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To cite this article: H.S. Warner Esq. (1846) XVI. On the connexion of the circle and hyperbola; and on the geometrical interpretation of imaginary exponentials , Philosophical Magazine Series 3, 29:192, 88-91, DOI: [10.1080/14786444608645584](https://doi.org/10.1080/14786444608645584)

To link to this article: <http://dx.doi.org/10.1080/14786444608645584>

Published online: 30 Apr 2009.

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XVI. *On the Connexion of the Circle and Hyperbola; and on the Geometrical Interpretation of Imaginary Exponentials.* By H. S. WARNER, Esq.*

THE following investigations relate to an interesting branch of modern science; but as it is dangerous to claim, at this time, anything as one's own without a careful examination of preceding writings, and as I am not acquainted with the recent progress of research in regard to what were called imaginary quantities, I send these observations for publication, merely on the chance that they are new; hoping that if they are not so, I may be informed of the fact through the columns of the Philosophical Magazine.

1. The assumption on which I proceed is that,—

The proper geometrical interpretation of $A\sqrt{-1}$ is a line of the length A, measured in a direction perpendicular to that in which it would have been measured if it had not been multiplied by $\sqrt{-1}$.

2. Taking the equation of the equilateral hyperbola (the centre being the origin, and, for simplicity, the semi-axis being the unit of length) $y^2 = x^2 - 1$, we find, that for a value of x (positive or negative) greater than unity, y is possible, determining the opposite hyperbolas; but that for a value of x less than unity, the equation assumes the form $y^2 = -1 \times (1 - x^2)$ or $y = \sqrt{-1} \times \sqrt{1 - x^2}$. Hence ($y = \sqrt{1 - x^2}$ being the equation of a circle whose radius = 1) by the assumption above, the hyperbola will be represented between $x = +1$ and $x = -1$, by a circle (radius 1) whose ordinates are measured perpendicular to the plane of xy ; that is by a circle standing upright, or a circle described on a plane at right angles to the plane of xy , the intersection of the two planes coinciding with the axis of x . The perpendicular plane I must, for want of a better name, call the *imaginary plane*.

3. Considering the circle in the same way, we shall find that for positive or negative values of x greater than unity, it is represented by a pair of opposite hyperbolas, drawn on the imaginary plane. Hence we arrive at this remarkable result, *that the equilateral hyperbola and the circle are similarly related to planes at right angles to one another, or that to change the one curve into the other, we have merely to take the imaginary plane as the real one, and vice versâ.* The figure will illustrate this: CA X is the axis of x , CY the axis of y , and CK a line perpendicular to the plane of xy , or what may be called the axis of imaginaries: CK and CL are the asymp-

* Communicated by the Author.

totes (from the perspective used the axis of imaginaries appears to coincide with one of the asymptotes), and CIAJ is the inscribed square of the asymptotes. The dotted lines (representing a circle as commonly drawn) are the lines on

the one plane, while the black ones (representing the common form of the hyperbola) are those on the other plane: hence if the black lines be considered as those on the real plane, the dotted circle will be on the imaginary plane, and the figure will be the locus of the *hyperbola*: if, on the other hand, the dotted lines be considered as being on the real plane, the black ones will be imaginary, and the figure will be the locus of the *circle*.

4. If we take any point P on the hyperbola (regarding for the present the black lines as those on the real plane) determined by the co-ordinates $CT=x$, $TP=y$, and draw the line PR parallel to CL, then by the property of the hyperbola we have

$$\frac{\text{area I A P R}}{\text{area C I A J}} = \log \frac{C R}{C I};$$

$$\text{but } I A P R = A C P, \quad C I A J = \frac{1}{2}, \quad C I = \sqrt{\frac{1}{2}};$$

$$\therefore 2 (A C P) = \log (C R \times \sqrt{2}). \quad \dots \dots (a.)$$

The equation of the hyperbola being $y = \sqrt{x^2 - 1}$, and of the asymptote $y = x$, we have

$$CQ = x\sqrt{2}, \quad QR = \sqrt{\frac{1}{2}} \times PQ = \sqrt{\frac{1}{2}} \times (x - \sqrt{x^2 - 1}),$$

$$\begin{aligned} CR &= CQ - QR = x\sqrt{2} - (x - \sqrt{x^2 - 1})\sqrt{\frac{1}{2}} \\ &= (x + \sqrt{x^2 - 1})\sqrt{\frac{1}{2}} \therefore CR\sqrt{2} = x + \sqrt{x^2 - 1} \\ &= CT + TP, \end{aligned}$$

hence we have

$$2(ACP) = \log(CT + TP) = \log(x + y). \quad \dots (b.)$$

5. Let us now observe the variations of the quantities that form this equation, as the point P moves along the curve, or is assumed at different points of it.

