estimated with an exactness of one one-hundredth of a millimeter; that is to say, a kilogramme weight may be weighed to within one hundred-thousandth of its value.

In another hall there is mounted a hydrostatic balance, which serves for determining densities. Here again all the details of the operations are of the most perfect character. The water which is to serve for hydrostatic weighing is first distilled in an ordinary retort, then redistilled by means of a platinum apparatus, and finally collected in a platinum vessel. This latter, which is placed under the balance, is employed for the weighings; and a series of ingenious apparatus permits of the weight being plunged into the water, and of the necessary manipulations being effected in such a way as to reduce to a minimum every chance of error that could intervene.

The section of weights possesses, in addition, a fine collection of iridized platinum and quartz weights for comparing weights of the first class, and of gilded brass ones for those of the second class.

Besides the fundamental apparatus just mentioned, the Bureau is provided with a large number of different instruments, some designed for certain special work, and others that are necessary for those special studies that are forcedly connected with the operations of comparing or weighing. Among the first, one of the most remarkable for the great delicacy of the method that it brings in play is the Fizeau apparatus, by means of which expansions are measured upon small specimens or fragments of a few millimeters thickness only, by the use of an optical process founded on the observation of a phenomenon of the interference of light. This apparatus permits of ascertaining and measuring variations in diameter between two points of a few thousandths of a millimeter.

The accessory instruments are cathetometers, spherometers, etc., as well as barometers, thermometers, and hygrometers. Fig. 2 shows, for example, the normal barometer of the section of weights, a large instrument in which are combined every improvement that could permit of measuring the pressure of the atmosphere with the highest degree of accuracy possible.

The ascertaining of the temperature plays an essential part in all the operations that have to be performed on standards, either of length or of weight. So the studies that relate to thermometry hold so important a position that it might be considered as constituting a section apart, comprising its special instruments also.—*La Nature*.

MATHEMATICS.

BRITISH ASSOCIATION, SOUTHPORT, 1883. INAUGURAL ADDRESS BY ARTHUR CAYLEY, M.A., D.C.L., LL.D., F.R.S., SADLERIAN PROFESSOR OF PURE MATHEMATICS IN THE UNIVERSITY OF CAMBRIDGE, PRESIDENT.

MATHEMATICS connect themselves on the one side with common life and the physical sciences; on the other side with philosophy, in regard to our notions of space and time; and in the questions which have arisen as to the universality and necessity of the truths of mathematics, and the foundation of our knowledge of them. I would remark here that the connection (if it exists) of arithmetic and algebra with the notion of time is far less obvious than that of geometry with the notion of space.

As to the former side, I am not making before you a defence of mathematics, but if I were I should desire to do it—in such manner as in the "Republic" Socrates was required to defend justice, quite irrespectively of the worldly advantages which may accompany a life of virtue and justice, and to show that, independently of all these, justice was a thing desirable in itself and for its own sake—not by speaking to you of the utility of mathematics in any of the questions of common life or of physical science. Still less would I speak of this utility before, I trust, a friendly audience interested or willing to appreciate an interest in mathematics in itself and for its own sake. I would, on the contrary, rather consider the obligations of mathematics to these different subjects as the sources of mathematical theories now as remote from them, and in as different a region of thought—for instance, geometry from the measurement of land, or the theory of numbers from arithmetic—as a river at its mouth is from its mountain source.

On the other side the general opinion has been and is that it is indeed by experience that we arrive at the truths of mathematics, but that experience is not their proper foundation; the mind itself contributes something. This is involved in the Platonic theory of reminiscence; looking at two things, trees or stones or anything else, which seem to us more or less equal, we arrive at the idea of equality; but we must have had this idea of equality before the time when first seeing the two things we were led to regard them as coming up more or less perfectly to this idea of equality; and the like as regards our idea of the beautiful, and in other cases.

The same view is expressed in the answer of Leibnitz, the *nisi intellectus ipse*, to the scholastic dictum, *nihil in intellectu quod non prius in sensu*: there is nothing in the intellect which was not first in the sensation, except (said Leibnitz) the intellect itself. And so again in the "Critique of Pure Reason," Kant's view is that, while there is no doubt but that all our cognition begins with experience, we are nevertheless in possession of cognitions *a priori*, independent, not of this or that experience, but absolutely so of all experience, and in particular that the axioms of mathematics furnish an example of such cognitions *a priori*. Kant holds further that space is no empirical conception which has been derived from external experiences, but that in order that sensations may be referred to something external, the representation of space must already lie at the foundation; and that the external experience is itself first only possible by this representation of space. And in like manner time is no empirical conception which can be deduced from an experience, but it is a necessary representation lying at the foundation of all intuitions.

And so in regard to mathematics, Sir W. R. Hamilton, in an introductory lecture on astronomy (1836), observes: "These purely mathematical sciences of algebra and geometry are sciences of the pure reason, deriving no weight and no assistance from experiment, and isolated or at least isolable from all outward and accidental phenomena. The idea of order, with its subordinate ideas of number and figure, we must not indeed call innate ideas, if that phrase be defined to imply that all men must possess them with equal clearness and fullness: they are, however, ideas which seem to be so far born with us that the possession of them in any conceivable degree is only the development of our original powers, the unfolding of our proper humanity."

