THE EQUATIONS OF THE THEORY OF ELECTRONS TRANSFORMED RELATIVE TO A SYSTEM IN ACCELERATED MOTION

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THE possibility of the exact transformation of the æthereal equations of electrodynamics from axes moving with a constant velocity to axes at rest was first proved by Larmor.* The extension to the case of accelerated motion has been rendered possible by recent investigations of Cunningham and Bateman, to the latter of whom is due the general theory of the transformation. So far the only rigorous mathematical investigation of the accelerated motion of a deformable electron, *i.e.*, an electron whose shape depends on its motion, has been given by Born,† who deals with a special case (*Hyperbelbewegungen*) of the motion discussed below. In the following paper the form of the transformation for a certain class of accelerated motions is investigated; but the applications are, for the most part, confined to the case of quasi-stationary motion, *i.e.*, motion which is very nearly uniform and in a straight line.

This is of considerable importance, as it enables us to verify and extend results obtained from the theory of quasi-stationary motion as developed by Abraham.‡

This latter theory is unsatisfactory, inasmuch as all the quantities involved are calculated for the uniform motion, and then differentiated with respect to the time in order to obtain their rates of increase for the accelerated motion. Further, from this theory Abraham has deduced the necessity of postulating for a deformable electron an energy other than electromagnetic, its value depending on the velocity of the electron.

* Larmor, *Æther and Matter*, Chap. xi.

† M. Born, "Die Theorie des Starren Elektrons in der Kinematik des Relativitätsprinzips," *Annalen der Physik*, Vol. 30, 1909.

‡ Abraham, "Die Dynamik des Elektrons," *Ann. der Physik*, 1903; *Theorie der Elektrizität*, Vol. II, 1st ed., pp. 136-214.

The necessity of introducing this extra energy has been admitted by Lorentz* and Poincaré, and is considered by the former to be an important objection to the hypothesis of a deformable electron.

The constants of the general transformation which have a kinematical interpretation include a constant δ , representing a uniform dilatation round the origin,† besides those interpretable in terms of the velocity and acceleration of the centre of the electron.

The exact theory of the transformation enables us to obtain for quasi-stationary motion the values of the electromagnetic masses, longitudinal and transverse, of a moving electron.

When δ is zero these values agree with those given by Lorentz for the two cases of volume and surface distribution of charge, an electron at rest being assumed to be a sphere and its charge to be distributed uniformly over its volume and surface respectively.

When the constant δ is so determined that $\delta + ua$ is zero, where u is the velocity of the centre of the electron and a proportional to its acceleration in the direction of motion, then we have a special case where there is no distinction between the longitudinal and transverse mass, and, further, for the general motion, the accurate values of the forces due to the reaction of the æther can be obtained.

Again, according to the transformation theory, it is shown that there is no need to assume the existence of an energy other than electromagnetic, the equation of energy being of necessity satisfied when the electron is in motion.

The type of motion considered in this paper is characterized by the fact that the path of the centre of the electron is always a conic, which cannot be either an ellipse or a circle. It is, however, the simplest case of accelerated motion of a deformable electron for which the electromagnetic disturbance can be accurately calculated, and as such is of considerable importance.

It must be noticed that this paper does not deal with any question involving the "equations of motion" of an electron. The motion is determined beforehand solely by the transformation, and the forces acting on the electron are then calculated.

The converse investigation, as to how an electron moves under the action of a given force, constant or otherwise, is not attempted in this paper. As at present developed, the transformation theory, or Theory of Relativity, does not appear to be sufficiently complete to deal with this

* Lorentz, *The Theory of Electrons*, Leipzig, 1909, p. 213.

† Used by Lorentz, *Amsterdam Proceedings*, 1901.

question, which has, for rigid electrons, been treated with various assumptions by G. W. Walker, Schott, and others.

Nor is there any necessary inconsistency between the results of this paper and those obtained by Schott* for deformable electrons, inasmuch as the assumptions lying as the basis of the work are in many cases quite different.

In fact, the results of any electron theory as at present developed depend, to a very large extent, on the assumptions, mathematical as well as physical, made in the course of the work, and these it is not always easy to detect.

In this paper it is assumed (i) that the Principle of Relativity, or (mathematically speaking) the transformation theory of the electrodynamical equations, holds for accelerated motion as well as for uniform motion†; (ii) that an electron *at rest* is a sphere with a uniform volume or surface distribution of charge, leaving out of account the question as to how this charge can exist; (iii) that the equations of the electromagnetic field are those used by Larmor and Lorentz; (iv) that the force per unit volume due to the reaction of the æther is given by the expression $\rho(E + [v, H])$, where the symbols are defined in § 1.

1. *General Theory of the Transformation.*

It has been shown by Hargreaves‡ that the equations of the theory of electrons can be expressed in the form of integrals in which use is made of the generalized form of Stokes' Theorem for four dimensions.

The fundamental equations of the theory of electrons are, using the vector notation,

$$\left. \begin{aligned} \text{curl } H &= \frac{\partial E}{\partial t} + \rho v \\ \text{curl } E &= -\frac{\partial H}{\partial t} \\ \text{div } E &= \rho, \quad \text{div } H = 0 \end{aligned} \right\}, \quad (1)$$

* G. A. Schott, "Electromagnetic Radiation," Appendixes F and G, and *passim*.

† From a mathematical point of view this appears to be absolutely necessary, since it is the variable motion of an electron which is of interest.

‡ "Integral Forms and their connexion with Physical Equations," *Camb. Phil. Trans.*, Vol. 21, 1908, p. 107.

whence it follows that $\frac{\partial \rho}{\partial t} + \text{div } \rho w = 0$,

which is the equation of continuity.

The electric units are those adopted by Lorentz, while the units of length and time have been so chosen that the velocity of radiation shall be unity.

We take the axes to form a right-handed system, so that curl A represents a vector whose components are

$$\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}, \quad \dots, \quad \dots$$

Also H is the magnetic force, E the electric force, ρ the volume density of electricity, and ρw the convection current.

These equations may be replaced by the two integral forms

$$\begin{aligned} \iint (H_x dy dz + H_y dz dx + H_z dx dy + E_x dx dt + E_y dy dt + E_z dz dt) &= 0, \\ \iint (E_x dy dz + E_y dz dx + E_z dx dy - H_x dx dt - H_y dy dt - H_z dz dt) \\ &= \iiint (\rho dx dy dz - \rho w_x dy dz dt - \rho w_y dz dx dt - \rho w_z dx dy dt), \end{aligned}$$

a result which follows by the rule given by Hargreaves, the equation of continuity

$$\frac{\partial \rho}{\partial t} + \text{div } (\rho w) = 0$$

being the condition that the above triple integral can be expressed as a double integral.

These forms of the equations are specially suitable for the theory of transformation, which is limited by the condition that the total charge on the system of particles remains constant.*

If it is possible under this limitation to transform the equations so as to leave them in the same form when referred to another system of axes x_1, y_1, z_1, t_1 , then we must have

$$E_x dy dz + \dots - H_x dx dt - \dots = E_{1x} dy_1 dz_1 + \dots - H_{1x} dx_1 dt_1 - \dots,$$

and $H_x dy dz + \dots + E_x dx dt + \dots$

$$= \theta [H_{1x} dy_1 dz_1 + \dots + E_{1x} dx_1 dt_1 + \dots],$$

where θ is a constant.

* Bateman, "The Transformation of the Electrodynamical Equations," *Proc. London Math. Soc.*, Ser. 2, Vol. 8, 1910.

