XXVIII. On the lateral vibration of loaded and unloaded bars

John Morrow M.Sc. M.Eng.

To cite this article: John Morrow M.Sc. M.Eng. (1906) XXVIII. On the lateral vibration of loaded and unloaded bars , Philosophical Magazine Series 6, 11:63, 354-374, DOI: [10.1080/14786440609463451](https://doi.org/10.1080/14786440609463451)

To link to this article: <http://dx.doi.org/10.1080/14786440609463451>

Published online: 16 Apr 2009.

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XXVIII. *On the Lateral Vibration of Loaded and Unloaded Bars.* By JOHN MORROW, M.Sc. (Vict.), M.Eng. (Liverpool), Lecturer in Engineering, University College, Bristol*.

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Section I. *Introduction and Notation.*

§ 1. **A** METHOD of calculating the frequency of the lateral vibration of bars has been described in a recent paper, "On the Lateral Vibration of Bars of Uniform and Varying Sectional Area" (see *Philosophical Magazine*, July 1905).

It is an important feature of this method that it gives, in a simple form, the equation of the elastic central line of the displaced bar, and thus provides data from which the stresses and strains in all parts may be readily calculated. The method lends itself to many cases of loaded bars which have not hitherto been solved, and the present paper, after dealing with some cases of unloaded bars under different end conditions, and bars of negligible mass carrying concentrated loads, gives more particularly the solutions for some important problems of loaded bars of appreciable mass.

It will be seen that a full consideration of the simpler cases treated first very materially lessens the labour involved in solving the more complex problems in which both the load and the mass of the bar itself are taken into account.

These solutions are, in general, obtained by a process of continuous approximation. Each approximation depends on the principle that, at any point in the length of the bar, the curvature is equal to the couple due to the reversed effective forces divided by the flexural rigidity.

To estimate the value of the couple a vibration-curve must be assumed. The above principle then gives an expression for the curvature at all points.

The process of continuous approximation to the exact solution is based on the fact that the expression for the deflexion, as obtained from that for the curvature, is a much closer approximation to the truth than is that originally assumed for the purpose of calculating the effective forces.

* Communicated by the Physical Society: read November 24, 1905.

A reference should be made here to the important papers by Professor Dunkerley * and by Dr. Chree † on the "Whirling" of Shafts. The relationship of the whirling to the vibrational problem was very clearly brought out by Chree, and under certain circumstances the two problems are identical.

§ 2. The notation used is similar to that of the paper previously cited; x and z are taken parallel, and y perpendicular, to the undisturbed position of the axis, bending occurring in the xy plane.

E = Young's Modulus for the material (assumed homogeneous and isotropic),

ρ = density of material of bar,

ω = sectional area,

$I = \omega k^2$ = Geometrical Moment of Inertia of cross-section about the neutral axis,

l = length or span of bar,

$U = \sqrt{\frac{E}{\rho}}$ = Velocity of transmission of longitudinal vibrations in the bar,

$y_1 y_a$ &c. = displacements of given points in the length of the bar,

$M_0 M_1$ = bending couples required to fix the ends.

We have also

$$\frac{\ddot{y}_1}{y_1} = \frac{\ddot{y}}{y};$$

and for the frequency

$$N = \frac{1}{2\pi} \sqrt{-\frac{\ddot{y}_1}{y_1}}.$$

The value of N is, however, not always recorded, as it is sufficient to find the expression for $\frac{\ddot{y}_1}{y_1}$.

Section II. Unloaded Bars.

§ 3. *Uniform Bar Clamped at Both Ends.*—Taking the origin at one end, the terminal conditions are

$$x=0, y=\frac{dy}{dx}=0; \quad x=l, y=\frac{dy}{dx}=0; \quad x=\frac{l}{2}, y=y_1, \frac{dy}{dx}=0;$$

* Phil. Trans. Roy. Soc. A 1894, p. 279.

† Phil. Mag. May 1904, p. 504.

and hence the equation to be assumed as the first approximation to the type of displacement is

$$y = \frac{16y_1}{l^2} \left(x^2 - \frac{2x^3}{l} + \frac{x^4}{l^2} \right).$$

The ordinary Euler-Bernoulli theory leads to

$$-EI \frac{d^2 y}{dx^2} = M_0 + \rho\omega \int_0^x (x-z) \ddot{y}_z dz - \frac{1}{2} \rho\omega x \int_0^l \ddot{y} dx.$$

Substituting for y , integrating, and determining M_0 by the condition that $\frac{dy}{dx} = 0$ when $x=l$, we find

$$\begin{aligned} -y = \frac{16\rho\omega\ddot{y}_1}{EI l^2} & \left(.178,571 l^4 x^2 - .27 l^3 x^3 + .27 x^6 - .238,095 \frac{x^7}{l} \right. \\ & \left. + .059,524 \frac{x^8}{l^2} \right); \\ \therefore \frac{\ddot{y}_1}{y_1} = -494.74 \frac{EI}{\rho\omega l^4}. \end{aligned}$$

Similarly, using these values of y and $\frac{\ddot{y}_1}{y_1}$, we arrive at a second approximation, in which

$$\begin{aligned} -y = \frac{.079158 \rho\omega\ddot{y}_1}{EI l^6} & \left(.35572 l^8 x^2 - .55113 l^7 x^3 + .49603 l^4 x^6 \right. \\ & \left. - .33069 l^3 x^7 + .05511 x^{10} - .03006 \frac{x^{11}}{l} + .00501 \frac{x^{12}}{l^2} \right) \\ \text{and} \quad & -\frac{\ddot{y}_1}{y_1} = 500.4 \frac{EI}{\rho\omega l^4}, \end{aligned}$$

agreeing well with the value of 500.6 obtained for the constant by the exact solution.