The general question of the ideas of space and time, the axioms and definitions of geometry, the axioms relating to number, and the nature of mathematical reasoning, are fully

and ably discussed in Whewell's "Philosophy of the Inductive Sciences" (1840), which may be regarded as containing an exposition of the whole theory.

But it is maintained by John Stuart Mill that the truths of mathematics, in particular those of geometry, rest on experience; and, as regards geometry, the same view is on very different grounds maintained by the mathematician Riemann.

It is not so easy as at first sight it appears to make out how far the views taken by Mill in his "System of Logic, Ratiocinative and Inductive" (ninth edition, 1879) are absolutely contradictory to those which have been spoken of; they profess to be so; there are most definite assertions (supported by argument), for instance, p. 263: "It remains to inquire what is the ground of our belief in axioms, what is the evidence on which they rest. I answer, they are experimental truths, generalizations from experience. The proposition, 'Two straight lines cannot inclose a space,' or, in other words, two straight lines which have once met cannot meet again, is an induction from the evidence of our senses." But I cannot help considering a previous argument (p. 259) as very materially modifying this absolute contradiction. After inquiring, "Why are mathematics by almost all philosophers . . . considered to be independent of the evidence of experience and observation, and characterized as systems of necessary truth?" Mill proceeds (I quote the whole passage) as follows: "The answer I conceive to be that this character of necessity ascribed to the truths of mathematics, and even (with some reservations to be hereafter made) the peculiar certainty ascribed to them, is a delusion, in order to sustain which it is necessary to suppose that those truths relate to and express the properties of purely imaginary objects. It is acknowledged that the conclusions of geometry are derived partly at least from the so-called definitions, and these definitions are assumed to be correct representations, so far as they go, of the objects with which geometry is conversant. Now we have pointed out that from a definition as such no proposition, unless it be one concerning the meaning of a word, can ever follow, and that what apparently follows from a definition follows in reality from an implied assumption that there exists a real thing conformable thereto. This assumption in the case of the definitions of geometry is not strictly true; there exist no real things exactly conformable to the definitions. There exist no real points without magnitude, no lines without breadth, nor perfectly straight, no circles with all their radii exactly equal, nor squares with all their angles perfectly right. It will be said that the assumption does not extend to the actual, but only to the possible existence of such things. I answer that according to every test we have of possibility they are not even possible. Their existence, so far as we can form any judgment, would seem to be inconsistent with the physical constitution of our planet at least, if not of the universal [sic]. To get rid of this difficulty, and at the same time to save the credit of the supposed system of necessary truths, it is customary to say that the points, lines, circles, and squares which are the subjects of geometry exist in our conceptions merely, and are parts of our minds; which minds, by working on their own materials, construct *a priori* science, the evidence of which is purely mental and has nothing to do with outward experience. By howsoever high authority this doctrine has been sanctioned, it appears to me psychologically incorrect. The points, lines, and squares which any one has in his mind are (as I apprehend) simply copies of the points, lines, and squares which he has known in his experience. Our idea of a point I apprehend to be simply our idea of the *minimum visible*, the small portion of surface which we can see. We can reason about a line as if it had no breadth, because we have a power which we can exercise over the operations of our minds; the power, when a perception is present to our senses or a conception to our intellects, of attending to a part only of that perception or conception instead of the whole. But we cannot conceive a line without a breadth; we can form no mental picture of such a line: all the lines which we have in our minds are lines possessing breadth. If any one doubts this, we may refer him to his own experience. I much question if any one who fancies that he can conceive of a mathematical line thinks so from the evidence of his own consciousness. I suspect it is rather because he supposes that unless such a perception be possible, mathematics could not exist as a science: a supposition which there will be no difficulty in showing to be groundless."

I think it may be at once conceded that the truths of geometry are truths precisely because they relate to and express the properties of what Mill calls "purely imaginary objects"; that these objects do not exist in Mill's sense, that they do not exist in nature, may also be granted; that they are "not even possible," if this means not possible in an existing nature, may also be granted. That we cannot "conceive" them depends on the meaning which we attach to the word conceive. I would myself say that the purely imaginary objects are the only realities, the *ὄντως ὄντα*, in regard to which the corresponding physical objects are as the shadows in the cave; and it is only by means of them that we are able to deny the existence of a corresponding physical object; if there is no conception of straightness, then it is meaningless to deny the existence of a perfectly straight line.

But at any rate the objects of geometrical truth are the so-called imaginary objects of Mill, and the truths of geometry are only true, and *a fortiori* are only necessarily true, in regard to these so-called imaginary objects; and these objects, points, lines, circles, etc., in the mathematical sense of the terms, have a likeness to and are represented more or less imperfectly, and from a geometer's point of view no matter how imperfectly, by corresponding physical points, lines, circles, etc. I shall have to return to geometry, and will then speak of Riemann, but I will first refer to another passage of the "Logic."