These two relations give two sets of equations connecting the vectors E , H with the vectors E_1 , H_1 , viz.,

$$\begin{aligned} E_x &= E_{1x} \frac{\partial(y_1, z_1)}{\partial(y, z)} + E_{1y} \frac{\partial(z_1, x_1)}{\partial(y, z)} + E_{1z} \frac{\partial(x_1, y_1)}{\partial(y, z)} \\ &\quad - H_{1x} \frac{\partial(x_1, t_1)}{\partial(y, z)} - H_{1y} \frac{\partial(y_1, t_1)}{\partial(y, z)} - H_{1z} \frac{\partial(z_1, t_1)}{\partial(y, z)}, \\ -H_x &= E_{1x} \frac{\partial(y_1, z_1)}{\partial(x, t)} + E_{1y} \frac{\partial(z_1, x_1)}{\partial(x, t)} + E_{1z} \frac{\partial(x_1, y_1)}{\partial(x, t)} \\ &\quad - H_{1x} \frac{\partial(x_1, t_1)}{\partial(x, t)} - H_{1y} \frac{\partial(y_1, t_1)}{\partial(x, t)} - H_{1z} \frac{\partial(z_1, t_1)}{\partial(x, t)}, \end{aligned}$$

$$\begin{aligned} \text{and } H_x &= \theta \left[H_{1x} \frac{\partial(y_1, z_1)}{\partial(y, z)} + H_{1y} \frac{\partial(z_1, x_1)}{\partial(y, z)} + H_{1z} \frac{\partial(x_1, y_1)}{\partial(y, z)} \right. \\ &\quad \left. + E_{1x} \frac{\partial(x_1, t_1)}{\partial(y, z)} + E_{1y} \frac{\partial(y_1, t_1)}{\partial(y, z)} + E_{1z} \frac{\partial(z_1, t_1)}{\partial(y, z)} \right], \\ E_x &= \theta \left[H_{1x} \frac{\partial(y_1, z_1)}{\partial(x, t)} + H_{1y} \frac{\partial(z_1, x_1)}{\partial(x, t)} + H_{1z} \frac{\partial(x_1, y_1)}{\partial(x, t)} \right. \\ &\quad \left. + E_{1x} \frac{\partial(x_1, t_1)}{\partial(x, t)} + E_{1y} \frac{\partial(y_1, t_1)}{\partial(x, t)} + E_{1z} \frac{\partial(z_1, t_1)}{\partial(x, t)} \right]. \end{aligned}$$

In order that these two sets of equations may be consistent, we must have $\theta^2 = +1$, and eighteen relations of the type

$$\begin{aligned} \frac{\partial(y_1, z_1)}{\partial(x, y)} &= \theta \frac{\partial(x_1, t_1)}{\partial(z, t)}, \\ \frac{\partial(y_1, z_1)}{\partial(x, t)} &= -\theta \frac{\partial(x_1, t_1)}{\partial(y, z)}. \end{aligned}$$

Bateman,* to whom this transformation theory is due, has shown that the two systems of coordinates must satisfy the equation

$$dx_1^2 + dy_1^2 + dz_1^2 - dt_1^2 = \lambda^2(dx^2 + dy^2 + dz^2 - dt^2), \quad (2)$$

where

$$\lambda^4 = \frac{\partial(x_1, y_1, z_1, it_1)}{\partial(x, y, z, it)},$$

a relation implied in the equation (2), and has further proved that, when the systems of axes are both right-handed and $\frac{\partial t_1}{\partial t} > 0$, θ must be positive so that its value is $+1$.

* *Loc. cit.*, p. 235.

Certain results which will be needed afterwards and which will enable us to change from differential coefficients $\frac{\partial x_1}{\partial x}$, ..., to differential coefficients $\frac{\partial x}{\partial x_1}$, ... may, perhaps, be best proved at this stage.

For any quantity V which can be expressed either as a function of x, y, z, t or of x_1, y_1, z_1, t_1 we have

$$\frac{\partial V}{\partial x_1} = \frac{\partial V}{\partial x} \frac{\partial x}{\partial x_1} + \frac{\partial V}{\partial y} \frac{\partial y}{\partial x_1} + \frac{\partial V}{\partial z} \frac{\partial z}{\partial x_1} + \frac{\partial V}{\partial t} \frac{\partial t}{\partial x_1},$$

$$\frac{\partial V}{\partial y_1} = \frac{\partial V}{\partial x} \frac{\partial x}{\partial y_1} + \dots,$$

$$\frac{\partial V}{\partial z_1} = \frac{\partial V}{\partial x} \frac{\partial x}{\partial z_1} + \dots,$$

$$\frac{\partial V}{\partial t_1} = \frac{\partial V}{\partial x} \frac{\partial x}{\partial t_1} + \dots,$$

whence
$$\frac{\partial V}{\partial t} \frac{\partial(x, y, z, t)}{\partial(x_1, y_1, z_1, t_1)} = \frac{\partial(x, y, z, V)}{\partial(x_1, y_1, z_1, t_1)}, \quad (3)$$

with corresponding expressions for $\frac{\partial V}{\partial x}$, $\frac{\partial V}{\partial y}$, $\frac{\partial V}{\partial z}$.

Now, from the transformation theory,

$$\frac{\partial(x, y, z, t)}{\partial(x_1, y_1, z_1, t_1)} = \frac{1}{\lambda^4}.$$

Also by reversing the process used by Bateman* to prove

$$\frac{\partial(x_1, y_1, z_1)}{\partial(x, y, z)} = \lambda^2 \frac{\partial t_1}{\partial t},$$

and formulæ analogous to this, it can be shown that

$$\frac{\partial(x, y, z)}{\partial(x_1, y_1, z_1)} = \frac{1}{\lambda^3} \frac{\partial t}{\partial t_1}. \quad (4)$$

Putting $V = t_1$ in (3), we find

$$\frac{\partial t_1}{\partial t} = \lambda^4 \frac{\partial(x, y, z)}{\partial(x_1, y_1, z_1)} = \lambda^2 \frac{\partial t}{\partial t_1},$$

by using equation (4).

* *Loc. cit.*, p. 237, equation (A).

In a similar way it may be shown that

$$\frac{\partial x_1}{\partial t} = -\lambda^2 \frac{\partial t}{\partial x_1}, \quad \frac{\partial t_1}{\partial y} = -\lambda^2 \frac{\partial y}{\partial t_1}, \quad \frac{\partial x_1}{\partial y} = \lambda^2 \frac{\partial y}{\partial x_1}, * \quad (5)$$

a minus sign occurring whenever the time t or t_1 occurs *once* on each side of the equation.

2. The General Transformation.

The general solution of (2) has been given by Darboux,[†] and may be written in the form

$$x = a + \frac{k}{\lambda'} \{ l_{11}(x_1 - \alpha) + l_{12}(y_1 - \beta) + l_{13}(z_1 - \gamma) + l_{14}(t_1 - \delta) \},$$

$$y = b + \frac{k}{\lambda'} \{ l_{21}(x_1 - \alpha) + l_{22}(y_1 - \beta) + l_{23}(z_1 - \gamma) + l_{24}(t_1 - \delta) \},$$

$$z = c + \frac{k}{\lambda'} \{ l_{31}(x_1 - \alpha) + l_{32}(y_1 - \beta) + l_{33}(z_1 - \gamma) + l_{34}(t_1 - \delta) \},$$

$$t = d + \frac{k}{\lambda'} \{ l_{41}(x_1 - \alpha) + l_{42}(y_1 - \beta) + l_{43}(z_1 - \gamma) + l_{44}(t_1 - \delta) \},$$

where $\lambda' = x_1^2 + y_1^2 + z_1^2 - t_1^2 - 2(\alpha x_1 + \beta y_1 + \gamma z_1 - \delta t_1) + (\alpha^2 + \beta^2 + \gamma^2 - \delta^2)$.

Also the quantities l form an orthogonal linear substitution in four dimensions, the coordinates being x, y, z, it and x_1, y_1, z_1, it_1 , and therefore contain six arbitrary constants, so that the above solution contains fifteen constants, reducing to eleven if we make $x_1 = y_1 = z_1 = t_1 = 0$ when $x = y = z = t = 0$.

If we keep x_1, y_1, z_1 constant the path of a point in the x, y, z space is obtained by eliminating t_1 from the first three of the above equations. From the way in which t_1 occurs it is clear that the path is always a conic, which cannot be an ellipse, since we can always obtain values of the parameter t_1 for which x, y, z are infinite.

To prove this we consider the value of λ' treated as a quadratic in t_1 ; the roots of the equation $\lambda' = 0$ are real if

$$\delta^2 - 2(\alpha x_1 + \beta y_1 + \gamma z_1) + r_1^2 + (\alpha^2 + \beta^2 + \gamma^2 - \delta^2)$$

is positive.

* Cf. Darboux, *Leçons sur les systèmes orthogonaux et les coordonnées curvilignes*, 2nd ed., Vol. I, § 91, p. 159.

† *Op. cit.*, Vol. I, § 96, p. 167. The notation has been slightly changed in order to conform to the electromagnetic notation.