§ 4. *Clamped-supported Bar*.—In this case let y_1 be the displacement at such a distance from the fixed end (taken as origin) that $\frac{dy}{dx} = 0$ there. The initial type is then

$$y = 7.6931 y_1 \left(\frac{3x^2}{2l^2} - \frac{5x^3}{2l^3} + \frac{x^4}{l^4} \right),$$

and

$$-EI \frac{d^2 y}{dx^2} = \frac{M_0(l-x)}{l} + \rho\omega \frac{\ddot{y}_1}{y_1} \int_0^x (x-z) y_z dz - \rho\omega \frac{x \ddot{y}_1}{l y_1} \int_0^l y(l-x) dx.$$

Integrating and determining M_0 and the arbitrary constants by the terminal conditions

$$-y = \frac{.076931 \rho \omega \ddot{y}_1}{EI l^4} (.565476 l^8 x^2 - .744048 l^5 x^3 + .416 l^2 x^6 - .297619 l x^7 + .059524 x^8),$$

and

$$-\frac{\ddot{y}_1}{y_1} = 235.9 \frac{EI}{\rho \omega l^4}.$$

A small error in the value of x for which $y = y_1$ makes no appreciable difference in the final result.

The next approximation gives

$$-y = \frac{.018146 \rho \omega \ddot{y}_1}{EI l^8} (2.36692 l^{10} x^2 - 3.10194 l^9 x^3 + 1.57077 l^6 x^6 - .88577 l^5 x^7 + .08267 l^2 x^{10} - .03758 l x^{11} + .00493 x^{12});$$

and putting

$$x = .5805 l,$$

$$-\frac{\ddot{y}_1}{y_1} = 238.0 \frac{EI}{\rho \omega l^4}$$

and

$$N = \frac{15.42}{2\pi} \frac{kU}{l^2}.$$

§ 5. *Clamped-Free Bar of Circular Section, Diameter varying as distance from Free End.*—Taking, as previously in similar cases, the origin at the free end,

$$\frac{d^2 y}{dx^2} = \frac{\rho}{EI} \frac{\ddot{y}_1}{y_1} \int_0^x \omega y_z (x-z) dz.$$

Let

$$y_z = y_1 \left(1 - \frac{2z}{l} + \frac{z^2}{l^2} \right),$$

and the diameter $d = Ax$. Then

$$-\frac{d^2 y}{dx^2} = \frac{16 \rho \ddot{y}_1}{EA^2 x^4} \int_0^x z^2 (x-z) \left(1 - \frac{2z}{l} + \frac{z^2}{l^2} \right) dz,$$

$$-y = \frac{16 \rho \ddot{y}_1}{EA^2} \left(.016 l^2 - .04 l x + .0416 x^2 - .016 \frac{x^3}{l} + .0027 \frac{x^4}{l^2} \right),$$

and

$$-\frac{\ddot{y}_1}{y_1} = 3.75 \frac{EA^2}{\rho l^2}.$$

In the second approximation,

$$-\frac{\ddot{y}_1}{y_1} = 4.5324 \frac{EA^2}{\rho l^2};$$

and in the third approximation,

$$\begin{aligned} -y = & \frac{.696176 \rho \ddot{y}_1}{EA^2 l^4} \left(.304931 l^6 - .874742 l^5 x + .957615 l^4 x^2 \right. \\ & - .542880 l^3 x^3 + .192,901 l^2 x^4 - .044091 l x^5 + .006889 x^6 \\ & \left. - .000656 \frac{x^7}{l} + .000033 \frac{x^8}{l^2} \right), \\ -\frac{\ddot{y}_1}{y_1} = & 4.7106 \frac{EA^2}{\rho l^2}, \quad \text{and} \quad N = \frac{2.170}{2\pi} \frac{AU}{l}. \end{aligned}$$

Kirchhoff* has obtained the solution for the frequency in this case, and his result may be written

$$N = \frac{2.179}{2\pi} \frac{AU}{l}.$$

In the Euler-Bernoulli theory of beams it is assumed that the greatest diameter is small compared with the total effective length. When the diameter varies as the distance from one end, therefore, it is necessary that the variation should be small. In other cases, so far as the mathematical theory is concerned, the solution can only be looked upon as a probable approximation.

Section III. Loaded Bars of Negligible Mass.

§ 6. When the bar carries a load at some point in its length, and the mass of the bar itself is negligible, it is not necessary to assume a type of displacement. The method then becomes an exact one, and gives at once the true type and frequency of the vibration.

In the following paragraphs, m is the mass with which the bar is loaded, and a is the distance of its centre of gravity from the point chosen as origin.

§ 7. *Clamped-Free Bar.*—If the origin be at the load and y_1 be the displacement there,

$$-EI \frac{d^2 y}{dx^2} = m \ddot{y}_1 x,$$

and the vibration-curve is

$$-y = \frac{m \ddot{y}_1}{EI} \left(\frac{x^3}{6} - \frac{l^2 x}{2} + \frac{l^3}{3} \right),$$

* See *Berliner Monatsberichte*, 1879, p. 815.

where l is the distance between the load and the fixed end.

$$-\frac{\ddot{y}_1}{y_1} = \frac{3EI}{ml^3}.$$

On the other side of the load the bar is straight.

§ 8. *Bar supported at Both Ends.*—Origin at one end. Mass divides length of bar into segments a and b .

Here, for $x < a$,

$$EI \frac{d^2 y}{dx^2} = \frac{b}{l} m \ddot{y}_a x;$$

whilst, for $x > a$,

$$EI \frac{d^2 y'}{dx^2} = m \ddot{y}_a \left(\frac{b}{l} x - x + a \right).$$

The constants of integration, expressing the inclinations of the bar at its ends, are obtained from the consideration that both $\frac{dy}{dx}$ and y must have the same value in each equation when $x = a$. The inclinations are, therefore,

$$-\frac{m \ddot{y}_a}{EI} \frac{ab}{l} \left(\frac{a}{6} + \frac{b}{3} \right) \quad \text{and} \quad -\frac{m \ddot{y}_a}{EI} \frac{a}{l} \left(\frac{a^2}{6} + \frac{l^2}{3} \right).$$

Hence

$$y = \frac{m \ddot{y}_a}{6EI} \frac{b}{l} (x^3 - 2alx + a^2x),$$

and

$$y' = \frac{m \ddot{y}_a}{6EI} \frac{l-x}{l} (a^3 - 2alx + ax^2).$$

In either case,

$$-\frac{\ddot{y}}{y} = \frac{3EI}{ma^2 b^3}.$$

§ 9. *Clamped-Clamped Bar.*—With the same notation as in the preceding case.

If $x < a$,

$$-EI \frac{d^2 y}{dx^2} = M_0 + (M_1 - M_0) \frac{x}{l} - m \ddot{y}_a \frac{bx}{l};$$

and if $x > a$ we must add $m \ddot{y}_a (x - a)$ to the above.

