Certain Mathematical Instruments for Graphically Indicating the Direction of Refracted and Reflected Light Rays. By J. R. Milne, B.Sc., Carnegie Scholar in Physics.

(MS. received March 10, 1905. Read June 19, 1905.)

The refraction equation $\sin i = \mu \sin r$, though simple in itself, is apt to give rise, in problems connected with refraction, to formulæ too involved for arithmetical computation. In such cases it may be necessary to trace the course through the optical system in question of a certain number of arbitrarily chosen rays, and thence to find the course of the other rays by interpolation. The

FIG. 1.—The divisions on the edge of the arm A G are not shown in the above figure.

linkage about to be described affords a rapid and accurate means of determining the paths of the rays through any optical system.

This linkage consists (see fig. 1) of five strips of metal, freely jointed at the points A B C D. If F A be the direction of

a ray of light incident on the surface (plane or spherical) of a medium of refractive index $= \mu$, at the point A ; then A G will be the direction in the refracted ray, provided that the division in the scale A G corresponding to the value of the refractive index μ be brought to the edge of the link C D. The proof is as follows. The divisions on A G are equally spaced, and are numbered from 0.50 to 2.00, and the length from the centre A to the division 1.00 is equal to the length of the links A D and B C. Accordingly, if H be the division on the scale corresponding to a refractive index μ , then we have the equation

$$A H = \mu \cdot A D \quad . \quad . \quad . \quad (1)$$

Further, the links A B and D C are of equal length, so that the perpendicular distances of the points D and H from the link A B are always equal. Accordingly we have, using equation (1),

$$\sin i = \frac{E D}{A D} = \frac{E D}{\frac{A H}{\mu}} = \mu \cdot \frac{E D}{A H} .$$

So

$$\sin i = \mu \sin r .$$

To use the linkage, the point A is brought to the point of intersection of the incident ray with the trace on the paper of the refracting surface. The edge of A L is laid along the trace if the latter be a straight line ; if it be the arc of a circle, then A L is made the tangent to that arc at the point A by arranging A B to pass through the centre of the circle (say at M). A F is then turned about A till it coincides with the incident ray, and the division on A G corresponding to μ is brought to the edge of C D by turning A G about A. The edge of A G then gives the direction of the ray after refraction. If the medium above the boundary surface have a refractive index different from 1, then μ must be taken as the ratio of the refractive indices of the two media, and so may be ≥ 1 , but as the scale on A G ranges from .500 to 2.00, the procedure is exactly the same whatever the value of μ . Should the incident ray lie on the other side of the line B N from that shown, the linkage is simply rotated through an angle of 180° about B N as axis. To avoid using values of $\mu < 1$, F may be moved to the other side of L, and the link A G then rotated till it

A B at their unit graduations. The point B is now to be marked, and the instrument placed as shown by the dotted lines, so that the points of intersection at O and B each lie at the division corresponding to μ . Then OD is the refracted ray; for $\sin POA = \cos AOC = OC/OA = \mu$. $OC/OA' = \mu$. $\cos A'OC = \sin DOP'$.

It is to be observed that in proceeding from a more to a less optically dense medium, the above procedure must be reversed, and the μ graduations used for the first step, and the unit graduations for the final one.

II. The instrument consists of a piece of transparent celluloid shaped as shown in fig. 3A. Its straight edge E being set along

FIG. 3A.

FIG. 3B.

the incident ray AO (fig. 3B), the intersection B of the inner semicircular edge (if the second medium be more dense than the first), or the outer semicircular edge (if the second medium be less dense than the first), with the surface BO is marked on the latter line. If the diameter of the semicircle OC be taken as unit length, then, because angle AOP = angle OCB, OB is the value of $\sin AOP$. The instrument is now to be so placed that the semicircle whose diameter is equal to μ takes the place of the semicircular edge with respect to the points O and B. We thus have, as before, $\sin P'OD = OB/OD = OB/\mu$, and therefore $\sin P'OD = \sin AOP/\mu$. Hence the straight edge of the instrument which now lies along OD gives the direction of the

refracted ray. For greater convenience, each semicircle drawn on the celluloid bears two numbers, one giving the ratio of its diameter to the diameter of the semicircle formed by the inner edge, and the other the ratio of its diameter to the diameter of the semicircle formed by the outer edge.

III. The instrument consists merely of an ordinary "45° set square," graduated in identical fashion along each of the two equal sides. The number attached to any graduation line gives the distance of the latter from the right angle corner in terms of an arbitrary unit, and the scale ranges from 1 up to 2. It is best to provide several such scales, each beginning where the preceding one ends. In use, one of the equal sides is laid along

FIG. 4A.

FIG. 4B.

the incident ray AO , fig. 4A, so that the surface BO intersects it at O at the graduation on one of the scales marked with a number equal to the value of μ . The intersection B of the other equal side with the line BO is then marked on the latter. If the square be now placed so that the equal sides still pass through O and B , but with O at the graduation 1 of the scale, OD gives the direction of the refracted ray. $\sin POA = \sin OBA = AO/OB = \mu$. $CO/OB = \mu$. $\sin DOP'$. When the ray passes to a less dense medium the procedure is reversed, the unit graduation being used for the first step and the μ graduation for the second.

When a large size square is being used on a blackboard for

teaching purposes, it simplifies the demonstration to apply the square as shown in fig. 4B. In this case the second step is taken with the right angle corner *below* the line OB. The lines OA, AB, BC, and CO are all drawn in; and the line OD is then drawn at right angles to CO by means of the square. In this case AO and OD are as before the rays, while AB and OC are the traces in the two media of the plane wave fronts, of which the relative velocities are represented by AO and BC.

An alternative way of using this instrument, which is preferable when the angle of incidence is small, is to employ not the surface but the normal to it, as shown in fig. 4C. Here $\sin POA = AP/PO = \mu$. $PC/PO = \mu$. $\sin P'OD$.

FIG. 4C.

Of these four instruments, each seems to have its particular advantages. The linkage is the most useful for class demonstration, because it shows simultaneously both of the rays and also the surface. The jointed strip arrangement is extremely portable, while the celluloid device gives directly the actual sines of the angles involved. The set square instrument is found to be the most convenient form for private use in the graphical solution of optical problems. In dealing with spherical surfaces, too, it is an advantage that with this instrument the normal to the surface may be used instead of the tangent, because the former can be obtained very easily by drawing a line through the centre of the circle.

In order to obtain the *reflected* ray in each case the instrument is to be re-applied after having been rotated through an angle of 180° about the line B O as axis (except in figure 4c, where the line P O must be used for this purpose).

(Issued separately July 7, 1905.)