As P approaches A, both the area ACP and CT + TP diminish till at A we have $2 \times 0 = \log 1$.

Suppose P to be taken on the branch AF, then ACP and TP will both be negative, and the algebraic sum of CT and TP will be their arithmetical difference, which will be always less than 1; hence the logarithm of a number less than unity is negative.

Let us now suppose that a point on the imaginary circle is taken, as at P_2 . Then $2(ACP_2) = \log(CT_2 + T_2P_2)$; but if the arc $AP_2 = \theta$, we have

$$2(ACP_2) = \theta, \quad CT_2 = \cos \theta, \quad T_2P_2 = \sin \theta;$$

but $CA P_2$ and T_2P_2 being both on the imaginary plane are affected with the symbol $\sqrt{-1}$, while CT_2 being on the real plane is unaffected with it; hence finally we have

$$\begin{aligned} \theta\sqrt{-1} &= \log(\cos \theta + \sqrt{-1} \cdot \sin \theta) \} \\ \text{or } \epsilon^{\theta\sqrt{-1}} &= \cos \theta + \sqrt{-1} \cdot \sin \theta \end{aligned} \quad \dots (c.)$$

Similarly, if P_2 be taken on the imaginary branch below the real plane so that θ is negative, we have

$$\epsilon^{-\theta\sqrt{-1}} = \cos \theta - \sqrt{-1} \cdot \sin \theta. \quad \dots (c'.)$$

Let θ be $= \frac{\pi}{2}$ so that P_2 falls on H, then CT_2 or $\cos \theta$ vanishes, so that

$$\frac{\pi}{2}\sqrt{-1} = \log \sqrt{-1} \text{ or } \epsilon^{\frac{\pi}{2}\sqrt{-1}} = \sqrt{-1}. \quad \dots (d.)$$

When P_2 reaches B,

$$\pi\sqrt{-1} = \log(-1) \text{ or } \epsilon^{\pi\sqrt{-1}} = -1. \quad \dots (e.)$$

And at G,

$$\frac{3\pi}{2} \sqrt{-1} = \log(-\sqrt{-1}) \text{ or } \varepsilon^{\frac{3\pi}{2}\sqrt{-1}} = -\sqrt{-1}. \quad (f.)$$

$$\text{At A, } 2\pi \sqrt{-1} = \log 1 \text{ or } \varepsilon^{2\pi\sqrt{-1}} = 1. \quad (g.)$$

From A we may suppose P_2 to run over the circle a second time and then a third time, and so on to an m th time; so that in the formulas (c.), (d.), &c. we may increase the coefficient of $\sqrt{-1}$ in the exponent of ε , by $2m\pi$ without changing the result; also we may suppose P_2 on arriving at A after the m th revolution, to proceed along the real branch of the curve so that the logarithm of any quantity X may be represented by tabular $\log X + 2m\pi\sqrt{-1}$.

6. Perhaps it may not be unimportant to remark, that in describing the properties of the hyperbola, the proper way of enunciating the connexion of hyperbolic spaces with logarithms is the one I have used, namely $2(A C P) = \log(\bar{C} T + T P)$; for this expression holds good as well for the imaginary as for the real branch of the hyperbola, as I have shown above, while the common expression $\frac{I A P R}{C I A J} = \log \frac{C R}{C I}$ is quite unintelligible when applied to the imaginary circle, being derived from the consideration of the asymptotes, which are no part of the curve, though usually drawn with it.

7. There are several other points arising out of the connexion of the circle and hyperbola which it would exceed my limits to touch upon here, but which I shall probably make the subject of another article, should it turn out that the preceding results are new.

8. In conclusion, I may remark that this investigation is as yet incomplete, or imperfect, for the most general formula arising from it is $\log X = Y + 2m\pi\sqrt{-1}$ (Y being the arithmetical logarithm of X), while the most general formula obtained by other means is

$$\log X = \frac{Y + n\pi\sqrt{-1}}{1 + 2m\pi\sqrt{-1}}.$$

See De Morgan's Calculus, p. 384.

Port of Spain, Trinidad,
April 2, 1846.