By integration we find

$$M_0 = m \ddot{y}_a \frac{ab^2}{l^2} \quad \text{and} \quad M_1 = m \ddot{y}_a \frac{a^2 b}{l^2},$$

$$-EI y = \frac{m \ddot{y}_a b^2 x^2}{6l^3} [3a(l-x) - bx]$$

and

$$-EI y' = \frac{m \ddot{y}_a a^2}{6l^3} [(a+3b)x^3 + 3(a-2b)lx^2 + 3l^2x - al^3].$$

In either case, $x=a$ gives

$$-\frac{\ddot{y}_a}{y_a} = \frac{3EI}{m} \left(\frac{l}{ab} \right)^3.$$

This result and those of §§ 7 and 8 are identical with the corresponding results obtained, otherwise, by Chree.

§ 10. *Supported Bar with two symmetrically placed Concentrated Loads.*—When there are two equal masses, placed at equal distances a from each end of a bar whose mass is negligible, an exact solution can be obtained in a precisely similar manner. Thus, for a bar supported at each end,

For $x < a$,

$$EI \frac{d^2 y}{dx^2} = m \ddot{y}_a x.$$

$$EI y = m \ddot{y}_a x \left(\frac{x^2}{6} - \frac{la}{2} + \frac{a^2}{2} \right).$$

For $x=a$ to $(l-a)$,

$$EI \frac{d^2 y'}{dx^2} = m \ddot{y}_a a,$$

$$EI y' = m \ddot{y}_a a \left(\frac{a^2}{6} - \frac{lx}{2} + \frac{x^2}{2} \right).$$

The curve between the two masses is a circular arc of radius $EI/m\ddot{y}_a a$, and the inclination of the extreme ends is $m\ddot{y}_a \frac{a}{2}(a-l)/EI$.

$$\frac{\ddot{y}_a}{y_a} = \frac{6EI}{ma^2(4a-3l)}.$$

§ 11. *Clamped-Clamped Bar with two Concentrated Loads.*—In the previous case, if the ends are clamped, we must subtract M_0 (*i. e.* the couple at each end required to fix the directions) from the expression for $\frac{d^2 y}{dx^2}$.

We find

$$M_0 = m \ddot{y}_a \frac{a}{l} (l-a),$$

$$EI y = m \ddot{y}_a \frac{x^2}{2l} \left(a^2 - la + \frac{lx}{3} \right),$$

$$EI y' = m \ddot{y}_a \frac{a^2}{2l} \left(x^2 - lx + \frac{la}{3} \right),$$

and

$$-\frac{\ddot{y}_a}{y_a} = \frac{6EI l}{ma^3(2l-3a)}.$$

§ 12. *Dynamical Method.*—Other methods for the solution of some of the problems of this section have been used by Dr. Chree, but they depend on the assumption of a curve of vibration.

A correct type having been assumed, expressions are obtained for the Kinetic and Potential Energies of the system. Employing these in Lagrange's Equations of Motion the frequency is readily obtained.

Taking, for example, a massless bar carrying a load m , as in § 8. Assume

$$\begin{aligned}y &= \eta bx(l^2 - b^2 - x^2), \\y' &= \eta ax'(l^2 - a^2 - x'^2),\end{aligned}$$

measuring x and x' from opposite ends of the bar.

The Kinetic Energy

$$T = \frac{1}{2}m\dot{y}_a^2 - 2ma^4b^4\eta^2.$$

The Potential Energy of Bending, V ,

$$\begin{aligned}&= \frac{1}{2}EI \left\{ \int_0^a \left(\frac{d^2y}{dx^2} \right)^2 dx + \int_0^b \left(\frac{d^2y'}{dx'^2} \right)^2 dx' \right\} \\&= 6EIa^2b^2(a+b)\eta^2.\end{aligned}$$

The Lagrangian Equation

$$\frac{d}{dt} \left(\frac{\partial T}{\partial \dot{\eta}} \right) - \frac{\partial T}{\partial \eta} + \frac{\partial V}{\partial \eta} = 0$$

becomes

$$4ma^4b^4\ddot{\eta} + 12EIa^2b^2(a+b)\eta = 0;$$

$$\therefore \frac{\ddot{\eta}}{\eta} = - \frac{3EI}{ma^2b^2}.$$

Section IV. Loaded Massive Bars.

§ 13. When the mass of the bar, in addition to that of the concentrated load, is taken into account, the expressions for the elastic curve and the frequency are more complicated. If, however, the position of the load and the ratio of the masses are given, the solutions are simple.

§ 14. *Clamped-Free Massive Bar, Load at End.*—In this case, if we assume

$$y = y_1 \left(1 - \frac{4}{3} \frac{x}{l} + \frac{1}{3} \frac{x^4}{l^4} \right),$$

the equation

$$-EI \frac{d^2 y}{dx^2} = \rho \omega \int_0^x \ddot{y}_z (x-z) dz + m x \ddot{y}_1$$

gives

$$\begin{aligned} -EI y = \rho \omega \ddot{y}_1 \left(.08194 l^4 - .11269841 l^3 x + .0416 x^4 - .01 \frac{x^5}{l} \right. \\ \left. + .00019841 \frac{x^8}{l^4} \right) + m \ddot{y}_1 (.16 x^3 - .5 l^2 x + .3 l^3) \\ - \frac{\ddot{y}_1}{y_1} = \frac{EI}{.08194 \rho \omega l^4 + .3 m l^3}. \end{aligned}$$

The next approximation gives

$$\begin{aligned} -EI y = \frac{\rho \omega \ddot{y}_1 10^{-3}}{.08194 \rho \omega l^4 + .3 m l^3} \left\{ \rho \omega \left(6.63084 l^8 - 9.12718 l^7 x \right. \right. \\ \left. \left. + 3.41435 l^4 x^4 - .93915 l^3 x^5 + .02480 x^8 - .00367 \frac{x^9}{l} \right. \right. \\ \left. \left. + .00002 \frac{x^{12}}{l^4} \right) + m (26.1905 l^7 - 36.1 l^6 x + 13.8 l^3 x^4 \right. \\ \left. - 4.16 l^2 x^5 + .1984 x^7) \right\} + m \ddot{y}_1 (.3 l^3 - .5 l^2 x + .16 x^3). \end{aligned}$$

Whence

$$- \frac{\ddot{y}_1}{y_1} = EI \left\{ \rho \omega l^4 \frac{.00663084 \rho \omega l + .0261905 m}{.08194 \rho \omega l + .3 m} + .3 m l^3 \right\}^{-1}$$

§ 15. *Supported Massive Bar, Load in any position.*—Let the load divide the bar into segments a and b . At each end

$$y = \frac{d^2 y}{dx^2} = 0;$$

and when $x=a$, $y=y_a$.