Speaking of the truths of arithmetic, Mill says (p. 297) that even here there is one hypothetical element: "In all propositions concerning numbers a condition is implied without which none of them would be true, and that condition is an assumption which may be false. The condition is that $1 = 1$; that all the numbers are numbers of the same or of equal units." Here at least the assumption may be absolutely true; one shilling = one shilling in purchasing power, although they may not be absolutely of the same weight and fineness; but it is hardly necessary; one coin + one coin = two coins, even if the one be a shilling and the other a half crown. In fact, whatever difficulty be raisable as to geometry, it seems to me that no similar difficulty applies to arithmetic; mathematician or not, we have each of us, in its most abstract form, the idea of a number; we can each of us appreciate the truth of a proposition in regard to numbers; and we cannot but see that a truth in regard to numbers is something different in kind from an experimental truth generalized from experience. Compare, for instance, the proposition that the sun, having already

risen so many times, will rise to-morrow, and the next day, and the day after that, and so on; and the proposition that even and odd numbers succeed each other alternately *ad infinitum*: the latter at least seems to have the characters of universality and necessity. Or, again, suppose a proposition observed to hold good for a long series of numbers, one thousand numbers, two thousand numbers, as the case may be: this is not only no proof, but it is absolutely no evidence, that the proposition is a true proposition, holding good for all numbers whatever; there are in the theory of numbers very remarkable instances of propositions observed to hold good for very long series of numbers, and which are nevertheless untrue.

I pass in review certain mathematical theories.

In arithmetic and algebra, or say in analysis, the numbers or magnitudes which we represent by symbols are in the first instance ordinary (that is, positive) numbers or magnitudes. We have also in analysis and in analytical geometry *negative* magnitudes; there has been in regard to these plenty of philosophical discussion, and I might refer to Kant's paper, "Ueber die negativen Grossen in die Weltweisheit" (1763), but the notion of a negative magnitude has become quite a familiar one, and has extended itself into common phraseology. I may remark that it is used in a very refined manner in bookkeeping by double entry.

But it is far otherwise with the notion which is really the fundamental one (and I cannot too strongly emphasize the assertion) underlying and pervading the whole of modern analysis and geometry, that of imaginary magnitude in analysis and of imaginary space (or space as a *locus in quo* of imaginary points and figures) in geometry. I use in each case the word imaginary as including real. This has not been, so far as I am aware, a subject of philosophical discussion or inquiry. As regards the older metaphysical writers, this would be quite accounted for by saying that they knew nothing, and were not bound to know anything about it; but at present, and considering the prominent position which the notion occupies—say even that the conclusion was that the notion belongs to mere technical mathematics, or has reference to nonentities in regard to which no science is possible, still it seems to me that (as a subject of philosophical discussion), the notion ought not to be thus ignored; it should at least be shown that there is a right to ignore it.

Although in logical order I should perhaps now speak of the notion just referred to, it will be convenient to speak first of some other quasi-geometrical notions; those of more than three dimensional space, and of non-Euclidian two and three dimensional space, and also of the generalized notion of distance. It is in connection with these that Riemann considered that our notion of space is founded on experience, or rather that it is only by experience that we know that our space is Euclidian space.

It is well known that Euclid's twelfth axiom, even in Playfair's form of it, has been considered as needing demonstration, and that Lobatschewsky constructed a perfectly consistent theory wherein this axiom was assumed not to hold good, or say a system of non-Euclidian plane geometry. There is a like system of non-Euclidian solid geometry. My own view is that Euclid's twelfth axiom in Playfair's form of it does not need demonstration, but is part of our notion of space, of the physical space of our experience—the space, that is, which we become acquainted with by experience, but which is the representation lying at the foundation of all external experience. Riemann's view before referred to may I think be said to be that, having in *intellectu* a more general notion of space (in fact, a notion of non-Euclidian space), we learn by experience that space (the physical space of our experience) is, if not exactly, at least to the highest degree of approximation, Euclidian space.

But, suppose the physical space of our experience to be thus only approximately Euclidian space, what is the consequence which follows? Not that the propositions of geometry are only approximately true, but that they remain absolutely true in regard to that Euclidian space which has been so long regarded as being the physical space of our experience.

It is interesting to consider two different ways in which, without any modification at all of our notion of space, we can arrive at a system of non-Euclidian (plane or two-dimensional) geometry; and the doing so will, I think, throw some light on the whole question.

First, imagine the earth a perfectly smooth sphere; understand by a plane the surface of the earth, and by a line the apparently straight line (in fact, an arc of great circle) drawn on the surface; what experience would in the first instance teach would be Euclidian geometry; there would be intersecting lines which produced a few miles or so would seem to go on diverging, and apparently parallel lines which would exhibit no tendency to approach each other; and the inhabitants might very well conceive that they had by experience established the axiom that two straight lines cannot inclose a space, and the axiom as to parallel lines. A more extended experience and more accurate measurements would teach them that the axioms were each of them false; and that any two lines if produced far enough each way would meet in two points; they would in fact arrive at a spherical geometry, accurately representing the properties of the two-dimensional space of their experience. But their original Euclidian geometry would not the less be a true system; only it would apply to an ideal space, not the space of their experience.

Secondly, consider an ordinary, indefinitely extended plane; and let us modify only the notion of distance. We measure distance, say, by a yard measure or a foot rule, anything which is short enough to make the fractions of it of no consequence (in mathematical language by an infinitesimal element of length); imagine, then, the length of this rule constantly changing (as it might do by an alteration of temperature), but under the condition that its actual length shall depend only on its situation on the plane and on its direction; viz., if for a given situation and direction it has a certain length, then whenever it comes back to the same situation and direction it must have the same length. The distance along a given straight or curved line between any two points could then be measured in the ordinary manner with this rule, and would have a perfectly determinate value; it could be measured over and over again, and would always be the same: but, of course, it would be the distance, not in the ordinary acceptance of the term, but in quite a different acceptance. Or in a somewhat different way: if the rate of progress from a given point in a given direction be conceived as depending only on the configuration of the ground, and the distance along a given path between any two points thereof be measured by the time required for traversing it, then in this way also the distance would have a perfectly determinate value; but it would be a distance not in the ordinary acceptance of the term but in quite a different ac-

ception. And corresponding to the new notion of distance, we should have a new, non-Euclidian system of plane geometry; all theorems involving the notion of distance would be altered.