This expression can be written in the form

$$\frac{1}{r_1^2} [r_1^2 - (ax_1 + \beta y_1 + \gamma z_1)]^2 + \frac{1}{r_1^2} [r_1^2 (\alpha^2 + \beta^2 + \gamma^2) - (ax_1 + \beta y_1 + \gamma z_1)^2],$$

which is never negative, becoming zero if

$$x_1 = \alpha, \quad y_1 = \beta, \quad z_1 = \gamma.$$

We see, therefore, that we are dealing with a particular type of accelerated motion in which the path described by any point of an electron is a conic which cannot be either an ellipse or a circle. For our purpose, however, we may build up the transformation in the following way:—

Consider the transformation*

$$x' = \frac{kx''}{r'^2 - t'^2}, \quad y' = \frac{ky''}{r'^2 - t'^2}, \quad z' = \frac{kz''}{r'^2 - t'^2}, \quad t' = \frac{kt''}{r'^2 - t'^2},$$

change x'' into $\bar{x} - a$, y'' into $\bar{y} - \beta$, z'' into $\bar{z} - \gamma$, and t'' into $\bar{t} - \delta$, and we obtain

$$x' = \frac{k(\bar{x} - a)}{r^2 - \bar{t}^2 - 2(a\bar{x} + \beta\bar{y} + \gamma\bar{z} - \delta\bar{t}) + \alpha^2 + \beta^2 + \gamma^2 - \delta^2}, \quad \dots, \quad \dots,$$

whence

$$x = a + l_{11}x' + l_{12}y' + l_{13}z' + l_{14}t',$$

and similar equations give Darboux's solution of the equation

$$d\bar{x}^2 + d\bar{y}^2 + d\bar{z}^2 - d\bar{t}^2 = \lambda^2(dx^2 + dy^2 + dz^2 - dt^2),$$

$k\lambda$ being equal to

$$\bar{r}^2 - \bar{t}^2 - 2(a\bar{x} + \beta\bar{y} + \gamma\bar{z} - \delta\bar{t}) + \alpha^2 + \beta^2 + \gamma^2 - \delta^2.$$

If we proceed further, and put

$$\bar{x} = \frac{x_1}{r_1^2 - t_1^2}, \quad \dots, \quad \dots,$$

$$\text{we find } x' = \frac{k[x_1 - \alpha(r_1^2 - t_1^2)]}{1 - 2(ax_1 + \beta y_1 + \gamma z_1 - \delta t_1) + (\alpha^2 + \beta^2 + \gamma^2 - \delta^2)(r_1^2 - t_1^2)}, \quad (6)$$

and similar expressions for $\bar{y}$, $\bar{z}$, and $\bar{t}$, as the solution of the equation

$$dx_1^2 + dy_1^2 + dz_1^2 - dt_1^2 = \lambda^2(dx'^2 + dy'^2 + dz'^2 - dt'^2),$$

* Used by E. Cunningham, "The Principle of Relativity in Electrodynamics, and an extension thereof," *Proc. London Math. Soc.*, Ser. 2, Vol. 8, 1910.

where $k\lambda$ is now equal to

$$1 - 2(\alpha x_1 + \beta y_1 + \gamma z_1 - \delta t_1) + (\alpha^2 + \beta^2 + \gamma^2 - \delta^2)(r_1^2 - t_1^2).$$

This transformation is of a different type to the one used by Cunningham, reducing to the form $\bar{x}' = kx_1$, ..., when $\alpha = \beta = \gamma = \delta = 0$.

In this paper this latter type of transformation of the form given by equation (6) is alone considered.

The general transformation of this type may now be written in the form

$$x = \frac{k}{D_1} \{ l_{11}[x_1 - \alpha(r_1^2 - t_1^2)] + l_{12}[y_1 - \beta(r_1^2 - t_1^2)] + l_{13}[z_1 - \gamma(r_1^2 - t_1^2)] \\ + l_{14}[t_1 - \delta(r_1^2 - t_1^2)] \},$$

with similar expressions for z , y , t , where

$$D_1 \equiv 1 - 2(\alpha x_1 + \beta y_1 + \gamma z_1 - \delta t_1) + (\alpha^2 + \beta^2 + \gamma^2 - \delta^2)(r_1^2 - t_1^2),$$

which contains eleven constants, the remaining four having been chosen so as to make $x = y = z = t = 0$ when $x_1 = y_1 = z_1 = t_1 = 0$.

The interpretation of these eleven constants can be best obtained by considering the infinitesimal transformation. If we suppose that the various constants, with the exception of k , are all of an order of magnitude whose square may be neglected, the values of the quantities l become

$$\begin{aligned} l_{11} &= 1, & l_{12} &= +2a, & l_{13} &= +2b, & l_{14} &= 2f, \\ l_{21} &= -2a, & l_{22} &= 1, & l_{23} &= +2c, & l_{24} &= 2g, \\ l_{31} &= -2b, & l_{32} &= -2c, & l_{33} &= 1, & l_{34} &= 2h, \\ l_{41} &= 2f, & l_{42} &= 2g, & l_{43} &= 2h, & l_{44} &= 1, \end{aligned}$$

and the infinitesimal transformation is

$$\begin{aligned} x &= k \{ x_1 + \alpha(x_1^2 - y_1^2 - z_1^2 + t_1^2) + 2\beta y_1 x_1 + 2\gamma z_1 x_1 - 2\delta t_1 x_1 + 2a y_1 + 2b z_1 + 2f t_1 \}, \\ y &= k \{ y_1 + 2a x_1 y_1 + \beta(y_1^2 - x_1^2 - z_1^2 + t_1^2) + 2\gamma y_1 z_1 - 2\delta t_1 y_1 - 2a x_1 + 2c z_1 + 2g t_1 \}, \\ z &= k \{ z_1 + 2a x_1 z_1 + 2\beta y_1 z_1 + \gamma(z_1^2 - x_1^2 - y_1^2 - t_1^2) - 2\delta t_1 z_1 - 2b x_1 - 2c y_1 + 2h t_1 \}, \\ t &= k \{ t_1 + 2a x_1 t_1 + 2\beta y_1 t_1 + 2\gamma z_1 t_1 - \delta(x_1^2 + y_1^2 + z_1^2 + t_1^2) - 2f x_1 - 2g y_1 - 2h z_1 \}. \end{aligned}$$

We may replace t_1 by t/k in the first three equations, whence, keeping $x_1 y_1 z_1$ constant,

$$\frac{dx}{dt} = \left[\frac{2a}{k} t - 2\delta x_1 + 2f \right] = 2 \left[\frac{at}{k} - \frac{\delta x}{k} + f \right],$$

with similar expressions for dy/dt and dz/dt .

Hence to a fixed point in the x_1, y_1, z_1 space there corresponds a point in motion in the x, y, z space, the motion being such that any point (x, y, z) has the velocities given above. The origin, therefore, moves with a velocity whose components at time $t = 0$ are $2f, 2g, 2h$, and with an acceleration whose components are $2a/k, 2\beta/k, 2\gamma/k$.

The motion of a point (x, y, z) relative to the origin is given by the velocities $-2x\delta/k, -2y\delta/k$, and $-2z\delta/k$, so that the constant δ may be said to express an expansion or contraction relative to the origin such that

$$-2\delta = k V/r,$$

where V is the radial velocity relative to the origin at a distance r .

The constant k clearly expresses a uniform change of volume, while the remaining constants a, b, c have no kinematical significance and correspond to a rotation in the theory of small deformations. The infinitesimal transformation, in fact, may be divided into two parts, a conformal transformation in three dimensions given by

$$x = k \{ x' + a(x'^2 - y'^2 - z'^2) + 2\beta y'x' + 2\gamma z'x' + 2ay' + 2bz' \},$$

$$y = k \{ y' + 2ax'y' + \beta(y'^2 - x'^2 - z'^2) + 2\gamma z'y' - 2ax' + 2cz' \},$$

$$z = k \{ z' + 2ax'z' + 2\beta y'z' + \gamma(z'^2 - x'^2 - y'^2) - 2bx' - 2cy' \},$$

and a kinematical part given by

$$x' = x_1 + at_1^2 - 2\delta t_1 x_1 + 2ft_1,$$

$$y' = y_1 + \beta t_1^2 - 2\delta t_1 y_1 + 2gt_1,$$

$$z' = z_1 + \gamma t_1^2 - 2\delta t_1 z_1 + 2ht_1.$$

It may further be noticed that the transformation for the time t contains only those constants which have a kinematical interpretation. If we retain only this class of constants and, therefore consider a, b , and c as zero, we may obtain the general transformation for a constant velocity by putting $\alpha = \beta = \gamma = \delta = 0$. In this case the values of the quantities l are given by

$$Bl_{11} = 1 + f^2 - g^2 - h^2,$$

$$Bl_{12} = 2fg,$$

$$Bl_{21} = 2fg,$$

$$Bl_{22} = 1 + g^2 - f^2 - h^2,$$

$$Bl_{31} = 2fh,$$

$$Bl_{32} = 2gh,$$

$$Bl_{41} = 2f,$$

$$Bl_{42} = 2g,$$

$$\begin{aligned}
 Bl_{13} &= 2fh, & Bl_{14} &= 2f, \\
 Bl_{23} &= 2gh, & Bl_{24} &= 2g, \\
 Bl_{33} &= 1+h^2-f^2-g^2, & Bl_{34} &= 2h, \\
 Bl_{43} &= 2h, & Bl_{44} &= 1+f^2+g^2+h^2,
 \end{aligned}$$