Hence, for the first approximation

$$y = \frac{y_a}{A} (l^3 x - 2 l x^3 + x^4),$$

where, for brevity,

$$A \equiv l^3 a - 2 l a^3 + a^4.$$

For $x < a$,

$$EI \frac{d^2 y}{dx^2} = \frac{b}{l} m \ddot{y}_a x + \frac{\rho \omega}{2} \frac{\ddot{y}_a}{y_a} x \int_0^l y_z dz - \rho \omega \frac{\ddot{y}_a}{y_a} \int_0^x (x-z) y_z dz.$$

When $x > a$ we must add $-m \ddot{y}_a (x-a)$ to the above expression. Determining the constants of integration by

equating the values for $\frac{dy}{dx}$ and y when $x=a$, we get for $x < a$,

$$y = \frac{\ddot{y}_a}{EI} \left\{ \frac{m}{6l} (lx^3 - ax^3 - a^3x - 2al^2x + 3a^2lx) + \frac{\rho\omega}{A} (-\cdot 01011905l^7x + \cdot 016\dot{l}^5x^3 - \cdot 0083\dot{l}^3x^5 + \cdot 00238095l^7x - \cdot 00059524x^8) \right\}.$$

Whilst for $x > a$ the first term in the brackets is

$$\frac{m}{6l} (la^3 - a^3x - ax^3 - 2al^2x + 3alx^2).$$

In either case

$$\frac{\ddot{y}_a}{y_a} = EI \cdot P;$$

where

$$P^{-1} = \frac{ma^2b^2}{3l} + \frac{\rho\omega}{A} (-\cdot 01011905l^7a + \cdot 016\dot{l}^5a^3 - \cdot 0083\dot{l}^3a^5 + \cdot 00238095la^7 - \cdot 00059524a^8).$$

§ 16. In the second approximation, for $x < a$

$$EI \frac{d^2y}{dx^2} = \frac{b}{l} m\ddot{y}_ax - \rho\omega \frac{\ddot{y}_a}{y_a} \int_0^x (x-z)y_z dz + \rho\omega \frac{\ddot{y}_a}{y_a} \frac{x}{l} \left[\int_0^a y(l-x)dx + \int_a^l y'(l-x)dx \right].$$

If $x > a$, the second expression on the right hand must be written

$$-\rho\omega \frac{\ddot{y}_a}{y_a} \left[\int_0^a (x-z)y_z dz + \int_a^x (x-z)y'_z dz \right],$$

and the term $-m\ddot{y}_a(x-a)$ must be added.

By the method already indicated we find the inclinations of the ends to be

at $x=0$:

$$\frac{dy}{dx} = \frac{\ddot{y}_a}{EI} \left[\frac{m}{l} \left(\frac{a^2l}{2} - \frac{a^3}{6} - \frac{al^2}{3} \right) + \frac{\rho\omega P}{A} \left\{ \cdot 00010386\rho\omega l^{11} + \frac{mA}{6l} \left(-\frac{17a^7}{420} + \frac{la}{30} + \frac{a^5l^2}{60} - \frac{a^3l^4}{45} + \frac{4al^6}{315} \right) \right\} \right];$$

at $x=l$:

$$\frac{dy'}{dx} = \frac{\ddot{y}_a}{EI} \left[\frac{m}{l} \left(-\frac{a^3}{6} - \frac{al^2}{3} \right) + \frac{\rho\omega P}{A} \left\{ \cdot 00010386\rho\omega l^{11} + \frac{mA}{6l} \left(-\frac{17a^7}{420} + \frac{a^5l^2}{60} - \frac{a^3l^4}{45} + \frac{4al^6}{315} \right) \right\} \right]$$

and the equations of the central line

$$y = \frac{\ddot{y}_a}{EI} \left[\frac{mbx}{6l} (x^2 - 2la + a^2) - \frac{\rho\omega P}{A} \left\{ \rho\omega 10^{-4} (-1.0386l^{11}x + 1.7085l^9x^3 - .8433l^7x^5 + .1984l^5x^7 - .0276l^3x^9 + .0030lx^{11} - .0005x^{12}) \right. \right. \\ \left. \left. + \frac{mA}{6l} \left[(l-a) \frac{x^7}{840} - \left(\frac{a^2}{6} - \frac{al}{2} + \frac{l^2}{3} \right) \frac{ax^5}{20} - \left(\frac{a^4}{20} - \frac{a^3l}{4} + \frac{a^2l^2}{3} - \frac{2l^4}{15} \right) \frac{ax^3}{6} - \left(\frac{la^6}{30} - \frac{17a^7}{420} + \frac{a^5l^2}{60} - \frac{a^3l^4}{45} + \frac{4al^6}{315} \right) x \right] \right\} \right].$$

$$y' = \frac{\ddot{y}_a}{EI} \left[\frac{ma}{6l} (l-x)(a^2 - 2lx + x^2) - \frac{\rho\omega P}{A} \left\{ \rho\omega 10^{-4} (-1.0386l^{11}x + 1.7085l^9x^3 - .8433l^7x^5 + .1984l^5x^7 - .0276l^3x^9 + .0030lx^{11} - .0005x^{12}) \right. \right. \\ \left. \left. + \frac{mA}{6l} \left[-\frac{ax^7}{840} + \frac{alx^6}{120} - \left(\frac{a^2}{6} + \frac{l^2}{3} \right) \frac{ax^5}{20} + \frac{a^3lx^4}{24} - \left(\frac{a^4}{20} + \frac{a^2l^2}{3} - \frac{2l^4}{15} \right) \frac{ax^3}{6} + \frac{a^5lx^2}{40} + \left(\frac{17}{420}a^7 - \frac{a^5l^2}{60} + \frac{a^3l^4}{45} - \frac{4al^6}{315} \right) x - \frac{17}{420}a^7l \right] \right\} \right].$$