We may proceed further. Suppose that as the rule moves away from a fixed central point of the plane it becomes shorter and shorter; if this shortening takes place with sufficient rapidity, it may very well be that a distance which in the ordinary sense of the word is finite will in the new sense be infinite; no number of repetitions of the length of the ever shortening rule will be sufficient to cover it. There will be surrounding the central point of a certain finite area such that (in the new acceptance of the term distance) each point of the boundary thereof will be at an infinite distance from the central point; the points outside this area you cannot by any means arrive at with your rule; they will form a *terra incognita*, or rather an unknowable land: in mathematical language, an imaginary or impossible space; and the plane space of the theory will be that within the finite area—that is, it will be finite instead of infinite.

We thus with a proper law of shortening arrive at a system of non-Euclidian geometry which is essentially that of Lobatschewsky. But in so obtaining it we put out of sight its relation to spherical geometry: the three geometries (spherical, Euclidian, and Lobatschewsky's) should be regarded as members of a system, viz., they are the geometries of a plane (two-dimensional) space of constant positive curvature, zero-curvature, and constant negative curvature respectively; or, again, they are the plane geometries corresponding to three different notions of distance; in this point of view they are Klein's elliptic, parabolic, and hyperbolic geometries respectively.

Next, as regards solid geometry: we can by a modification of the notion of distance (such as has just been explained in regard to Lobatschewsky's system) pass from our present system to a non-Euclidian system; for the other mode of passing to a non-Euclidian system it would be necessary to regard our space as a flat three-dimensional space existing in a space of four dimensions (i. e., as the analogue of a plane existing in ordinary space); and to substitute for such flat three-dimensional space a curved three-dimensional space, say of constant positive or negative curvature. In regarding the physical space of our experience as possibly non-Euclidian, Riemann's idea seems to be that of modifying the notion of distance, not that of treating it as a locus in four-dimensional space.

I have just come to speak of four-dimensional space. What meaning do we attach to it? or can we attach to it any meaning? It may be at once admitted that we cannot conceive of a fourth dimension of space: that space as we conceive of it, and the physical space of our experience, are alike three-dimensional; but we can, I think, conceive of space as being two or even one dimensional; we can imagine rational beings living in a one-dimensional space (a line) or in a two-dimensional space (a surface), and conceiving of space accordingly, and to whom, therefore, a two-dimensional space, or (as the case may be) a three dimensional space, would be as inconceivable as a four-dimensional space is to us. And very curious speculative questions arise. Suppose the one-dimensional space a right line, and that it afterward becomes a curved line; would there be any indication of the change? Or, if originally a curved line, would there be anything to suggest to them that it was not a right line? Probably not, for a one-dimensional geometry hardly exists. But let the space be two-dimensional, and imagine it originally a plane, and afterward bent (converted, that is, into some form of developable surface) or converted into a curved surface; or imagine it originally a developable or curved surface. In the former case there should be an indication of the change, for the geometry originally applicable to the space of their experience (our own Euclidian geometry) would cease to be applicable; but the change could not be apprehended by them as a bending or deformation of the plane, for this would imply the notion of a three-dimensional space in which this bending or deformation could take place. In the latter case their geometry would be that appropriate to the developable or curved surface which is their space: viz., this would be their Euclidian geometry; would they ever have arrived at our own more simple system? But take the case where the two-dimensional space is a plane, and imagine the beings of such a space familiar with our own Euclidian plane geometry; if, a third dimension being still inconceivable by them, they were by their geometry or otherwise led to the notion of it, there would be nothing to prevent them from forming a science such as our own science of three-dimensional geometry.

Evidently all the foregoing questions present themselves in regard to ourselves, and to three-dimensional space as we conceive of it, and as the physical space of our experience. And I need hardly say that the first step is the difficulty, and that granting a fourth dimension we may assume as many more dimensions as we please. But whatever answer be given to them, we have, as a branch of mathematics, potentially if not actually, an analytical geometry of n -dimensional space. I shall have to speak again upon this.

Coming now to the fundamental notion already referred to, that of imaginary magnitude in analysis and imaginary space in geometry: I connect this with two great discoveries in mathematics made in the first half of the seventeenth century—Harriot's representation of an equation in the form $f(x) = 0$, and the consequent notion of the roots of an equation as derived from the linear factors of $f(x)$ (Harriot, 1560–1621: his "Algebra," published after his death, has the date 1631), and Descartes' method of co-ordinates, as given in the "Geometrie," forming a short supplement to his "Traité de la Méthode, etc." (Leyden, 1637).

I show how by these we are led analytically to the notion of imaginary points in geometry; for instance, we arrive at the theorem that a straight line and circle in the same plane intersect *always* in two points, real or imaginary. The conclusion as to the two points of intersection cannot be contradicted by experience: take a sheet of paper and draw on it the straight line and circle, and try. But you might say, or at least be strongly tempted to say, that it is meaningless. The question of course arises, What is the meaning of an imaginary point? and, further, In what manner can the notion be arrived at geometrically?

