A GENERAL THEOREM ON INTEGRAL FUNCTIONS OF FINITE ORDER

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[Received October 30th, 1907.—Read November 14th, 1907.—Revised December, 1907.]

1. In a memoir* published in 1903, Prof. Wiman put forward the following conjecture.

Let $F(z)$ be any integral function of order ρ less than $\frac{1}{2}$; then, if ϵ be an arbitrarily small positive number, there are values of r as large as we please, such that, for all points z of the circle $|z| = r$,

$$|F(z)| > \exp(r^{\rho-\epsilon}).$$

In the present paper I give a proof of an analogous theorem, expressing a relation between the maximum and the minimum modulus of $F(z)$ on the circle $|z| = r$. This theorem is given in § 4; the two articles now following are devoted to the proofs of certain results which are required in the proof of the main theorem.

2. LEMMA.—Let $\beta_1, \beta_2, \ldots$ be a sequence of real positive numbers such that for all values of s, $\beta_{s+1} \geqslant \beta_s$, and such that

$$\operatorname*{Lt}_{s=\infty} \beta_s s^{-2} = \infty.$$

We define the function $\beta(x)$, for values of x greater than 1, to be $\beta_s + (\beta_{s+1} - \beta_s)(x - s)$, when $s \leqslant x \leqslant s+1$. Then $\beta(x)$ is a continuous non-decreasing† function of x. It is easily seen, moreover, that

$$\operatorname*{Lt}_{x=\infty} \beta(x)\, x^{-2} = \infty. \tag{1}$$

Let us denote the curve $y = \beta(x)$ by C. This curve C starts at the point $[1, \beta(1)]$.

We shall show that, if any positive number h, however large, be

* A. Wiman, *Arkiv. för Math. Astr. och Fysik*, Band I., 1903. "Ueber die angenäherte Darstellung von ganzen Funktionen."

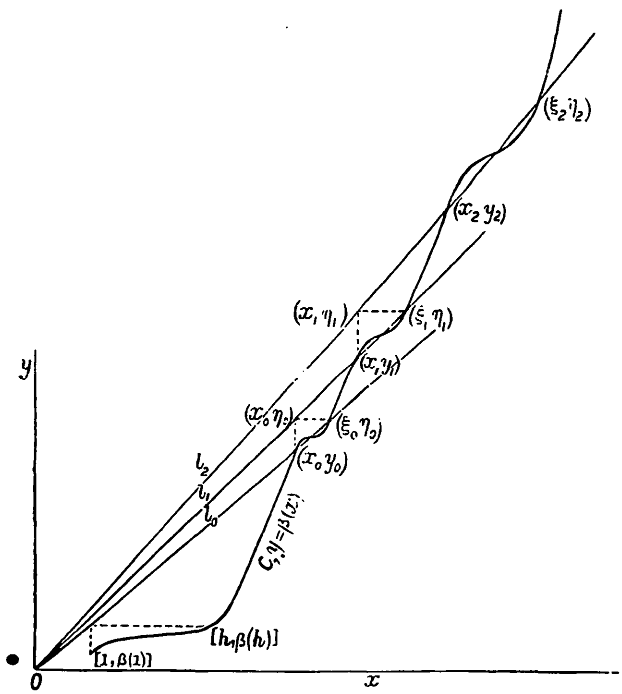
† By this phrase I mean that $\beta(x+y) \geqslant \beta(x)$, when $y > 0$.

assigned, there is a ray l through the origin lying in the first quadrant, such that if (X, Y) be that intersection of C and l which is nearest to the origin, the following state of things obtains.

(i.) $X > h$.

(ii.) When $x < X$, $Yx/X > \beta(x)$.

(iii.) When $x > X + \frac{1}{2}$, $Yx/X < \beta(x)$.



In the first place, if l, $y = mx$, be any ray* which passes above the point $[1, \beta(1)]$, there is at least one intersection of l and C. For the distant part of C is clearly above l [since $\underset{x=\infty}{\text{Lt}}\, \beta(x)\, x^{-2} = \infty$, and consequently $\beta(x) > mx$ when x is sufficiently large], while the point $[1, \beta(1)]$ of C is below l.

Now let us suppose that, for a given h, there is no ray l such that (i.), (ii.), and (iii.) hold simultaneously. Then it is clear from the figure that the following must be the state of things.

Let l be any ray which passes above $[1, \beta(1)]$, and let (X, Y) be the

* I use the word "ray" throughout to mean a ray through the origin, lying in the first quadrant, and not coincident with the axis of y.

intersection of l and C which is nearest to the origin. Then, if $X > h$, there is a second intersection (X', Y') of l and C such that $X' > X+\frac{1}{2}$.