Putting $x=a$ in either of these, we get the following relation between y_a and $\ddot{y}_a$:

$$-\left(\ddot{y}_a\right)^{-1} = \frac{1}{EI} \left[\frac{ma^2b^2}{3l} + \rho\omega P \left\{ (-1.0386l^{11}a + 1.7085l^9a^3 - .8433l^7a^5 + .1984l^5a^7 - .0276l^3a^9 + .0030la^{11} - .0005a^{12}) \frac{\rho\omega 10^{-4}}{A} \right. \right. \\ \left. \left. + \frac{ma^2}{90l} (.3392857a^6 + .5178572a^5l - 1.3a^4l^2 + .6a^2l^4 - .1904762l^6) \right\} \right].$$

§ 17. In particular, if $a = \frac{l}{2}$,

$$A = \frac{5}{16}l^4, \quad \text{and} \quad P^{-1} = -\frac{ml^3}{48} - .0103051\rho\omega l^4;$$

hence

$$-\frac{\ddot{y}_a}{y_a} = EI \left[\frac{ml^3}{48} + \frac{3.306\rho\omega l + 7.436m}{322.038\rho\omega l + 651.042m} \rho\omega l^4 \right]^{-1};$$

and further, if the masses of the load and bar be equal,

$$-\frac{\ddot{y}_a}{y_a} = \frac{31.375 EI}{ml^3}.$$

§ 18. *Clamped-Clamped Massive Bar, Load in any position.*

The end conditions lead to the preliminary assumption

$$y = y_a \left(\frac{l}{ab} \right)^2 \left(x^2 - \frac{2x^3}{l} + \frac{x^4}{l^2} \right).$$

For $x < a$,

$$-EI \frac{d^2 y}{dx^2} = M_0 + (M_1 - M_0) \frac{x}{l} + \rho \omega \frac{\ddot{y}_a}{y_a} \int_0^x (x-z) y_z dz \\ - \frac{\rho \omega x}{2} \frac{\ddot{y}_a}{y_a} \int_0^l y_x dx - m \ddot{y}_a \frac{bx}{l}.$$

If $x > a$, add $m \ddot{y}_a (x-a)$.

By putting $x=a$, as in previous cases, we find

$$M_0 = m \ddot{y}_a \frac{ab^3}{l^2} + .00357143 \frac{\rho \omega \ddot{y}_a l^6}{a^2 b^2},$$

$$M_1 = m \ddot{y}_a \frac{a^2 b}{l^2} + .00357143 \frac{\rho \omega \ddot{y}_a l^6}{a^2 b^2};$$

and hence for $x < a$,

$$-EI y = \rho \omega \ddot{y}_a \left(\frac{l}{ab} \right)^2 \left(2.7 x^6 - 2.38095 \frac{x^7}{l} + .59524 \frac{x^8}{l^2} - 2.7 l^3 x^3 \right. \\ \left. + 1.78572 l^4 x^2 \right) 10^{-3} + m \ddot{y}_a \frac{bx^2}{6 l^3} (3abl - l^2 x - abx + a^2 x);$$

and for $x > a$ the latter portion of the expression is changed to

$$m \ddot{y}_a \frac{a^2}{6 l^3} (3lx^3 - 2ax^3 - 6l^2 x^2 + 3alx^2 + 3l^3 x - al^3).$$

When $x=a$, either of the above gives

$$-\frac{\ddot{y}_a}{y_a} = EI \cdot P,$$

where

$$P^{-1} = \frac{\rho \omega}{b^2} 10^{-3} (1.78572 l^6 - 2.7 al^5 + 2.7 a^4 l^2 - 2.38095 a^5 l \\ + .59524 a^6) + \frac{m}{3} \left(\frac{ab}{l} \right)^3.$$

§ 19. In the next approximation, for $x < a$:

$$EI \frac{d^2 y}{dx^2} = M_0 + (M_1 - M_0) \frac{x}{l} + \rho \omega \frac{\ddot{y}_a}{y_a} \int_0^x (x-z) y_z dz - \frac{\rho \omega \ddot{y}_a}{l} \frac{y_a}{y_a} x \int_0^a y_x (l-x) dx \\ - \frac{\rho \omega \ddot{y}_a}{l} \frac{y_a}{y_a} x \int_0^l y'_x (l-x) dx - \frac{m \ddot{y}_a bx}{l}; \\ = M_0 + (M_1 - M_0) \frac{x}{l} + \rho \omega P \ddot{y}_a \left[\frac{\rho \omega l^2 10^{-4}}{a^2 b^2} (-.33068 l^7 x + 1.48809 l^4 x^4 \right. \\ \left. - 1.38 l^3 x^5 + .49603 x^8 - .33068 \frac{x^9}{l} + .06614 \frac{x^{10}}{l^2}) \right. \\ \left. + \frac{m}{6 l^3} \left\{ \left(\frac{3}{20} a^2 l - \frac{l^3}{20} - \frac{a^3}{10} \right) x^5 + \left(\frac{al^3}{4} - \frac{a^2 l^2}{2} + \frac{a^3 l}{4} \right) x^4 \right. \right. \\ \left. \left. - \left(\frac{3}{20} a^2 l^5 - \frac{7}{20} a^3 l^4 + \frac{a^4 l^3}{4} - \frac{a^5 l^2}{20} \right) x \right\} \right] - m \ddot{y}_a \frac{bx}{l}.$$

Similarly for $x > a$:

$$\begin{aligned}
 -EI \frac{d^2 y'}{dx^2} &= M_0 + (M_1 - M_0) \frac{x}{l} + \rho \omega \frac{\ddot{y}_a}{y_a} \left[\int_0^a (x-z) y_z dz + \int_a^x (x-z) y_z' dz \right] \\
 &\quad - \rho \omega \frac{\ddot{y}_a}{y_a} \frac{x}{l} \left[\int_0^a y_x (l-x) dx + \int_a^l y_x' (l-x) dx \right] - m \ddot{y}_a \left(\frac{bx}{l} - x + a \right); \\
 &= M_0 + (M_1 - M_0) \frac{x}{l} + \rho \omega P \ddot{y}_a \left[\frac{\rho \omega l^2}{a^2 b^2} 10^{-4} \left(-\cdot 33068 l^7 x \right. \right. \\
 &\quad \left. \left. + 1\cdot 48809 l^4 x^4 - 1\cdot 38 l^3 x^5 + \cdot 49603 x^8 - \cdot 33068 \frac{x^9}{l} + \cdot 06614 \frac{x^{10}}{l^2} \right) \right. \\
 &\quad \left. + \frac{m}{6 l^3} \left\{ \left(\frac{3}{20} a^2 l - \frac{a^3}{10} \right) x^5 + \left(\frac{a^5 l}{4} - \frac{a^2 l^2}{2} \right) x^4 + \frac{a^2 l^3 x^3}{2} - \frac{a^3 l^2 x^2}{2} \right. \right. \\
 &\quad \left. \left. - \left(\frac{3}{20} l^3 - \frac{7}{20} a l^2 - \frac{a^3}{20} \right) a^2 l^2 x - \frac{a^5 l^3}{20} \right\} \right] - m \ddot{y}_a \left(\frac{bx}{l} - x + a \right).
 \end{aligned}$$