where

$$B = -f^2 - g^2 - h^2.$$

If we put

$$\begin{aligned}
 1+f^2+g^2+h^2 &= 2f/u = 2g/v = 2h/w, \\
 s^2 &= u^2+v^2+w^2, \quad \epsilon^2 = (1-s^2)^{-1},
 \end{aligned}$$

we find that, when k is unity,

$$\begin{aligned}
 x &= x_1 + \frac{\epsilon^2 u}{1+\epsilon} (ux_1 + vy_1 + wz_1) + u\epsilon t_1, \\
 y &= y_1 + \frac{\epsilon^2 v}{1+\epsilon} (ux_1 + vy_1 + wz_1) + v\epsilon t_1, \\
 z &= z_1 + \frac{\epsilon^2 w}{1+\epsilon} (ux_1 + vy_1 + wz_1) + w\epsilon t_1, \\
 t &= \epsilon (ux_1 + vy_1 + wz_1) + \epsilon t_1,
 \end{aligned}$$

which is the general transformation for a point moving with a constant velocity whose components are u, v, w .* If v and w are zero, so that the point moves along the axis of x with a constant velocity u , the above transformation reduces to the well known equations given by Larmor and Lorentz.

3. A Special Type of Transformation.

We will now consider in detail the transformation in which the constants a, b, c are zero, the other constants being retained in the greater part of the work.

If we have a transformation

$$x = x', \quad y = y', \quad z = z', \quad t = t'$$

which satisfies equation (2), x', y', z', t' being functions of x_1, y_1, z_1, t_1 , it can be easily shown that the transformation

$$x = \epsilon(x' + ut'), \quad y = y', \quad z = z', \quad t = \epsilon(t' + ux')$$

* Recently given by G. Herglotz, "Über die Mechanik des deformierbaren Körpers vom Standpunkte der Relativitätstheorie," *Annalen der Physik*, Vol. 36, 1911, p. 497.

where $\epsilon^2 \equiv (1-u^2)^{-1}$ also satisfies equation (2), the Jacobian λ^4 being independent of u .

Using the values of x', y', z', t' given in equation (6), we finally obtain the transformation

$$\left. \begin{aligned} x &= \frac{\epsilon k}{D_1} \{x_1 - a(r_1^2 - t_1^2) + u[t_1 - \delta(r_1^2 - t_1^2)]\} \\ t &= \frac{\epsilon k}{D_1} \{t_1 - \delta(r_1^2 - t_1^2) + u[x_1 - a(r_1^2 - t_1^2)]\} \\ y &= \frac{k}{D_1} [y_1 - \beta(r_1^2 - t_1^2)], \quad z = \frac{k}{D_1} [z_1 - \gamma(r_1^2 - t_1^2)] \end{aligned} \right\}, \quad (7)$$

where we have now to interpret the constants $a, \beta, \gamma, \delta, u$.

Consider the origin of a system of moving axes which at time $t = 0$ coincides with a system of fixed axes x, y, z .

Then the component velocities of the origin of the moving axes are the values of

$$\frac{dx}{dt}, \quad \frac{dy}{dt}, \quad \frac{dz}{dt} \quad \text{for} \quad x = y = z = t = 0,$$

and its accelerations the values of

$$\frac{d^2x}{dt^2}, \quad \frac{d^2y}{dt^2}, \quad \frac{d^2z}{dt^2} \quad \text{for} \quad x = y = z = t = 0.$$

In finding the velocity and acceleration, x, y, z are to be considered as expressed in terms of x_1, y_1, z_1 , and t ; further x_1, y_1, z_1 are to be kept constant, since we wish to transform from a system at rest in the x_1, y_1, z_1 space to a system in motion in the x, y, z space.

Hence
$$\frac{dx}{dt} = \frac{\partial x}{\partial t_1} / \frac{\partial t}{\partial t_1},$$

and when $x = y = z = t = 0, \quad x_1 = y_1 = z_1 = t_1 = 0.$

Now

$$\begin{aligned} \frac{\partial x}{\partial t_1} &= \frac{\epsilon k}{D_1^2} \{ D_1 [u + 2(a + u\delta)t_1] \\ &\quad - 2[x_1 + ut_1 - (a + u\delta)(r_1^2 - t_1^2)][\delta - (a^2 + \beta^2 + \gamma^2 - \delta^2)t_1] \}, \end{aligned}$$

$$\begin{aligned} \frac{\partial t}{\partial t_1} &= \frac{\epsilon k}{D_1^2} \{ D_1 [1 + 2(\delta + ua)t_1] \\ &\quad - 2[t_1 + ux_1 - (\delta + ua)(r_1^2 - t_1^2)][\delta - (a^2 + \beta^2 + \gamma^2 - \delta^2)t_1] \}, \end{aligned}$$

so that $\frac{dx}{dt} = u$, for $x_1 = y_1 = z_1 = t_1 = 0$.

Similarly $\frac{dy}{dt} = \frac{dz}{dt} = 0$, for $x_1 = y_1 = z_1 = t_1 = 0$.

Again, $\frac{d^2x}{dt^2} = \left(\frac{\partial}{\partial t_1}\right)^{-1} \frac{\partial}{\partial t_1} \left(\frac{dx}{dt}\right) = \frac{2a(1-u^2)}{k\epsilon}$,

for the origin of the moving axes, and similarly

$$\frac{d^2y}{dt^2} = 2\beta/k\epsilon^2, \quad \frac{d^2z}{dt^2} = 2\gamma/k\epsilon^2.$$

Therefore, if p, q, r be the component accelerations of the origin,

$$2a = kp\epsilon^3, \quad 2\beta = kq\epsilon^2, \quad 2\gamma = kr\epsilon^2.$$

Hence the above transformation (7) gives the motion of a point which at time $t=0$ has a velocity $(u, 0, 0)$ and an acceleration (p, q, r) . As regards the constant δ , it is not difficult to show that when u is zero δ is defined by

$$-2\delta = k \lim_{r \rightarrow 0} \left(\frac{V}{r}\right),$$

where V is the radial velocity relative to the origin at a distance r from the origin, giving an interpretation to the constant δ of a nature similar to that adopted in the case of the infinitesimal transformation, when $\delta, \alpha, \beta, \gamma$ were all small.

When u is not zero, *i.e.*, when the centre of the electron is not instantaneously at rest, we have to distinguish between the velocity of a point relative to the centre in different directions, and it is the velocity of a point relative to the centre in its direction of motion which enables us to give an interpretation to the constant δ , which now, however, is grouped together with other constants in the form $\delta + ua$.

It is, in fact, not difficult to prove that

$$-2(\delta + ua) = \epsilon k \lim \left(\frac{V_x}{x}\right),$$

where V_x is the velocity relative to the centre of the electron in the direction of motion, x is measured from the centre in the direction of motion, and in performing the limiting process indicated we first make y and z tend to zero, and then make x tend to zero.

Thus the motion in which $\delta + ua$ vanishes is a special type character-

ized by the fact that there is no relative expansion or contraction in the direction of motion.

When the constants are such that $\delta + ua = 0$ we shall say that we are dealing with the motion of a non-expanding electron.

