_{s+1}, x_s) = \text{value of } (u+iv) \text{ at } x_{s+1} - \text{value of } (u+iv) \text{ at } x_s,$$

where the values of u, v will depend on the path of the variable x round the circuit.

16. Let us now examine

$$\begin{aligned}\chi(x, x_r) &= (x-x_r) \left[\frac{df(x | x_1 | \dots | x_{r-1})}{dx} \right]_{x=x_r} \\ &\quad + \frac{(x-x_r)^2}{2!} \left[\frac{d^2f(x | x_1 | \dots | x_r)}{dx^2} \right]_{x=x_r} \\ &\quad + \dots\end{aligned}$$

Suppose that the circuit has been taken round $x = 1$, but not round $x = -1$; and that x_r coincides with x_1 .

Then $\chi(x, x_r)$, where x_r coincides with x_1 , is, by Art. 12,

$$\begin{aligned}&Q(x_1) Q\left(\frac{x_2-x_1}{1-x_1}\right) \dots Q\left(\frac{x_1-x_{r-1}}{1-x_{r-1}}\right) \\ &\times R(x_1) R\left(\frac{x_2-x_1}{1-x_1}\right) \dots R\left(\frac{x_1-x_{r-1}}{1-x_{r-1}}\right) \\ &\times \sum_{n=0}^{\infty} \left\{ \frac{(x-x_1)^{n+1} s^n}{n+1} \sum_{s=0}^{n+1} \frac{\left(-\frac{1}{2}\right)_{n-s}}{(n-s)!} \frac{\left(-\frac{1}{2}\right)_s}{s!} \frac{1}{(x_1-1)^{n-s}} \frac{1}{(x_1+1)^s} \right\} \\ &= -Q(x_1) R(x_1) \sum_{n=0}^{\infty} \left\{ \frac{(x-x_1)^{n+1} s^n}{n+1} \sum_{s=0}^{n+1} \frac{\left(-\frac{1}{2}\right)_{n-s}}{(n-s)!} \frac{\left(-\frac{1}{2}\right)_s}{s!} \frac{1}{(x_1-1)^{n-s}} \frac{1}{(x_1+1)^s} \right\},\end{aligned}$$

by Art. 5.

17. *Continuation of the Power Series for arcsin x round a Circuit enclosing $x = +1$, but not $x = -1$.*

$$\begin{aligned} & f(x|x_1| \dots |x_{r-1}|x_1) \\ &= f(x_1) + \chi(x_2, x_1) + \chi(x_3, x_2) + \dots + \chi(x_{r-1}, x_r) + \chi(x, x_r) \\ & \qquad \qquad \qquad [\text{where } x_r = x_1] \\ &= (u_1 + iv_1) + (u_2 + iv_2 - u_1 - iv_1) + \dots + (u_r + iv_r - u_{r-1} - iv_{r-1}) \\ & \quad - Q(x_1) R(x_1) \sum_{n=0}^{\infty} \left\{ \frac{(x-x_1)^{n+1}}{n+1} \sum_{s=0}^{n+1} \frac{(-\frac{1}{2})_{n-s}}{(n-s)!} \frac{(-\frac{1}{2})_s}{s!} \frac{1}{(x_1-1)^{n-s}} \frac{1}{(x_1+1)^s} \right\}. \end{aligned}$$

But $u_r = \pi - u_1$ and $v_r = -v_1$;

therefore

$$\begin{aligned} & f(x|x_1| \dots |x_{r-1}|x_1) \\ &= \pi - u_1 - iv_1 \\ & \quad - Q(x_1) R(x_1) \sum_{n=0}^{\infty} \frac{(x-x_1)^{n+1}}{n+1} \sum_{s=0}^{n+1} \frac{(-\frac{1}{2})_{n-s}}{(n-s)!} \frac{(-\frac{1}{2})_s}{s!} \frac{1}{(x_1-1)^{n-s}} \frac{1}{(x_1+1)^s} \\ &= \pi - f(x_1) \\ & \quad - Q(x_1) R(x_1) \sum_{n=0}^{\infty} \frac{(x-x_1)^{n+1}}{n+1} \sum_{s=0}^{n+1} \frac{(-\frac{1}{2})_{n-s}}{(n-s)!} \frac{(-\frac{1}{2})_s}{s!} \frac{1}{(x_1-1)^{n-s}} \frac{1}{(x_1+1)^s}. \end{aligned}$$

But $f(x|x_1) = f(x_1)$

$$+ Q(x_1) R(x_1) \sum_{n=0}^{\infty} \frac{(x-x_1)^{n+1}}{n+1} \sum_{s=0}^{n+1} \frac{(-\frac{1}{2})_{n-s}}{(n-s)!} \frac{(-\frac{1}{2})_s}{s!} \frac{1}{(x_1-1)^{n-s}} \frac{1}{(x_1+1)^s};$$

therefore $f(x|x_1| \dots |x_{r-1}|x_1) = \pi - f(x|x_1)$

for a circuit in the positive direction round $x = +1$, which does not include $x = -1$.

18. In like manner for a circuit in the positive direction round $x = -1$, which does not include $x = +1$, the result would be $-\pi - f(x|x_1)$.

For a circuit in the positive direction round both $x = +1$ and $x = -1$ the result would be $-2\pi + f(x|x_1)$, provided that the point x_1 is situated in such a manner that the circuit is equivalent to a circuit round $x = -1$, followed by a circuit round $x = +1$, both these circuits being described in the positive direction.

Other circuits may be treated in a similar way by resolving them into circuits round $x = +1$ and $x = -1$.