Integrating and equating when $x=a$, we find

$$\begin{aligned}
 M_0 &= \rho \omega P \ddot{y}_a \left[\cdot 0000071145 \frac{\rho \omega l^{10}}{a^2 b^2} + \frac{m}{6 l^3} \left(\frac{a^7 l}{140} - \frac{a^6 l^2}{30} + \frac{a^5 l^3}{20} \right. \right. \\
 &\quad \left. \left. - \frac{11}{210} a^3 l^5 + \frac{a^2 l^6}{35} \right) \right] + m \ddot{y}_a \frac{a b^2}{l^2}; \\
 M_1 &= \rho \omega P \ddot{y}_a \left[\cdot 0000071145 \frac{\rho \omega l^{10}}{a^2 b^2} + \frac{m}{6 l^3} \left(-\frac{a^7 l}{140} + \frac{a^6 l^2}{60} - \frac{13}{420} a^3 l^5 \right. \right. \\
 &\quad \left. \left. + \frac{3}{140} a^2 l^6 \right) \right] + m \ddot{y}_a \frac{a^2 b}{l^2}.
 \end{aligned}$$

Whence the form of the centre-line is

$$\begin{aligned}
 -EI y &= \rho \omega P \ddot{y}_a \left[\rho \omega \left(\frac{l}{ab} \right)^2 10^{-6} \left(3\cdot 5573 l^8 x^2 - 5\cdot 5113 l^7 x^3 \right. \right. \\
 &\quad \left. \left. + 4\cdot 9603 l^4 x^6 - 3\cdot 3069 l^3 x^7 + \cdot 5511 x^{10} - \cdot 3006 \frac{x^{11}}{l} + \cdot 0501 \frac{x^{12}}{l^2} \right) \right. \\
 &\quad \left. + \frac{m}{6 l^3} \left\{ (3a^2 l - l^3 - 2a^3) \frac{x^7}{840} + (a l^3 - 2a^2 l^2 + a^3 l) \frac{x^6}{120} + (7a^6 l - 2a^7 \right. \right. \\
 &\quad \left. \left. - 35a^4 l^3 + 52a^3 l^4 - 22a^2 l^5) \frac{x^3}{840} + (3a^7 l - 14a^6 l^2 + 21a^5 l^3 - 22a^3 l^5 \right. \right. \\
 &\quad \left. \left. + 12a^2 l^6) \frac{x^2}{840} \right\} \right] + m \ddot{y}_a \left\{ \frac{a b^2}{2} \frac{x^2}{l^2} - (2a + l) \frac{b^2}{6} \frac{x^3}{l^3} \right\}
 \end{aligned}$$

and

$$\begin{aligned}
 -EIy' = \rho\omega Pj_a \left[\rho\omega \left(\frac{l}{ab} \right)^2 10^{-6} \left(3\cdot5573l^8x^2 + 5\cdot5113l^7x^3 + 4\cdot9603l^6x^4 \right. \right. \\
 \left. \left. - 3\cdot3069l^5x^7 + 5\cdot511x^{10} - 3\cdot006 \frac{x^{11}}{l} + 0\cdot501 \frac{x^{12}}{l^2} + \frac{m}{6l^3} \left\{ (3a^2l - 2a^3) \frac{x^7}{840} \right. \right. \right. \\
 \left. \left. + (a^3l - 2a^2l^2) \frac{x^6}{120} + \frac{a^2l^3x^5}{40} - \frac{a^3l^3x^4}{24} + (52a^3l^4 - 22a^2l^5 - 2a^7 + 7a^6l) \frac{x^3}{840} \right. \right. \\
 \left. \left. + (3a^7l - 14a^6l^2 - 22a^3l^5 + 12a^2l^6) \frac{x^2}{840} + \frac{a^6l^3x}{120} - \frac{a^7l^3}{840} \right\} \right] \\
 \left. + \frac{m\ddot{y}_a}{6l^3} \{ (3a^2l - 2a^3)x^3 + (3a^3l - 6a^2l^2)x^2 + 3a^2l^3x - a^3l^3 \} \right].
 \end{aligned}$$

When $x=a$ these give

$$\begin{aligned}
 -\left(\frac{\ddot{y}_a}{y_a} \right)^{-1} = \frac{\rho\omega P}{EI} \left[\rho\omega \frac{l^2}{l^2} 10^{-6} \left(3\cdot5573l^8 - 5\cdot5113l^7a + 4\cdot9603l^6a^4 \right. \right. \\
 \left. \left. - 3\cdot3069l^5a^5 + 5\cdot511a^8 - 3\cdot006 \frac{a^9}{l} + 0\cdot501 \frac{a^{10}}{l^2} \right) \right. \\
 \left. + \frac{ma^4}{1260l^3} (3l^6 - 11l^5a + 13l^4a^2 - 2l^3a^3 - 7l^2a^4 + 5la^5 - a^6) \right] \\
 + \frac{a^3l^3}{3l^3} \frac{m}{EI}.
 \end{aligned}$$

§ 20. As a particular case, if the load be at the centre

$$P^{-1} = \cdot0020213\rho\omega l^4 + \cdot0052083ml^3$$

and

$$-\frac{\ddot{y}_a}{y_a} = EI \left[\frac{ml^3}{192} + \frac{4\cdot0398\rho\omega l + 10\cdot076m}{2021\cdot3\rho\omega l + 5208\cdot3m} \rho\omega l^4 \right]^{-1}.$$

If the masses of load and bar be equal we have a further simplification to

$$-\frac{\ddot{y}_a}{y_a} = \frac{139\cdot65 EI}{ml^3}.$$

Section V. Practical Formulæ for Loaded Bars.