This result is evident when we observe that l is the ray through (X, Y), so that Yx/X is the ordinate of the point of l whose abscissa is x; that the part of the curve C for which $1 \leqslant x < X$ is below the corresponding part of l, while the distant part of C is above the corresponding part of l; and when we remember that if (i.) and (ii.) are true, on our present hypothesis (iii.) must be false.

We shall now show that the existence of this point $(X'Y')$, which we shall call the result (A), leads to a consequence incompatible with (1).

Let l_0 be the ray through $[1, \beta(h)]$. Since $\beta(x)$ is a non-decreasing function, it is evident from the figure that if $(x_0 y_0)$ be the intersection of l_0 and C which is nearest to the origin, we have $x_0 > h$.

Then, by the result (A), there must be a second intersection (ξ_0, η_0) of l_0 and C, such that $\xi_0 - x_0 > \frac{1}{2}$.

Let l_1 be the ray through $(x_0 \eta_0)$, and let $(x_1 y_1)$ be its intersection with C which is nearest to the origin.

It is again evident, from the figure and from the fact that $\beta(x)$ is a non-decreasing function, that $x_1 > \xi_0$.

Hence, by (A), there is a second intersection $(\xi_1 \eta_1)$ of l_1 and C, such that $\xi_1 - x_1 > \frac{1}{2}$.

Let l_2 be the ray through $(x_1 \eta_1)$, and let $(x_2 y_2)$ be its intersection with C nearest to the origin, and so on.

Then we have the following relations :—

$$\xi_n - x_n > \tfrac{1}{2}, \quad \xi_n > x_n > \xi_{n-1}, \tag{2}$$

and, since $(x_{n-1} \eta_{n-1})$, $(x_n y_n)$, $(\xi_n \eta_n)$ lie on the ray l_n,

$$\frac{\eta_{n-1}}{x_{n-1}} = \frac{y_n}{x_n} = \frac{\eta_n}{\xi_n}. \tag{3}$$

Hence we have

$$\begin{aligned}
\frac{\eta_n}{\xi_n} &= \frac{\eta_{n-1}}{\xi_{n-1}} \frac{\xi_{n-1}}{x_{n-1}} \\
&= \frac{\eta_{n-2}}{\xi_{n-2}} \frac{\xi_{n-2}}{x_{n-2}} \frac{\xi_{n-1}}{x_{n-1}} \\
&\cdots \quad \cdots \quad \cdots \quad \cdots \\
&= \frac{\eta_0}{\xi_0} \frac{\xi_0}{x_0} \cdots \frac{\xi_{n-2}}{x_{n-2}} \frac{\xi_{n-1}}{x_{n-1}} \\
&= \frac{\eta_0}{\xi_0 x_0} \frac{\xi_0}{x_1} \frac{\xi_1}{x_2} \cdots \frac{\xi_{n-2}}{x_{n-1}} \xi_{n-1} \\
&< \frac{\eta_0}{\xi_0 x_0} \xi_n, \quad \text{by means of (2).}
\end{aligned}$$

Thus $$\eta_n < [\eta_0/(\xi_0 x_0)]\, \xi_n^2. \tag{4}$$

Now $$\xi_n > x_n + \tfrac{1}{2} > \xi_{n-1} + \tfrac{1}{2} > \dots > \frac{n-1}{2},$$

so that by sufficiently increasing n we can make ξ_n as large as we please.

Again, x_0, ξ_0, η_0 are finite constants depending on h, and $(\xi_n \eta_n)$ is a point of the curve C.

But then the result (4) is incompatible with (1). Hence the result (A) is impossible, and the Lemma is proved.

3. We shall require the two following definite integrals :—

I. If $0 < \lambda < 1$,

$$\int_0^1 \log\left(\frac{1}{1-x}\right)(x^\lambda + x^{-\lambda})\frac{\lambda\, dx}{x} = 1/\lambda - \pi \cot(\pi\lambda).$$

II. If $0 < \lambda < 1$,

$$\int_0^1 \log(1+x)(x^\lambda + x^{-\lambda})\frac{\lambda\, dx}{x} = \pi \operatorname{cosec}(\pi\lambda) - 1/\lambda.$$

The integrals are easily seen to be convergent.