There is a well known construction in perspective for drawing lines through the intersection of two lines which are so nearly parallel as not to meet within the limits of the sheet of paper. You have two given lines which do not meet, and you draw a third line, which, when the lines are all of them produced, is found to pass through the intersection of the given lines. If instead of lines we have two circular arcs not meeting each other, then we can, by means of these arcs, construct a line; and if on completing the circles it is found that the circles intersect each other in two real points, then it will be found that the line passes through these two points. If the circles appear not to intersect, then

the line will appear not to intersect either of the circles. But the geometrical construction being in each case the same, we say that in the second case also the line passes through the two intersections of the circles.

Of course it may be said in reply that the conclusion is a very natural one, provided we assume the existence of imaginary points; and that, this assumption not being made, then, if the circles do not intersect, it is meaningless to assert that the line passes through their points of intersection. The difficulty is not got over by the analytical method before referred to, for this introduces difficulties of its own. Is there in a plane a point, the co-ordinates of which have given imaginary values? As a matter of fact, we do consider in plane geometry imaginary points introduced into the theory analytically or geometrically as above.

The like considerations apply to solid geometry, and we thus arrive at the notion of imaginary space as a *locus in quo* of imaginary points and figures.

I have used the word imaginary rather than complex, and I repeat that the word has been used as including real. But, this once understood, the word becomes in many cases superfluous, and the use of it would even be misleading. Thus, "a problem has so many solutions;" this means so many imaginary (including real) solutions. But if it were said that the problem had "so many imaginary solutions," the word "imaginary" would here be understood to be used in opposition to real. I give this explanation the better to point out now wide the application of the notion of the imaginary is, viz. (unless expressly or by implication excluded), it is a notion implied and presupposed in all the conclusions of modern analysis and geometry. It is, as I have said, the fundamental notion underlying and pervading the whole of these branches of mathematical science.

I consider the question of the geometrical representation of an imaginary variable. We represent the imaginary variable $x + iy$ by means of a point in a plane, the co-ordinates of which are x, y . This idea, due to Gauss, dates from about the year 1831. We thus picture to ourselves the succession of values of the imaginary variable $x + iy$ by means of the motion of the representative point: for instance, the succession of values corresponding to the motion of the point along a closed curve to its original position. The value $X + iY$ of the function can, of course, be represented by means of a point (taken for greater convenience in a different plane), the co-ordinates of which are X, Y .

We may consider in general two points, moving each in its own plane, so that the position of one of them determines the position of the other, and consequently the motion of the one determines the motion of the other; for instance, the two points may be the tracing point and the pencil of a pantograph. You may with the first point draw any figure you please, there will be a corresponding figure drawn by the second point. For a good pantograph a copy on a different scale (it may be); for a badly adjusted pantograph, a distorted copy; but the one figure will always be a sort of copy of the first, so that to each point of the one figure there will correspond a point in the other figure.

In the case above referred to, where one point represents the value $x + iy$ of the imaginary variable and the other the value $X + iY$ of some function $\varphi(x + iy)$ of that variable, there is a remarkable relation between the two figures: this is the relation of orthomorphic projection, the same which presents itself between a portion of the earth's surface and the representation thereof by a map on the stereographic projection or on Mercator's projection—viz., any indefinitely small area of the one figure is represented in the other figure by an indefinitely small area of the same shape. There will possibly be for different parts of the figure great variations of scale, but the shape will be unaltered; if for the one area the boundary is a circle, then for the other area the boundary will be a circle; if for one it is an equilateral triangle, then for the other it will be an equilateral triangle.

I have been speaking of an imaginary variable ($x + iy$), and of a function $\varphi(x + iy) = X + iY$ of that variable, but the theory may equally well be stated in regard to a plane curve; in fact, the $x + iy$ and the $X + iY$ are two imaginary variables connected by an equation; say their values are u and v , connected by an equation $F(u, v) = 0$; then, regarding u , v as the co-ordinates of a point *in plano*, this will be a point on the curve represented by the equation. The curve, in the widest sense of the expression, is the whole series of points, real or imaginary, the co-ordinates of which satisfy the equation, and these are exhibited by the foregoing corresponding figures in two planes; but in the ordinary sense the curve is the series of real points, with co-ordinates u, v , which satisfy the equation.

In geometry, it is the curve, whether defined by means of its equation, or in any other manner, which is the subject for contemplation and study. But we also use the curve as a representation of its equation—that is, of the relation existing between two magnitudes, x, y , which are taken as the co-ordinates of a point on the curve. Such employment of a curve for all sorts of purposes—the fluctuations of the barometer, the Cambridge boat races, or the Funds—is familiar to most of you. It is in like manner convenient in analysis, for exhibiting the relations between any three magnitudes, x, y, z , to regard them as the co-ordinates of a point in space; and, on the like ground, we should at least wish to regard any four or more magnitudes as the co-ordinates of a point in space of a corresponding number of dimensions. Starting with the hypothesis of such a space, and of points therein, each determined by means of its co-ordinates, it is found possible to establish a system of n -dimensional geometry analogous in every respect to our two and three-dimensional geometries, and to a very considerable extent serving to exhibit the relations of the variables.