§ 21. The formulæ obtained in the last section for the frequency of the lateral vibrations of loaded bars of appreciable mass are too cumbersome for general use. In every case, however, given the position and mass of the concentrated load and the mass of the bar (assumed uniform throughout

its length), the frequency is given by a formula of the type

$$N = \frac{1}{2\pi} \sqrt{-\frac{\ddot{y}_a}{y_a}},$$

and the ratio $\ddot{y}_a/y_a$ is given by

$$-\frac{\ddot{y}_a}{y_a} = \frac{\beta EI}{\rho \omega l^4}.$$

In the following paragraphs the values of β are given for all positions of the load and for ratios of mass of load to mass of bar varying from zero to 1.0.

§ 22. *Bar supported at both ends.*—In this case the values of β have been obtained from the last equation of § 16. They are given in Table I. and represented graphically in figure 1. If the ratio of mass of load to that of bar cannot be expressed

exactly in tenths, further interpolation is required for accuracy. This may easily be obtained graphically by plotting the values of β given by the curves for the required position of the load.

In the tables l is the effective length of the bar, and a the distance of the centre of gravity of the load from either end. The numbers on the curves represent the ratio of mass of load to that of effective length of bar.

TABLE I.
Values of β for Bars Supported at Each End.

Ratio.	Values of β for different positions of Load.				
Mass of Load. Mass of Bar.	$a/l=.1.$	$a/l=.2.$	$a/l=.3.$	$a/l=.4.$	$a/l=.5.$
0.1	95.56	90.98	85.91	82.03	79.80
0.2	93.74	85.20	76.71	70.85	67.73
0.3	91.95	80.02	69.22	62.35	58.99
0.4	90.21	75.36	63.01	55.68	52.31
0.5	88.69	71.17	57.81	50.30	47.03
0.6	86.86	67.39	53.91	45.86	42.74
0.7	85.24	63.96	49.56	42.15	39.18
0.8	83.67	60.84	46.25	38.99	36.17
0.9	82.15	57.99	43.35	36.27	33.60
1.0	80.66	55.39	40.78	33.91	31.38

§ 23. *Bar Clamped at Both Ends.*—The values of β are here found from the last equation of § 19. They are given in Table II. and fig. 2.

TABLE II.
Values of β for Clamped-Clamped Bar.

Ratio.	Values of β for different positions of Load.				
Mass of Load Mass of Bar.	$a/l=.1.$	$a/l=.2.$	$a/l=.3.$	$a/l=.4.$	$a/l=.5.$
0.1	499.9	480.9	443.9	411.2	399.0
0.2	497.9	460.8	396.7	347.9	331.3
0.3	495.9	441.3	357.4	301.1	283.0
0.4	493.9	422.6	324.6	265.2	247.0
0.5	491.8	404.7	296.9	236.7	219.0
0.6	489.6	387.3	273.3	213.8	196.7
0.7	487.3	371.9	253.0	194.8	178.5
0.8	484.9	357.0	235.4	178.9	163.3
0.9	482.6	342.9	217.8	165.4	150.6
1.0	480.1	329.8	206.4	153.7	139.7

Section VI. Correction for Rotatory Inertia.

§ 24. Hitherto the terms depending on the angular motion, both of the concentrated masses and of the sections of the bars themselves, have been neglected. This is equivalent to supposing the inertia of each element to be concentrated at its centre.

The effect of the angular acceleration of the bar is sufficiently small to be neglected in practice, but in the case of a concentrated mass the additional terms may be important.

There are various methods of procedure. An important approximate one being to assume in the first place that the type of vibration is unaltered by rotatory inertia and then to find the correction due to this cause.

A general method has been given by Lord Rayleigh. It is applied to a special case in the next paragraph, in which

I' = Moment of Inertia of concentrated mass about an axis perpendicular to the plane of bending ($=mk'^2$).
 T and V are the Kinetic and Potential Energies of the system.

§ 25. For a Fixed-Free Massless Bar (*cf.* Rayleigh's 'Sound,' Art. 183). Origin at fixed end and mass m at free end, the equation of the bar is

$$y = (3y_1 - l\theta)\left(\frac{x}{l}\right)^2 + (l\theta - 2y_1)\left(\frac{x}{l}\right)^3,$$

where y_1 and θ are the values of y and $\frac{dy}{dx}$ at the point where the load is attached. This equation is deduced from

$$EI \frac{d^2y}{dx^2} = m\ddot{y}_1(l-x) + I'\ddot{\theta}.$$

Integrating, and determining $\ddot{y}_1$ and $\ddot{\theta}$ from

$$\left. \begin{aligned} EI y_1 &= m\ddot{y}_1 \frac{l^3}{3} + I'\ddot{\theta} \frac{l^2}{2} \\ EI \theta &= m\ddot{y}_1 \frac{l^2}{2} + I'\ddot{\theta} l \end{aligned} \right\}$$

we get the desired result.

Following Rayleigh's solution, the equations of motion are

$$\left. \begin{aligned} m\ddot{y}_1 + \frac{2EI}{l^3} (6y_1 - 3l\theta) &= 0 \\ I'\ddot{\theta} + \frac{2EI}{l^3} (-3ly_1 + 2l^2\theta) &= 0 \end{aligned} \right\}$$

whence

$$-\frac{\ddot{y}}{y_1} = \frac{2EI}{I'} \left\{ 1 + \frac{3k'^2}{l^2} \pm \sqrt{1 + \frac{3k'^2}{l^2} + \frac{9k'^4}{l^4}} \right\},$$

answering to the two different periods.

§ 26. The solution can be simplified if we assume to start with that the effect of I' is small. This is usually the case in practice, and the method has been fully investigated by Dr. Chree.