For some purposes it is useful to express x_1, y_1, z_1, t_1 in terms of x, y, z, t , which can be done by reversing the equations (7), giving

$$\left. \begin{aligned} x_1 &= \frac{\epsilon k(x - ut) + a(r^2 - t^2)}{D} \\ t_1 &= \frac{\epsilon k(t - ux) + \delta(r^2 - t^2)}{D} \\ y_1 &= \frac{ky + \beta(r^2 - t^2)}{D}, \quad z_1 = \frac{kz + \gamma(r^2 - t^2)}{D} \end{aligned} \right\}, \quad (8)$$

where

$$D \equiv k^2 + 2k[\beta y + \gamma z + a\epsilon(x - ut) - \delta\epsilon(t - ux)] + (\alpha^2 + \beta^2 + \gamma^2 - \delta^2)(r^2 - t^2).$$

If we consider an electron at rest in the x_1, y_1, z_1 space to be bounded by the sphere $x_1^2 + y_1^2 + z_1^2 = a^2$, we see that at time $t = 0$ its boundary when in motion will be given by

$$\begin{aligned} \alpha^2 [k^2 + 2k(\beta y + \gamma z + a\epsilon x + \delta\epsilon ux) + (\alpha^2 + \beta^2 + \gamma^2 - \delta^2)r^2]^2 \\ = k^2(\epsilon^2 x^2 + y^2 + z^2) + (\alpha^2 + \beta^2 + \gamma^2)r^4 + 2kr^2(a\epsilon x + \beta y + \gamma z). \end{aligned}$$

If the electron is moving with a uniform velocity, this reduces to the spheroid

$$\epsilon^2 x^2 + y^2 + z^2 = k^2 a^2,$$

giving the Fitzgerald-Lorentz contraction in the direction of motion.*

To discuss the particular type of motion given by (7) it is convenient to take the centre of the electron at rest in the x_1, y_1, z_1 space to be the origin in that space and also to choose the axes of y and z in the x, y, z space so that the acceleration of the centre of the electron at the time $t = 0$ shall be in the plane of xy . Under these circumstances $x_1 = y_1 = z_1 = \gamma = 0$, so that the path of the centre is given by eliminating t_1 from the first two of the equations

$$\begin{aligned} x &= \frac{\epsilon k}{D_1} \{ut_1 + (a + u\delta)t_1^2\}, & y &= \frac{k\beta t_1^2}{D_1}, \\ t &= \frac{\epsilon k}{D_1} \{t_1 + (\delta + ua)t_1^2\}, & z &= 0, \end{aligned}$$

* It is important to notice that this contraction is only defined for the case of uniform motion.

where

$$D_1 = 1 + 2\delta t_1 - (\alpha^2 + \beta^2 - \delta^2) t_1^2.$$

The path is, therefore, the hyperbola

$$\beta^2 x^2 + y^2 (\alpha^2 - \beta^2 u^2 \epsilon^2) - 2\alpha\beta\epsilon xy - k\beta\epsilon^2 u^2 y = 0,$$

and the centre of the electron describes that branch which passes through the origin, the tangent at this latter point being the axis of x .

The values of x and y in terms of the time t are given by

$$\alpha^2 (t^2 - x^2) - \beta^2 \epsilon^2 (x - ut)^2 - \epsilon k \alpha (x - ut) = 0,$$

$$y^2 (\alpha^2 + \epsilon^2 \beta^2) + 2\beta\epsilon u \alpha t y + k\beta\epsilon^2 y - t^2 \beta^2 = 0,$$

δ being automatically eliminated, from which the position of the centre of the electron can be found for different values of t , that branch of the hyperbola being taken for which $x = y = 0$ when $t = 0$.

A particular case is obtained by making $\beta = 0$; this gives $y = 0$, so the centre moves along the axis of x . If we make it pass through the origin

$$x = \frac{1}{2\alpha} \{ \sqrt{(\epsilon^2 k^2 + 4\epsilon k u \alpha t + 4\alpha^2 t^2) - \epsilon k} \}.$$

Tracing the position of the centre of the electron from $t = -\infty$ to $t = +\infty$, we find that it starts with a velocity equal to that of light from $x = +\infty$, reaching the origin at a time $-\epsilon u k / \alpha$ with a velocity $-u$; it comes to rest at a time $-\epsilon u k / 2\alpha$, and then retraces its path, passing through the origin again at time $t = 0$, when it has a velocity u and an acceleration $2\alpha / k\epsilon^3$, its greatest distance from the origin in the negative direction being $k(\epsilon - 1) / 2\alpha$.

4. *Expressions for the Forces acting on an Electron.*

For the purpose of calculating the forces on an electron it will be easiest to avail ourselves of the invariant theory of the general transformation as developed by Bateman, though the results can be obtained by a calculation of the electric and magnetic vectors from the formulæ connecting their values in the two spaces.

The following invariants of the theory will be found useful:—

$$\lambda^2 (dx^2 + dy^2 + dz^2 - dt^2) = dx_1^2 + dy_1^2 + dz_1^2 - dt_1^2, \quad (2)$$

$$\lambda^4 dx dy dz dt = dx_1 dy_1 dz_1 dt_1, \quad (9)$$

$$H_x dy dz + H_y dz dx + H_z dx dy + E_x dx dt + E_y dy dt + E_z dz dt, \quad (10)$$

$$E_x dy dz + E_y dz dx + E_z dx dy - H_x dx dt - H_y dy dt - H_z dz dt, \quad (11)$$

$$\rho w_x dy dz dt + \rho w_y dz dx dt + \rho w_z dx dy dt - \rho dx dy dz. \quad (12)$$

From these invariants others can be derived by processes analogous to those of multiplication and reciprocation.

In this way we may deduce the following :—

$$\frac{1}{\lambda^2} (\rho w_x dx + \rho w_y dy + \rho w_z dz - \rho dt), \quad (13)$$

$$\frac{\rho^2}{\lambda^2} (1 - w^2) dx dy dz dt, \quad (14)$$

$$\rho dx dy dz dt [(E_x + w_y H_z - w_z H_y) \delta x + \dots + \dots - (w_x E_x + w_y E_y + w_z E_z) \delta t], * \quad (15)$$

where $\delta x, \delta y, \delta z, \delta t$ are arbitrary increments of the variables. For our purpose we seek to express the various quantities involved in terms of x_1, y_1, z_1 , and t , where x_1, y_1, z_1 are independent of the time t , and to integrate always over the x_1, y_1, z_1 space where the electron is assumed to be spherical and the integrations are simple.

In this case we have

$$\frac{dx}{dt} = w_x, \dots, \frac{dx_1}{dt} = 0, \dots,$$

$$\text{so that (2) becomes} \quad \lambda^2 (1 - w^2) dt^2 = dt_1^2,$$

$$\text{or} \quad \lambda \sqrt{1 - w^2} dt = dt_1, \quad (16)$$

which is an important result, since from (9), (14), and (16), we now deduce that

$$\rho dx dy dz = \rho_1 dx_1 dy_1 dz_1. \quad (17)$$

This expresses the condition, already assumed, that the total charge is invariant and can be verified by using Jacobians, only that we must take care to express x, y, z in terms of x_1, y_1, z_1 , and the ordinary time t .

Since the electron is at rest in the x_1, y_1, z_1 space $H_1 = 0$, while E_1 is simply the electrostatic force due to the electron itself.

* These and other invariants are given by Bateman, *loc. cit.*, p. 252, together with an account of the processes used in obtaining them.

From (15) and (16) we therefore deduce

$$\rho dx dy dz [(E_x + w_y H_z - w_z H_y) \delta x + \dots - (w_x E_x + w_y E_y + w_z E_z) \delta t] \\ = \rho_1 dx_1 dy_1 dz_1 \lambda \sqrt{1-w^2} [E_{1x} \delta x_1 + E_{1y} \delta y_1 + E_{1z} \delta z_1],$$

so that

$$\rho dx dy dz F_x = \rho_1 dx_1 dy_1 dz_1 \lambda \sqrt{1-w^2} \left[E_{1x} \frac{\partial x_1}{\partial x} + E_{1y} \frac{\partial y_1}{\partial x} + E_{1z} \frac{\partial z_1}{\partial x} \right], \\ \rho dx dy dz F_y = \rho_1 dx_1 dy_1 dz_1 \lambda \sqrt{1-w^2} \left[E_{1x} \frac{\partial x_1}{\partial y} + E_{1y} \frac{\partial y_1}{\partial y} + E_{1z} \frac{\partial z_1}{\partial y} \right], \\ \rho dx dy dz F_z = \rho_1 dx_1 dy_1 dz_1 \lambda \sqrt{1-w^2} \left[E_{1x} \frac{\partial x_1}{\partial z} + E_{1y} \frac{\partial y_1}{\partial z} + E_{1z} \frac{\partial z_1}{\partial z} \right], \\ -\rho dx dy dz (w, F) = \rho_1 dx_1 dy_1 dz_1 \lambda \sqrt{1-w^2} \left[E_{1x} \frac{\partial x_1}{\partial t} + E_{1y} \frac{\partial y_1}{\partial t} + E_{1z} \frac{\partial z_1}{\partial t} \right],$$

where $F \equiv E + [w, H]$ is the force per unit charge acting on the electron due to the reaction of the æther.

By integration on both sides of these equations we obtain from the first three the components of the force on the electron, and from the last equation the rate of work done by the electromagnetic forces. Calling these X, Y, Z and dA/dt respectively, we shall calculate, as far as possible, their values at time $t = 0$.