It is to be borne in mind that the space, whatever its dimensionality may be, must always be regarded as an imaginary or complex space, such as the two or three-dimensional space of ordinary geometry; the advantages of the representation would otherwise altogether fail to be obtained.

I omit some further developments in regard to geometry; and all that I have written as to the connection of mathematics with the notion of time.

I said that I would speak to you, not of the utility of the mathematics in any of the questions of common life or of physical science, but rather of the obligations of mathematics to these different subjects. The consideration which thus presents itself is in a great measure that of the history of the development of the different branches of mathematical science in connection with the older physical sciences, astronomy and mechanics; the mathematical theory is in the first instance suggested by some question of common life or of physical science, is pursued and studied quite independently thereof, and perhaps after a long interval comes in contact with it, or with quite a different question.

Geometry and algebra must, I think, be considered as each of them originating in connection with objects or questions of common life—geometry, notwithstanding its name, hardly in the measurement of, and, but rather from the contemplation of such forms as the straight line, the circle, the ball, the top (or sugar-loaf); the Greek geometers appropriated for the geometrical forms corresponding to the last two of these, the words *σφαῖρα* and *κωνος*, our cone and sphere, and they extended the word cone to mean the complete figure obtained by producing the straight lines of the surface both ways indefinitely. And so algebra would seem to have arisen from the sort of easy puzzles in regard to numbers which may be made, either in the picturesque forms of the Bija Ganita with its maiden with the beautiful locks, and its swarms of bees amid the fragrant blossoms, and the one queen bee left humming around the lotus flower; or in the more prosaic form in which a student has presented to him in a modern text-book a problem leading to a simple equation.

The Greek geometry may be regarded as beginning with Plato (B.C. 430–347): the notions of geometrical analysis, loci, and the conic sections are attributed to him, and there are in his "Dialogues" many very interesting allusions to mathematical questions; in particular the passage in the "Theætetus," where he affirms the incommensurability of the sides of certain squares. But the earliest extant writings are those of Euclid (B.C. 285); there is hardly anything in mathematics more beautiful than his wondrous fifth book; and he has also in the seventh, eighth, ninth, and tenth books fully and ably developed the first principles of the theory of numbers, including the theory of incommensurables. We have next Apollonius (about B.C. 247) and Archimedes (B.C. 287–212), both geometers of the highest merit, and the latter of them the founder of the science of statics (including therein hydrostatics): his dictum about the lever, his "*Εὐρηκα*," and the story of the defense of Syracuse are well known. Following these we have a worthy series of names, including the astronomers Hipparchus (B.C. 150) and Ptolemy (A.D. 125), and ending, say, with Pappus (A.D. 400), but continued by their Arabian commentators, and the Italian and other European geometers of the sixteenth century and later, who pursued the Greek geometry.

The Greek arithmetic was, from the want of a proper notation, singularly cumbersome and difficult; and it was for astronomical purposes superseded by the sexagesimal arithmetic, attributed to Ptolemy, but probably known before his time. The use of the present so-called Arabic figures became general among Arabian writers on arithmetic and astronomy about the middle of the tenth century, but it was not introduced into Europe until about two centuries later. Algebra among the Greeks is represented almost exclusively by the treatise of Diophantus (A.D. 150), in fact a work on the theory of numbers containing questions relating to square and cube numbers, and other properties of numbers, with their solutions; this has no historical connection with the later algebra introduced into Italy from the East by Leonardo Bonacci of Pisa (A.D. 1202–1208), and successfully cultivated in the fifteenth and sixteenth centuries by Lucas Pacioli or De Burgo, Tartaglia, Cardan, and Ferrari. Later on we have Vieta (1540–1603), Harriot, already referred to, Wallis, and others.

Astronomy is of course intimately connected with geometry; the most simple facts of observation of the heavenly bodies can only be stated in geometrical language; for instance, that the stars describe circles about the Pole star, or that the different positions of the sun among the fixed stars in the course of the year form a circle. For astronomical calculations it was found necessary to determine the arc of a circle by means of its chord; the notion is as old as Hipparchus, a work of whom is referred to as consisting of twelve books on the chords of circular arcs; we have (A.D. 125) Ptolemy's "Almagest," the first book of which contains a table of arcs and chords with the method of construction; and among other theorems on the subject he gives the theorem afterward inserted in Euclid (Book VI., Prop. D) relating to the rectangle contained by the diagonals of a quadrilateral inscribed in a circle. The Arabians made the improvement of using in place of the chord of an arc the sine, or half chord of double the arc, and so brought the theory into the form in which it is used in modern trigonometry: the before-mentioned theorem of Ptolemy, or rather a particular case of it, translated into the notation of sines, gives the expression for the sine of the sum of two arcs in terms of the sines and cosines of the component arcs; and it is thus the fundamental theorem on the subject. We have in the fifteenth and sixteenth centuries a series of mathematicians who with wonderful enthusiasm and perseverance calculated tables of the trigonometrical or circular functions, Purbach, Müller or Regiomontanus, Copernicus, Reinhold, Maurolycus, Vieta, and many others; the tabulations of the functions tangent and secant are due to Reinhold and Maurolycus respectively.