Adopting this simpler method (*cf.* Chree, *l.c.* p. 511, § 9)

the vibration curve in § 7 can be written

$$y = y_1 \left(\frac{x^3}{2l^3} - \frac{3x}{l} + 1 \right),$$

and we can take

$$\theta = \left[\frac{dy}{dx} \right]_{x=0} = - \frac{3y_1}{2l}.$$

Then, since

$$T = \frac{1}{2} m \dot{y}_1^2 + \frac{1}{2} I' \dot{\theta}^2 = \frac{1}{2} m \dot{y}_1^2 + \frac{9}{8} \frac{I'}{l^2} y_1^2,$$

and

$$V = \frac{1}{2} EI \int_0^l \left(\frac{d^2 y}{dx^2} \right)^2 dx = \frac{3}{2} \frac{EI}{l^3} y_1^2,$$

the equation of motion

$$\frac{d}{dt} \frac{\partial T}{\partial \dot{y}_1} - \frac{\partial T}{\partial y_1} + \frac{\partial V}{\partial y_1} = 0$$

becomes

$$m \ddot{y}_1 + \frac{9}{4} \frac{I'}{l^2} \dot{y}_1 + \frac{3EI}{l^3} y_1 = 0;$$

$$\therefore -\frac{\ddot{y}_1}{y_1} = \frac{3EI}{m l^3 \left(1 + \frac{9}{4} \frac{k'^2}{l^2} \right)}.$$

§ 27. The same problems worked by the methods of this paper do not involve Lagrange's Equations and are as follows:—

Taking the origin at the free end,

$$-EI \frac{d^2 y}{dx^2} = m \ddot{y}_1 x - I' \ddot{\theta}.$$

Whence

$$\begin{aligned} -EI y_1 &= m \dot{y}_1 \frac{l^3}{3} - I' \ddot{\theta} \frac{l^2}{2} \\ -EI \theta &= -m \ddot{y}_1 \frac{l^2}{2} + I' \ddot{\theta} l \end{aligned}$$

Putting $\frac{\ddot{\theta}}{\theta} = \frac{\ddot{y}_1}{y_1}$ we get

$$\frac{m l^2}{6} \left(\frac{\ddot{y}_1}{y_1} \right)^2 + 2 \left(\frac{1}{l} + \frac{m l}{3 I'} \right) EI \frac{\ddot{y}_1}{y_1} = - \frac{2}{l'} \left(\frac{EI}{l} \right)^2;$$

$$\therefore -\frac{\ddot{y}_1}{y_1} = \frac{2EI}{I' l} \left(1 + \frac{3I'}{m l^2} \pm \sqrt{1 + \frac{3I'}{m l^2} + \frac{9I'^2}{m^2 l^4}} \right).$$

Or, for an approximate solution, starting with

$$-EI \frac{d^2 y}{dx^2} = m \ddot{y} \quad x - l' \ddot{\theta},$$

we may put

$$\theta = -\frac{3}{2} \frac{y_1}{l}.$$

Whence

$$-EI y = \ddot{y}_1 (x-l)^2 \left\{ \frac{m}{6} (x+2l) + \frac{3I'}{4l} \right\},$$

and

$$-\frac{\ddot{y}_1}{y_1} = \frac{3EI}{m l^3 \left(1 + \frac{9}{4} \frac{k'^2}{l^2} \right)}.$$

§ 28. When the mass of the bar is taken into account, and Rayleigh's method is used, the kinetic energy is

$$T = \frac{1}{2} \left\{ m \dot{y}_1^2 + I' \left[\frac{d^2 y}{dx dt} \right]^2 + \int \rho \omega \dot{y}^2 dx + \int \rho \omega k^2 \left(\frac{d^2 y}{dx dt} \right)^2 dx \right\},$$

where the part in square brackets is to be taken for the position of the load, and those under the integral signs are between the limits. The potential energy of bending is, as before,

$$V = \frac{1}{2} \int EI \left(\frac{d^2 y}{dx^2} \right) dx.$$

In general the evaluation of these, though perfectly simple, is tedious. The values of y and $\frac{dy}{dx}$ are taken from earlier parts of this paper, and the energy expressions then used in the Lagrangian Equation just as in the example already given.

§ 29. It is, however, simpler merely to add the terms due to Rotatory Inertia to the previous solutions. A general investigation of these terms would occupy too much space, but in a numerical example the work is not difficult.

Referring to § 14 for the case of a loaded massive bar, the term to be added to the value of $-EI \frac{d^2 y}{dx^2}$ for the rotatory inertia of m is

$$-I \frac{d^2}{dt^2} \left(\frac{dy}{dx} \right)_{x=0};$$

and for an approximate solution we can write

$$\left(\frac{dy}{dx} \right)_{x=0} = -\frac{4}{3} \frac{y_1}{l}.$$

Hence the term to be added to $-EI\gamma$ is

$$\frac{4}{3} \frac{I'}{l} \left(\frac{x^2}{2} - lx + \frac{l^2}{2} \right);$$

and for a first approximation,

$$-\frac{\ddot{\gamma}_1}{\gamma_1} = \frac{EI}{\cdot 08194 \rho \omega l^4 + \cdot 3 ml^3 + \cdot 6 I'l}.$$

This example is sufficient to indicate the procedure to be adopted in all such cases.

University College, Bristol,
October 1905.

XXIX. On the Range of Stokes's Deep-Water Waves.

By R. F. GWYTHER*.

THE waves with which I propose to deal are the waves of finite amplitude of which the investigation was initiated by Sir George Stokes† in his paper on the Theory of Oscillatory Waves, and continued in the Supplement to that paper. The object of the paper is to establish the correctness of the opinion expressed in the paper quoted (p. 227) "After careful consideration I feel satisfied ... that we may approach as near as we please to the form in which the curvature at the vertex becomes infinite, and the vertex becomes a multiple point where the two branches ... enclose an angle of 120° ." The method of the Supplement is adhered to as closely as is convenient. The result is to establish that the velocity in all waves of the series, small as well as great, is represented by a function which possesses poles which as the amplitude increases approach nearer to the fluid surface, and in the limiting form is identical with that investigated by Mr. Michell‡ in his paper on the "Highest Wave in Water," in which the poles lie on the water-surface.

§ 1. Making a change in the notation used by Stokes in the Supplement, in order to make the analytical form of the velocity more obvious, I write

$$x + iy = \frac{\phi + i\psi}{c} - i \sum \frac{h_n}{nk} e^{\frac{ink(\phi + i\psi)}{c}},$$

where n is an integer, and k is a constant defining the wavelength. Also let $\psi=0$ define the free surface of the water.

* Communicated by the Author.

† Mathematical and Physical Papers, vol. i. p. 197, and Supplement, p. 314.

‡ Phil. Mag. November 1893.