Replacing w_x by $\frac{\partial x}{\partial t_1} / \frac{\partial t}{\partial t_1}$, ..., and remembering that

$$\lambda^2 \left[\left(\frac{\partial t}{\partial t_1} \right)^2 - \left(\frac{\partial x}{\partial t_1} \right)^2 - \left(\frac{\partial y}{\partial t_1} \right)^2 - \left(\frac{\partial z}{\partial t_1} \right)^2 \right] = 1,$$

we can substitute $\left(\frac{\partial t}{\partial t_1} \right)^{-1}$ for $\lambda \sqrt{1-w^2}$, giving a more convenient form for our purpose.

We can also replace $\frac{\partial x_1}{\partial x}, \frac{\partial y_1}{\partial x}, \frac{\partial z_1}{\partial x}$ by $\lambda^2 \frac{\partial x}{\partial x_1}, \lambda^2 \frac{\partial x}{\partial y_1}, \lambda^2 \frac{\partial x}{\partial z_1}$ from equations (5).

Again, at time $t = 0$, we must have, from equations (7),

$$t_1 + ux_1 - (\delta + ua)(r_1^2 - t_1^2) = 0,$$

$$\text{whence } t_1 = \frac{1}{2(\delta + ua)} \{ \sqrt{[1 - 4(\delta + ua)ux_1 + 4(\delta + ua)^2 r_1^2]} - 1 \}, \quad (18)$$

the positive sign being given to the square root, since $t_1 = 0$ for

$$x_1 = y_1 = z_1 = 0.$$

Further, for $t = 0$, we have

$$\frac{\partial t}{\partial t_1} = \frac{k\epsilon}{D_1} [1 + 2(\delta + ua)t_1] = \frac{k\epsilon}{D_1} \sqrt{[1 - 4(\delta + ua)ux_1 + 4(\delta + ua)^2 r_1^2]},$$

using the last equation.

We are now in a position to calculate the forces acting on the electron, which we shall consider to be when at rest a spherical shell of radii b and a , the charge e_1 being uniformly distributed over its volume. We can pass from this case to the limiting cases of volume and surface distribution by making $b = 0$ and $b = a$ respectively.

$$\text{Hence } E_{1x} = \frac{1}{3}\rho_1 x_1 \left(1 - \frac{b^3}{r_1^3}\right), \quad E_{1y} = \frac{1}{3}\rho_1 y_1 \left(1 - \frac{b^3}{r_1^3}\right),$$

$$E_{1z} = \frac{1}{3}\rho_1 z_1 \left(1 - \frac{b^3}{r_1^3}\right),$$

where r_1 is the distance of the point (x_1, y_1, z_1) from the origin.

Making use of these simplifications, we finally obtain

$$\begin{aligned} X &= \iiint \rho F_x dx dy dz \\ &= \iiint \rho_1 dx_1 dy_1 dz_1 \lambda^3 \left(\frac{\partial t}{\partial t_1}\right)^{-1} \left[E_{1x} \frac{\partial x}{\partial x_1} + E_{1y} \frac{\partial x}{\partial y_1} + E_{1z} \frac{\partial x}{\partial z_1} \right] \\ &= \iiint \frac{1}{3} \frac{\rho_1^2 dx_1 dy_1 dz_1 D_1^3 (1 - b^3/r_1^3)}{\epsilon k^3 \sqrt{[1 - 4(\delta + ua)ux_1 + 4(\delta + ua)^2 r_1^2]}} \left[x_1 \frac{\partial x}{\partial x_1} + y_1 \frac{\partial x}{\partial y_1} + z_1 \frac{\partial x}{\partial z_1} \right] \\ &= \iiint \frac{1}{3} \rho_1^2 \left(1 - \frac{b^3}{r_1^3}\right) \frac{D_1 dx_1 dy_1 dz_1 N_x}{k^3 \sqrt{[1 - 4(\delta + ua)ux_1 + 4(\delta + ua)^2 r_1^2]}}, \end{aligned}$$

$$\begin{aligned} \text{where } N_x &\equiv x_1 + 2x_1 t_1 \delta - x_1 \kappa^2 (r_1^2 + t_1^2) - 2(a + u\delta)r_1^2 - 2ut_1 \kappa^2 r_1^2 \\ &\quad - 4\delta t_1 (a + u\delta)r_1^2 + 2ut_1 (ax_1 + \beta y_1 + \gamma z_1) \\ &\quad + 2(a + u\delta)(r_1^2 + t_1^2)(ax_1 + \beta y_1 + \gamma z_1), \end{aligned}$$

and

$$\kappa^2 \equiv a^2 + \beta^2 + \gamma^2 - \delta^2.$$

Similarly,

$$\begin{aligned} Y &= \iiint \frac{1}{3} \rho_1^2 \left(1 - \frac{b^3}{r_1^3}\right) \frac{D_1 dx_1 dy_1 dz_1 N_y}{\epsilon k^3 \sqrt{[1 - 4(\delta + ua)ux_1 + 4(\delta + ua)^2 r_1^2]}}, \\ Z &= \iiint \frac{1}{3} \rho_1^2 \left(1 - \frac{b^3}{r_1^3}\right) \frac{D_1 dx_1 dy_1 dz_1 N_z}{\epsilon k^3 \sqrt{[1 - 4(\delta + ua)ux_1 + 4(\delta + ua)^2 r_1^2]}}, \end{aligned}$$

where

$$\begin{aligned}
 N_y \equiv & y_1 [1 - \kappa^2 (r_1^2 + t_1^2)] - 2\beta r_1^2 - 4\beta r_1^2 t_1 \delta + 2y_1 t_1 \delta \\
 & + 2\beta (ax_1 + \beta y_1 + \gamma z_1)(r_1^2 + t_1^2), \\
 N_z \equiv & z_1 [1 - \kappa^2 (r_1^2 + t_1^2)] - 2\gamma r_1^2 - 4\gamma r_1^2 t_1 \delta + 2z_1 t_1 \delta \\
 & + 2\gamma (ax_1 + \beta y_1 + \gamma z_1)(r_1^2 + t_1^2),
 \end{aligned}$$

and in all cases t_1 must be expressed in terms of x_1, y_1, z_1 by equation (18) before the integration is performed.

5. Calculations of the Forces.

It does not seem possible to perform the integrations in the general case, so that we shall be content with discussing some simple cases.

I. $\delta = 0$. *Lorentz Electrons.*

The case of an electron whose shape depends on its motion was first investigated by Lorentz. This type is an obvious extension of his idea, and gives, as we shall see, for the case of quasi-stationary motion the same values of the electromagnetic masses as deduced by him, proving that in this case the application of the theory of quasi-stationary motion is justified.

There are in this type of electron two special cases where the integration can be easily effected.

The first case is when $u = 0$, *i.e.*, the centre of the electron is instantaneously at rest at the instant when the force is calculated; t is then zero from (18), so that the integrals are all of the same kind, and we find for a volume distribution ($b = 0$)

$$\left. \begin{aligned}
 X, Y, Z = & -\frac{e_1^2}{\pi k^2} (2\alpha, 2\beta, 2\gamma) \left[\frac{1}{5a} + \frac{1}{7} (\alpha^2 + \beta^2 + \gamma^2) a \right] \\
 \text{and for the case of a surface distribution} \\
 X, Y, Z = & -\frac{e_1^2}{6\pi k^2} (2\alpha, 2\beta, 2\gamma) \left[\frac{1}{a} + (\alpha^2 + \beta^2 + \gamma^2) a \right]
 \end{aligned} \right\}. \quad (19)$$

In terms of the acceleration (p, q, r) the forces are

$$\left. \begin{aligned} X, Y, Z &= -(p, q, r) \frac{e_1^2}{\pi k} \left[\frac{1}{5a} + \frac{k^2}{28} (p^2 + q^2 + r^2) a \right]^* \\ \text{for a volume distribution of charge, and} \\ &= -(p, q, r) \frac{e_1^2}{6\pi k} \left[\frac{1}{a} + \frac{k^2}{4} (p^2 + q^2 + r^2) a \right] \\ \text{for a surface distribution.} \end{aligned} \right\} \quad (19)$$

The second case in which the integration can be easily performed is when $a = 0$, *i.e.*, the acceleration of the centre of the electron in the direction of motion is instantaneously zero. In this case we find from (18) that $t_1 = -ux_1$, and it follows that X is zero, while

$$\left. \begin{aligned} Y, Z &= -(2\beta, 2\gamma) \frac{e_1^2}{6\pi k^2} \left[\frac{6}{5a} + (6-u^2)(\beta^2 + \gamma^2) \frac{a}{7} \right] \\ \text{for a volume distribution, and} \\ Y, Z &= -(2\beta, 2\gamma) \frac{e_1^2}{4\pi k^2} \left[\frac{2}{3} a + (6-u^2)(\beta^2 + \gamma^2) \frac{a}{9} \right] \\ \text{for a surface distribution.} \end{aligned} \right\} \quad (20)$$

In terms of the accelerations q, r the forces are

$$\left. \begin{aligned} Y, Z &= -(q, r) \frac{e_1^2 \epsilon}{6\pi k} \left[\frac{6}{5a} + \frac{(6-u^2)}{28} (q^2 + r^2) k^2 \epsilon^4 a \right] \\ Y, Z &= -(q, r) \frac{e_1^2 \epsilon}{4\pi k} \left[\frac{2}{3} a + \frac{(6-u^2)}{36} (q^2 + r^2) k^2 \epsilon^4 a \right] \end{aligned} \right\}, \quad (20a)$$

for volume and surface distribution respectively.