Consider the first integral. We have

$$I_1 = \int_0^1 \log\left(\frac{1}{1-x}\right)(x^\lambda + x^{-\lambda})\frac{\lambda\, dx}{x} = \int_0^{1-\theta} \log\left(\frac{1}{1-x}\right)(x^\lambda + x^{-\lambda})\frac{\lambda\, dx}{x} + \epsilon(\theta), \tag{1}$$

where θ is a small positive number, and where

$$\underset{\theta=0}{\mathrm{Lt}}\, |\,\epsilon(\theta)\,| = 0.$$

The first term in the above expression

$$= \int_0^{1-\theta}\left[\sum_{s=1}^{\infty}\frac{x^{\lambda+s-1}}{s} + \sum_{s=1}^{\infty}\frac{x^{-\lambda+s-1}}{s}\right]\lambda\, dx$$

$$= \sum_{s=1}^{\infty}\int_0^{1-\theta}\frac{x^{\lambda+s-1}}{s}\lambda\, dx + \sum_{s=1}^{\infty}\int_0^{1-\theta}\frac{x^{-\lambda+s-1}}{s}\lambda\, dx$$

(since the infinite series are uniformly convergent when $x \leqslant 1-\theta$)

$$=\sum_{s=1}^{\infty} \frac{\lambda}{s(s+\lambda)}(1-\theta)^{\lambda+s}+\sum_{s=1}^{\infty} \frac{\lambda}{s(s-\lambda)}(1-\theta)^{-\lambda+s}.$$

Now $\Sigma\lambda/s(s\pm\lambda)$ is convergent.

Hence, by a well known theorem of Abel, the above expression

$$=\sum_{s=1} \frac{\lambda}{s(s+\lambda)}+\sum_{s=1}^{\infty} \frac{\lambda}{s(s-\lambda)}+\epsilon'(\theta), \tag{2}$$

where $$\operatorname*{Lt}_{\theta=0} |\epsilon'(\theta)| = 0.$$

From (1) and (2) we see that we must have

$$\begin{aligned} I_1 &= \sum_{s=1}^{\infty}\left[\frac{\lambda}{s(s+\lambda)}+\frac{\lambda}{s(s-\lambda)}\right] \\ &= -\sum_{s=1}^{\infty}\left(\frac{1}{s+\lambda}+\frac{1}{\lambda-s}\right) \\ &= \frac{1}{\lambda}-\pi\cot(\pi\lambda). \end{aligned}$$

In a similar manner we obtain

$$\begin{aligned} I_2 &= \sum_{s=1}^{\infty}(-)^{s-1}\left[\frac{1}{s+\lambda}+\frac{1}{\lambda-s}\right] \\ &= \pi\operatorname{cosec}\pi\lambda-\frac{1}{\lambda}. \end{aligned}$$

4. THEOREM.—*Let $F(z)$ be an integral function of order ρ, where $\frac{1}{2} > \rho \geqslant 0$. Let $M(r)$, $m(r)$ denote the maximum and the minimum moduli of $F(z)$ on the circle $|z| = r$.*

Then there is a sequence $r_1, r_2, \ldots$, where $r_1 < r_2 < \ldots$, and $\operatorname*{Lt}_{s=\infty} r_s = \infty$, with the following properties.

If ϵ be any assigned positive number there is a finite μ, such that when $s > \mu$, we have

$$m(r_s) > [M(r_s)]^{\cos(2\pi\rho)-\epsilon}.$$

Moreover the sequence $r_1, r_2, \ldots$ is independent of the arguments of the zeros of $F(z)$, depending only on their moduli. Thus, if $F_1(z)$ be any other integral function of order ρ, the sequence of the moduli of whose zeros is the same as the corresponding sequence for $F(z)$, we have, when $s >$ a finite μ',

$$m_1(r_s) > [M_1(r_s)]^{\cos(2\pi\rho)-\epsilon}.$$

Further, when $s >$ *a finite* μ'', *we have*

$$m(r_s) > [M_1(r_s)]^{\cos(2\pi\rho)-\epsilon}, \qquad m_1(r_s) > [M(r_s)]^{\cos(2\pi\rho)-\epsilon}.$$

Suppose first that $$\tfrac{1}{2} > \rho > 0.$$

Let $$F(z) = Cz^p . \prod_{s=1}^{\infty} (1+z/a_s),$$

when a_1, a_2, ... are arranged in order of non-decreasing moduli. Let $|a_s| = \alpha_s$. Let ϵ_1 be a (small) positive number less than $\frac{1}{2}-\rho$, which we shall presently choose suitably.