Logarithms were invented, not exclusively with reference to the calculation of trigonometrical tables, but in order to facilitate numerical calculations generally; the invention is due to John Napier of Merchiston, who died in 1618 at sixty-seven years of age; the notion was based upon refined mathematical reasoning on the comparison of the spaces described by two points, the one moving with a uniform velocity, the other with a velocity varying according to a given law. It is to be observed that Napier's logarithms were nearly but not exactly those which are now called (sometimes Napierian, but more usually) hyperbolic logarithms—those to the base e ; and that the change to the base 10 (the great step by which the invention was perfected for the object in view) was indicated by Napier but actually made by Henry Briggs, afterward Savilian Professor at Oxford (d. 1630). But it is the hyperbolic logarithm which is mathematically important. The direct function e^x or $\exp. x$, which has for its inverse the hyperbolic logarithm, presented itself, but not in a prominent way. Tables were calculated of the logarithms of numbers, and of those of the trigonometrical functions.

The circular function and the logarithm were thus invented each for a practical purpose, separately and without any proper connection with each other. The functions are connected through the theory of imaginaries, and form together a group of the utmost importance throughout mathematics; but this is mathematical theory; the obligation of mathematics is for the discovery of the functions.

Forms of spirals presented themselves in Greek architecture, and the curves were considered mathematically by Archimedes; the Greek geometers invented some other curves, more or less interesting, but recondite enough in their origin. A curve which might have presented itself to anybody, that described by a point in the circumference of a rolling carriage wheel, was first noticed by Mersenne in 1615, and is the curve afterward considered by Roberval, Pascal, and others, under the name of the Roulette, other-

wise the Cycloid. Pascal (1623-1662) wrote at the age of seventeen his "Essais pour les Coniques," in seven short pages, full of new views on these curves, and in which he gives, in a paragraph of eight lines, his theory of the inscribed hexagon.

Kepler (1571-1630), by his empirical determination of the laws of planetary motion, brought into connection with astronomy one of the forms of conic, the ellipse, and established a foundation for the theory of gravitation. Contemporary with him, for most of his life, we have Galileo (1564-1642), the founder of the science of dynamics; and closely following upon Galileo, we have Isaac Newton (1643-1727): the "Philosophiæ naturalis Principia Mathematica," known as the "Principia," was first published in 1687.

The physical, statical, or dynamical questions which presented themselves before the publication of the "Principia" were of no particular mathematical difficulty, but it is quite otherwise with the crowd of interesting questions arising out of the theory of gravitation, and which, in becoming the subject of mathematical investigation, have contributed very much to the advance of mathematics. We have the problem of two bodies, or what is the same thing, that of the motion of a particle about a fixed center of force, for any law of force; we have also the (mathematically very interesting) problem of the motion of a body attracted to two or more fixed centers of force; and, next preceding that of the actual solar system, the problem of three bodies; this has ever been and is far beyond the power of mathematics, and it is in the lunar and planetary theories replaced by what is mathematically a different problem, that of the motion of a body under the action of a principal central force and a disturbing force; or (in one mode of treatment) by the problem of disturbed elliptic motion. I

I may mention a few other instances where a practical or physical question has connected itself with the development of mathematical theory. I have spoken of two map projections—the stereographic, dating from Ptolemy; and Mercator's projection, invented by Edward Wright about the year 1600; each of these, as a particular case of the orthomorphic projection, belongs to the theory of the geometrical representation of an imaginary variable. I have spoken also of perspective, and (in an omitted paragraph) of the representation of solid figures employed in Monge's descriptive geometry. Monge, it is well known, is the author of the geometrical theory of the curvature of surfaces and of curves of curvature: he was led to this theory by a problem of earthwork—from a given area, covered with earth of uniform thickness, to carry the earth and distribute it over an equal given area, with the least amount of cartage. For the solution of the corresponding problem in solid geometry he had to consider the intersecting normals of a surface, and so arrived at the curves of curvature (see his "Memoire sur les Deblais et les Remblais," *Mem. de l'Acad.*, 1781). The normals of a surface are, again, a particular case of a doubly infinite system of lines, and are so connected with the modern theories of congruences and complexes.

The undulatory theory of light led to Fresnel's wave surface, a surface of the fourth order, by far the most interesting one which had then presented itself. A geometrical property of this surface, that of having tangent planes each touching it along a plane curve (in fact, a circle), gave to Sir W. R. Hamilton the theory of conical refraction. The wave surface is now regarded in geometry as a particular case of Kummer's quartic surface, with sixteen conical points and sixteen singular tangent planes.

My imperfect acquaintance as well with the mathematics as the physics prevents me from speaking of the benefits

to theories outside of ordinary mathematics is still on the text of the vast extent of modern mathematics.

In conclusion I would say that mathematics have steadily advanced from the time of the Greek geometers. Nothing is lost or wasted; the achievements of Euclid, Archimedes, and Apollonius are as admirable now as they were in their own days. Descartes' method of co-ordinates is a possession for ever. But mathematics have never been cultivated more zealously and diligently, or with greater success, than in this century—in the last half of it or at the present time; the advances made have been enormous, the actual field is boundless, the future full of hope. In regard to pure mathematics we may most confidently say:

"Yet I doubt not through the ages one increasing purpose runs,
And the thoughts of men are widened with the process of the suns."

CHAMPONNOIS'S STEAM STARCH PLANT.