II. $\delta + ua = 0$. *Non-expanding Electrons.*

This case has some features of special interest, the first being that the values of the forces can be accurately worked out, since, when $\delta + ua = 0$,

* The special case of $q = r = 0$ is treated by Born, "Die Theorie des Starren Elektrons in der Kinematik des Relativitätsprinzips," *Annalen der Physik*, Vol. 30, 1909. He finds the first term $-\frac{e_1^2 p}{5\pi a}$, ($k = 1$), and shows that there is no term in p^2 .

we have, from (18), $t_1 = -ux_1$, and the integration is easy, the results being

$$\left. \begin{aligned} X &= -\frac{2a(1-u^2)e_1^2}{\pi k^2} \left\{ \frac{1}{5a} + \frac{a}{7.3} [a^2(1-u^2)(3-2u^2) + (\beta^2 + \gamma^2)(3+u^2)] \right\} \\ Y, Z &= -\frac{(2\beta, 2\gamma)e_1^2}{\pi \epsilon k^2} \left\{ \frac{1}{5a} + \frac{a}{7.6} [a^2(1-u^2)(6-7u^2) + (\beta^2 + \gamma^2)(6-u^2)] \right\} \end{aligned} \right\} \quad (21)$$

for volume distribution, while for a surface distribution of charge the factors $\frac{1}{5}$ and $\frac{1}{7}$ are both replaced by $\frac{1}{6}$.

When either u or a is zero the above values of the forces agree with those given in equations (19) and (20).

III. Approximations, δ and a small.

If in the general case δ and a are both small, we may expand in terms of these quantities and obtain the following results.

Let the expansions have the following form

$$X = X_0 + aX_1 + \delta X' + \frac{1}{2}(a^2 X_2 + 2a\delta \bar{X} + \delta^2 X'') + \dots,$$

with similar expressions for Y and Z .

Then X_0 is zero,

$$\left. \begin{aligned} X_1 &= -\frac{2e_1^2}{\pi k^2} \left\{ \frac{1}{5a} + \frac{a}{7} \frac{3+4u^2-3u^4}{3} (\beta^2 + \gamma^2) \right. \\ &\quad \left. + \frac{a^3}{9} \frac{u^2(3-u^2)(1-u^2)}{3} (\beta^2 + \gamma^2)^2 \right\} \\ X' &= -\frac{2ue_1^2}{\pi k^2} \left\{ \frac{1}{5a} + \frac{a}{7} \frac{6-2u^2}{3} (\beta^2 + \gamma^2) \right. \\ &\quad \left. + \frac{a^3}{9} \frac{(3-u^2)(1-u^2)}{3} (\beta^2 + \gamma^2)^2 \right\} \end{aligned} \right\} \quad (22)$$

for volume distribution of charge, while for a surface distribution the factors $\frac{1}{5}$, $\frac{1}{7}$, $\frac{1}{9}$ are each replaced by $\frac{1}{6}$.

Further, the terms of the second order X_2 , $\bar{X}$, X'' all vanish. As regards Y and Z , Y_0 and Z_0 are given by their values in (20), Y_1 , Y' , Z_1 , Z' all vanish, and

$$\left. \begin{aligned} Y_2 &= -\frac{2\beta e_1^2}{\pi k^2 \epsilon} \left\{ \frac{6-19u^2+10u^4}{3} \frac{a}{7} - \frac{2u^2(1-u^2)(9-2u^2)}{3} \frac{a^3}{9} (\beta^2 + \gamma^2) \right\} \\ \bar{Y} &= -\frac{2u\beta e_1^2}{\pi k^2 \epsilon} \left\{ \frac{u^2}{3} \frac{a}{7} - \frac{2(1-u^2)(6-u^2)}{3} \frac{a^3}{9} (\beta^2 + \gamma^2) \right\} \\ Y'' &= -\frac{2\beta e_1^2}{\pi k^2 \epsilon} \left\{ \frac{6-u^2}{3} \frac{a}{7} - \frac{6(1-u^2)}{3} \frac{a^3}{9} (\beta^2 + \gamma^2) \right\} \end{aligned} \right\} \quad (23)$$

with similar expressions for the coefficients in the expansion of Z .

Again, for a surface distribution of charge the numerical factors are changed as before.

If we put $\delta = -ua$ in the expansions for X , Y , and Z we obtain the values given in (21), thus affording a verification of the various calculations.

6. *Quasi-Stationary Motion and Electromagnetic Mass.*

The most interesting application of the above results is to the case of quasi-stationary motion; in this case the accelerations (p , q , r) are small, and the values of X , Y , and Z for Lorentz electrons ($\delta = 0$) are given by

$$Y, Z = -(q, r) \frac{e_1^2 \epsilon}{5\pi k a} \quad [\text{from (20)}],$$

and

$$\begin{aligned} X &= {}_aX_1 = -\frac{2ae_1^2}{5\pi k^2 a} \quad [\text{from (22)}] \\ &= -\frac{pe_1^2 \epsilon^3}{5\pi k a}, \end{aligned}$$

in terms of the acceleration (p , q , r).

The external force necessary in order to make the electron describe its path must be equal and opposite to the force exerted on the electron by the æther. If X_e , Y_e , Z_e are the components of this external force,

$$X_e = -X, \quad Y_e = -Y, \quad Z_e = -Z.$$

Hence

$$X_e = \frac{e_1^2}{5\pi k a} p \epsilon^3, \quad Y_e, Z_e = (q, r) \frac{e_1^2 \epsilon}{5\pi k a},$$

so that the external force is *to this order* proportional to the acceleration, the coefficients

$$\frac{e_1^2}{5\pi k a} (\epsilon^3, \epsilon, \epsilon) \quad [\epsilon^2 \equiv (1-u^2)^{-1}],$$

being, therefore, interpreted as electromagnetic masses. When $k = 1$ these values agree with those given by Lorentz and Abraham.

Lorentz at first retains the constant k , afterwards seeking to show by the comparison of two expressions for the electromagnetic mass that it has the value unity. This proof cannot be considered as necessarily correct, since it rests on the general quasi-stationary principle that the quantities

involved, in this case the "electromagnetic momentum," can be calculated for the uniform motion and not for the variable motion.*

It has been suggested by Bucherer that an electron has the same volume whether at rest or in motion. For *uniform* motion this would give $k = \epsilon^{\frac{1}{2}}$, and, using this value of k , we obtain the values for the electromagnetic masses given by him.†

It does, not, however, appear possible to extend this idea of constant volume to the general case of accelerated motion. If $dx dy dz = dx_1 dy_1 dz_1$ we must have $\rho = \rho_1$ from (17), and hence from (9) and (14) $(1 - w^2)^{\frac{1}{2}} = \lambda^3$, or $\lambda^4 \frac{\partial t}{\partial t_1} = 1$, which cannot be satisfied for all values of x_1, y_1, z_1, t_1 , even for the infinitesimal transformation of § 2, in which $\lambda \frac{\partial t}{\partial t_1} = 1$.

For surface distribution of charge the factor $\frac{1}{k}$ in the electromagnetic mass must be replaced by $\frac{1}{\epsilon}$. For higher powers of the accelerations each component of the external force will still be proportional to the acceleration in its own direction, the coefficients being, however, functions of the accelerations in all three directions of the axes.