Let $$\beta_s = \alpha_s^{2(\rho+\epsilon_1)} = \alpha_s^{2\rho'}.$$

Since $F(z)$ is of order ρ, we have

$$\operatorname*{Lt}_{s=\infty} [\alpha_s s^{-1/(\rho+\epsilon_1)}] = \infty.$$

Hence $$\operatorname*{Lt}_{s=\infty} \beta_s s^{-2} = \infty.$$

Moreover β_s is clearly a non-decreasing function of s. We define the function $\beta(x)$ as in the Lemma, and then the result of the Lemma holds for this function.

Let $$\alpha(x) = [\beta(x)]^{1/2(\rho+\epsilon_1)} = [\beta(x)]^{1/2\rho'}.$$

Then $\alpha(s) = \alpha_s$, and $\alpha(x)$ is a non-decreasing function.

Now, let h be a (large) positive number to be chosen presently. Then by the Lemma there is a ray $y = \mu x$, such that, if (X, Y) be the intersection with $y = \beta(x)$ which is nearest to the origin, we have

$$\beta(x) < x\beta(X)/X, \quad \text{when } x < X;$$

and $$\beta(x) > x\beta(X)/X, \quad \text{when } x > X+\tfrac{1}{2}.$$

Choose $$r = |z| = [X''\beta(X)/X]^{1/2\rho'},$$

where $$X+\tfrac{1}{2} \geqslant X'' \geqslant X,$$

and where we shall determine X'' more precisely later.

Let m be the greatest integer not greater than X. Then we have

$$\log F(z) = \log|C| + p\log r + \sum_{s=1}^{\infty} \log\left|1+\frac{z}{a_s}\right|,$$

and $$\log F(z) = P+R+S+T_1+T_2+T_3+[\log|C|+p\log r],$$

where*

$$\left.\begin{aligned}
P &= \log\left|\frac{z^m}{a_1 a_2 \ldots a_m}\right| = \log\left[\frac{r^m}{a_1 a_2 \ldots a_m}\right]\\
R &= \sum_{s=3}^{\infty} \log\left|1+\frac{z}{a_{m+s}}\right|\\
S &= \sum_{s=1}^{m-1} \log\left|1+\frac{a_s}{z}\right|\\
T_1 &= \log\left|1+\frac{z}{a_{m+1}}\right|\\
T_2 &= \log\left|1+\frac{z}{a_{m+2}}\right|\\
T_3 &= \log\left|1+\frac{a_m}{z}\right|
\end{aligned}\right\} \tag{1}$$

We proceed to find a lower limit for P, and upper limits for R, $|R|$, S, $|S|$, ...

By the result of the Lemma, when $s \leqslant m < X$, we have

$$\beta_s = \beta(s) < \frac{s}{X}\beta(X).$$

Hence

$$a_s < \left(\frac{s}{X''}\right)^{1/(2\rho')}\left[\frac{X''}{X}\beta(X)\right]^{1/(2\rho')} < r\left(\frac{s}{X''}\right)^{\sigma'}, \tag{2}$$

where

$$\sigma' = 1/(2\rho').$$

Again, when $s \geqslant m+3 \geqslant X+\frac{1}{2}$, we have $\beta_s > s\beta(X)/X$, and hence

$$a_s > r[s/X'']^{\sigma'}. \tag{3}$$

First consider P. We have

$$\begin{aligned}
P &= \sum_{s=1}^{m} \log\left(\frac{r}{a_s}\right)\\
&> \sum_{s=1}^{m} \log\left(\frac{X''}{s}\right)^{\sigma'} \quad \text{[from (2)]}\\
&> \sigma'[m\log m - \log(m!)] \quad (\text{since } X'' > m)\\
&> \sigma'\Big[m\log m - \{m\log m - m[(1+\epsilon(m))]\}\Big]^{\dagger}\\
&> \sigma'.m[1+\epsilon(m)].
\end{aligned} \tag{4}$$

* The idea of dividing $\log F(z)$ into (practically) the three parts P, R, S, was employed by Wiman (*loc. cit.*) in the particular case $F(z) = \Pi(1+z/n^\rho)$.

† I shall always use $\epsilon(m)$, $\epsilon(x)$, ... for functions which tend to zero as their argument tends to its limit. I shall, moreover, use the same symbol for all the functions of this type with the same argument. The symbol $\epsilon(m)$ may be considered as an abbreviation for "some function which tends to zero with $1/m$."