THE manufacture of starch requires, as well known, very many operations, which succeed one another, almost all of them, uninterruptedly and require constant surveillance. In the first place, it is the washing and cleaning of the tubers, which is required to be done with a certain energy, very rapidly, and not without the use of considerable water; and then it is the rasping and the elutriation, which, to be satisfactory, must be repeated in order to extract the maximum of product. It is necessary after this to collect the starch, to wash it in water, bleach it, dry it, bolt it, and put it into bags, while, on the other hand, it is necessary to receive the pulps, mix them, and wash and press them in order to extract therefrom a portion of the water that they contain, so as to render them fit to be used for feeding cattle.

In order to co-ordinate these different operations ration-

PLATE I.—DETAILS OF STEAM STARCH PLANT AT MAGNEUX-LE-GABION, FRANCE.

would remark that we have here an instance in which an astronomical fact, the observed slow variation of the orbit of a planet, has directly suggested a mathematical method, applied to other dynamical problems, and which is the basis of very extensive modern investigations in regard to systems of differential equations. Again, immediately arising out of the theory of gravitation, we have the problem of finding the attraction of a solid body of any given form upon a particle, solved by Newton in the case of a homogeneous sphere, but which is far more difficult in the next succeeding cases of the spheroid of revolution (very ably treated by Maclaurin) and of the ellipsoid of three unequal axes: there is perhaps no problem of mathematics which has been treated by as great a variety of methods, or has given rise to so much interesting investigation, as this last problem of the attraction of an ellipsoid upon an interior or exterior point. It was a dynamical problem, that of vibrating strings, by which Lagrange was led to the theory of the representation of a function as the sum of a series of multiple sines and cosines; and connected with this we have the expansions in terms of Legendre's functions P_n , suggested to him by the question just referred to of the attraction of an ellipsoid; the subsequent investigations of Laplace on the attractions of bodies differing slightly from the sphere led to the functions of two variables called Laplace's functions. I have been speaking of ellipsoids, but the general theory is that of attractions, which has become a very wide branch of modern mathematics. Associated with it we have in particular the names of Gauss, Lejeune-Dirichlet, and Green; and I must not omit to mention that the theory is now one relating to n -dimensional space. Another great problem of celestial mechanics, that of the motion of the earth about its center of gravity, in the most simple case, that of a body not acted upon by any forces, is a very interesting one in the mathematical point of view.

which the theory of partial differential equations has received from the hydrodynamical theory of vortex motion, and from the great physical theories of electricity, magnetism, and energy.

It is difficult to give an idea of the vast extent of modern mathematics. This word "extent" is not the right one. I mean extent crowded with beautiful detail—not an extent of mere uniformity, such as an objectless plain, but of a tract of beautiful country seen at first in the distance, but which will bear to be rambled through and studied in every detail of hillside and valley, stream, rock, wood, and flower. But, as for anything else, so for a mathematical theory—beauty can be perceived, but not explained. As for mere extent, I might illustrate this by speaking of the dates at which some of the great extensions have been made in several branches of mathematical science.

And, in fact, in the address as written, I speak at considerable length of the extensions in geometry since the time of Descartes, and in other specified subjects since the commencement of the century; these subjects are the general theory of the function of an imaginary variable; the leading known functions, viz., the elliptic and single theta-functions and the Abelian and multiple theta-functions; the theory of equations and the theory of numbers. I refer also to some theories outside of ordinary mathematics: the multiple algebra or linear associative algebra of the late Benjamin Peirce; the theory of Argand, Warren, and Peacock in regard to imaginaries in plane geometry; Sir W. R. Hamilton's quaternions, Clifford's biquaternions, the theories developed in Grassmann's "Ausdehnungslehre" with recent extensions thereof to non-Euclidian space by Mr. Homersham Cox; also Boole's "Mathematical Logic," and a work connected with logic, but primarily mathematical and of the highest importance, Schubert's "Abzählende Geometrie" (1878). I remark that all this in regard

ally, in a given place, with little manual labor, it becomes indispensable to study in advance the best arrangements to be adopted, so as to avoid all miscalculations that might work to the injury of the manufacturer.

A properly arranged factory may make the fortune of its proprietor, while if certain points be neglected it may cause his ruin. To-day, in fact, competition is such in the different industries, that the majority of them are enabled to exist only by force of economy, and by the minutest care taken in regard to the least details.

What we state in regard to the *matériel*, properly so called, of the factory, applies equally to the motor itself, which should not only be of a sufficient power to run all the machinery without trouble and with safety and regularity, but should also be so constructed as to consume a minimum of fuel and to require but slight cost to keep it in repair.

It is well to look ahead, too, at the time of starting the factory, to its possible enlargement, and to calculate the motive power as a consequence thereof. A well constructed steam engine of large proportions, with condensation and variable expansion, permits this end to be attained within certain limits, especially if care be taken to give the generator such dimensions that the heating surface notably exceeds that which corresponds to the nominal power.

In presenting herewith general views of the apparatus used in the starch works of Mr. Joseph Gaudet, at Magneux-le-Gabion, France, and devised by Mr. Champonnois, we deem it necessary to accompany them with some observations and accurate information that may prove useful in factories like this that are designed to work automatically in nearly all their parts, and with the co-operation of as few men as possible.

General Description of the Plant.—(Figs. 1 to 6.) Fig. 1 gives a longitudinal view of the building and a general elevation of most of the apparatus. Fig. 2 gives a general plan,