In the case of the non-expanding electrons defined by $\delta + u\alpha = 0$ we obtain from (21) the accurate values of X, Y, Z . In terms of the acceleration (p, q, r) we therefore have

$$\left. \begin{aligned} X_e &= \frac{p\epsilon e_1^2}{\pi k} \left\{ \frac{1}{5a} + \frac{a}{7} \frac{\epsilon^4}{12} [(3 - 2u^2)p^2 + (3 + u^2)(q^2 + r^2)] \right\} \\ Y_e, Z_e &= (q, r) \frac{\epsilon e_1^2}{\pi k} \left\{ \frac{1}{5a} + \frac{a\epsilon^4}{7 \cdot 24} [(6 - 7u^2)p^2 + (6 - u^2)(q^2 + r^2)] \right\} \end{aligned} \right\}. \quad (24)$$

So that to the *first order* in (p, q, r) the external force is again proportional to the acceleration, the coefficients for the three components being, however, in this case the same, viz.,

$$\frac{e_1^2 \epsilon}{5\pi a k},$$

for a volume distribution of charge.

There is, therefore, for quasi-stationary motion no distinction in the

* Lorentz, *Theory of Electrons*, pp. 205 ff. Also §§ 27 and 37.

† *Mathematische Einführung in die Elektronentheorie*, Leipzig, 1904, pp. 57, 58. See also Lorentz, *op. cit.*, pp. 205 ff.

case of non-expanding electrons ($\delta + u\alpha = 0$) between longitudinal and transverse mass, as is the case for Lorentz electrons ($\delta = 0$); this is another feature of special interest attaching to the electrons in which $\delta + u\alpha$ vanishes.

In the general case from (24) we see that the components of the external forces are still proportional to the component accelerations; but the coefficients depend on the acceleration and are different along the direction of motion and perpendicular to it.

7. *The Equation of Energy and the Rate of Work done by the Electromagnetic Forces.*

It seems worth while to notice that, in the treatment of the theory of electrons as given by Lorentz, Abraham, and others, where the circuital relations form the basis of the theory, together with an expression for the electrodynamic force on a unit charge, the energy and the radiation are subsidiary quantities. In fact, the rate at which work is done by the electrodynamical forces is expressed by means of the circuital relations as the sum of a certain surface integral and the rate of decrease of a certain volume integral. The former integral is interpreted as rate of radiation of energy, and the latter as electromagnetic energy.

This point is clearly brought out in the above transformation theory, where it is found to be impossible to obtain invariants which separate the electromagnetic energy from the radiation.

The equation of energy in the usual form, viz.,

$$\iiint \rho (w, F) dx dy dz = - \frac{d}{dt} \iiint \frac{1}{2} (E^2 + H^2) dx dy dz \\ - \iint (lS_x + mS_y + nS_z) d\sigma,$$

where S_x, S_y, S_z are the components of the Poynting vector $[E, H]$, is a direct consequence of the circuital relations and the definition of F . We can add on to the Poynting vector dU/dt , where U is any vector, and subtract $\frac{d}{dt} \text{div } U$ from the electromagnetic energy $\frac{1}{2}(E^2 + H^2)$ without altering the whole equation. In particular, if $U = [A, H]$, where $\text{curl } A \equiv H$, we should find Macdonald's expression for the flow of energy based on Maxwell's first expression for the electromagnetic energy.*

* Macdonald, *Electric Waves*, Chapter VIII.

Similar considerations apply to the total force on a system of electrons expressed in terms of the Maxwell stress system and the rate of decrease of a volume integral, this latter being interpreted as a momentum.

When the circuital relations are satisfied in the x_1, y_1, z_1, t_1 space the corresponding relations in the x, y, z, t space are also satisfied, so that the energy equation in this latter space is also true. Since for a system at rest in the former space, as in the case here contemplated, the energy equation is certainly true, it follows that for the moving system in the x, y, z, t space the equation is also satisfied. Thus there is no need to assume the existence of any energy other than electromagnetic, as done by Abraham and Lorentz. This general conclusion could be verified in any special case by a calculation of the values of the electromagnetic energy, rate of radiation, and rate of work done by the electromagnetic forces, though, except in special cases, it would be a long piece of work to calculate the two former quantities. Here we shall restrict ourselves to the calculation of the rate of work done by the electromagnetic forces, which we called $\frac{dA}{dt}$ and which is given by the formulæ of § 3, deduced from the theory of integral invariants.

We have

$$\begin{aligned} \frac{dA}{dt} &= \iiint \rho dx dy dz (w, F) \\ &= - \iiint \rho_1 dx_1 dy_1 dz_1 \left(\frac{\partial t}{\partial t_1} \right)^{-1} \left[E_{1x} \frac{\partial x_1}{\partial t} + E_{1y} \frac{\partial y_1}{\partial t} + E_{1z} \frac{\partial z_1}{\partial t} \right] \\ &= \iiint \frac{\rho_1^2}{3} \frac{dx_1 dy_1 dz_1 D_1 \left(1 - \frac{b^2}{r_1^2} \right) \lambda^2 \left(x_1 \frac{\partial t}{\partial x_1} + y_1 \frac{\partial t}{\partial y_1} + z_1 \frac{\partial t}{\partial z_1} \right)}{k \epsilon \sqrt{\{ 1 - 4(\delta + u\alpha) u x_1 + 4(\delta + u\alpha)^2 r_1^2 \}}} . \end{aligned}$$

Now $\lambda^2 \left(x_1 \frac{\partial t}{\partial x_1} + y_1 \frac{\partial t}{\partial y_1} + z_1 \frac{\partial t}{\partial z_1} \right)$ for $t = 0$

is equal to $\frac{\epsilon D_1}{k} [u x_1 - 2(\delta + u\alpha) r_1^2],$

since

$$k\lambda = D_1,$$

so that

$$\frac{dA}{dt} = \iiint \frac{\rho_1^2}{3} \frac{dx_1 dy_1 dz_1 D_1^2 \left(1 - \frac{b^2}{r_1^2} \right) [u x_1 - 2(\delta + u\alpha) r_1^2]}{k^2 \sqrt{\{ 1 - 4(\delta + u\alpha) u x_1 + 4(\delta + u\alpha)^2 r_1^2 \}}} ,$$

where, again, t_1 is to be expressed in terms of x_1, y_1, z_1 by (18) before the integration is performed.

For non-expanding electrons ($\delta + u\alpha = 0$) the integration is quite simple, giving in terms of α , β , γ

$$\frac{dA}{dt} = -\frac{e_1^2}{\pi k^2} u\alpha(1-u^2) \left[\frac{1}{5a} + \frac{a}{7} \frac{(3-u^2)}{3} \{\beta^2 + \gamma^2 + \alpha^2(1-u^2)\} \right] \quad (25)$$

for a volume distribution of charge.

In the general case we may expand in powers of α and δ , so that

$$\frac{dA}{dt} = \alpha \frac{dA_1}{dt} + \delta \frac{dA'}{dt} + \frac{1}{2} \left(\alpha^2 \frac{dA_2}{dt} + 2\alpha\delta \frac{d\bar{A}}{dt} + \delta^2 \frac{dA''}{dt} \right) + \dots,$$

and we find

$$\left. \begin{aligned} \frac{dA_1}{dt} &= -\frac{ue_1^2}{2\pi k^2} \left[\frac{(5-u^2)}{5a} + \frac{4}{3} (\beta^2 + \gamma^2)(3-u^2)^2 \frac{a}{7} \right. \\ &\quad \left. + \frac{1}{3} (\beta^2 + \gamma^2)^2 (1-u^2)^2 (3-u^2)^2 \frac{a^3}{9} \right] \\ \frac{dA'}{dt} &= -\frac{e_1^2}{2\pi k^2} \left[\frac{(3+u^2)}{5a} + \frac{2}{3} (\beta^2 + \gamma^2)(3-u^2)(5-u^2) \frac{a}{7} \right. \\ &\quad \left. + \frac{1}{3} (\beta^2 + \gamma^2)^2 (3-u^2)^2 (1-u^2) \frac{a^3}{9} \right] \end{aligned} \right\}, \quad (26)$$

while

$$\frac{dA_2}{dt} = \frac{d\bar{A}}{dt} = \frac{dA''}{dt} = 0.$$

If we make $\delta = -u\alpha$ in the expansion for $\frac{dA}{dt}$, we may verify the result of (25) up to the first order in α .

It will be noticed that in all the values found for the forces X , Y , Z and the rate of work done $\frac{dA}{dt}$ there is no term independent of the radius a of the electron.

There is thus no term in the expression for the force corresponding to the radiation pressure as obtained by Abraham, nor a term in the expression for the rate of work done corresponding to energy lost by radiation, since both these terms are independent of the dimensions of the electron and of the distribution of charge.*

These results, of course, apply only to the particular case of accelerated motion considered, but suggest that there are fundamental differences between accelerated motion of a deformable electron, *i.e.*, accelerated motion according to the Theory of Relativity, and accelerated motion treated from other standpoints.

* Cf. Schott, *Electromagnetic Radiation*, Appendix D, §§ 225, 231, 235.