Next consider R and S. When $s \geqslant 3$,

$$\left|\frac{z}{a_{m+s}}\right| = \frac{r}{a_{m+s}} < \left[\frac{X''}{m+s}\right]^{\sigma'} \text{ [from (3)]}$$

$$< \left[\frac{m+2}{m+s}\right]^{\sigma'}.$$

Hence $$R = \sum_{s=3}^{\infty} \log\left|1+\frac{z}{a_{m+s}}\right|$$

$$\leqslant \sum_{s=3}^{\infty} \log\left[1+\left|\frac{z}{a_{m+s}}\right|\right] \text{ (algebraically), since } \left|\frac{z}{a_{m+s}}\right| < 1,$$

$$< \sum_{s=3}^{\infty} \log\left[1+\left(\frac{m+2}{m+s}\right)^{\sigma'}\right]$$

$$< \sum_{s=3}^{\infty} \left\{\int_{s-1}^{s} \log\left[1+\left(\frac{m+2}{m+x}\right)^{\sigma'}\right] dx\right\}$$

$$< \int_{2}^{\infty} \log\left[1+\left(\frac{m+2}{m+x}\right)^{\sigma'}\right] dx$$

$$< \int_{0}^{1} \log(1+t)\,\frac{m+2}{\sigma'}\,t^{-1-1/\sigma'}\,dt, \text{ on putting } \left(\frac{m+2}{m+x}\right)^{\sigma'} = t. \quad (5)$$

Again, when $s \leqslant m-1$,

$$\left|\frac{a_s}{z}\right| = \frac{a_s}{r} < \left[\frac{s}{X''}\right]^{\sigma'} < \left[\frac{s}{m}\right]^{\sigma'}.$$

Hence $$S = \sum_{s=1}^{m-1} \log\left[1+\frac{a_s}{z}\right]$$

$$\leqslant \sum_{s=1}^{m-1} \log\left[1+\frac{a_s}{r}\right] \text{ (algebraically)}$$

$$< \sum_{s=1}^{m-1} \log\left[1+\left(\frac{s}{m}\right)^{\sigma'}\right]$$

$$< \sum_{s=1}^{m-1} \left\{\int_{s}^{s+1} \log\left[1+\left(\frac{x}{m}\right)^{\sigma'}\right] dx\right\}$$

$$< \int_{0}^{m-1} \log\left[1+\left(\frac{x}{m}\right)^{\sigma'}\right] dx$$

$$< \int_{0}^{1} \log(1+t)\,\frac{m}{\sigma'}\,t^{-1+1/\sigma'}\,dt, \text{ on putting } \left(\frac{x}{m}\right)^{\sigma'} = t,$$

$$< \frac{m+2}{\sigma'} \int_{0}^{1} \log(1+t)\,t^{-1+1/\sigma'}\,dt. \quad (6)$$

From (5) and (6) we have

$$R+S < (m+2)\int_0^1 \log(1+t)(t^{1/\sigma'}+t^{-1/\sigma'})\,\frac{1/\sigma'\,dt}{t}$$

$$< (m+2)\left(\pi \operatorname{cosec}\frac{\pi}{\sigma'} - \sigma'\right)^{*}, \tag{7}$$

by the second result of § 3, since $1/\sigma' < 1$.

Again, we have

$$R > \sum_{s=3}^{\infty}\left\{-\log\left(1-\frac{r}{a_{m+s}}\right)\right\} \quad \text{(algebraically)}$$

$$> \sum_{s=3}^{\infty}\left\{-\log\left[1-\left(\frac{m+2}{m+s}\right)^{\sigma'}\right]\right\}$$

$$> \sum_{s=3}^{\infty}\left\{\int_{s-1}^{s}\log\left[1-\left(\frac{m+2}{m+x}\right)^{\sigma'}\right]^{-1} dx\right\}$$

$$> -\int_0^1 \log(1-t)^{-1}\,\frac{m+2}{\sigma'}\,t^{-1/\sigma'-1}\,dt, \quad \text{on putting } \left(\frac{m+2}{m+x}\right)^{\sigma'} = t.$$

Similarly $$S > -\int_0^1 \log(1-t)^{-1}\,\frac{m}{\sigma'}\,t^{1/\sigma'-1}\,dt,$$

and hence $$R+S > -(m+2)\int_0^1 \log(1-t)^{-1}(t^{1/\sigma'}+t^{-1/\sigma'})\,\frac{1/\sigma'\,dt}{t}$$

$$> -(m+2)\left(\sigma' - \pi\cot\frac{\pi}{\sigma'}\right), \tag{8}$$

by the second result of § 3.

We shall now show that it is possible to choose X'' subject to the limitations so far imposed, and such that

$$[T_1+T_2+T_3+|\log|C||+p\log r]/P = \epsilon(m). \tag{9}$$

We have $$m \leqslant X < X+\tfrac{1}{2} < m+2. \tag{10}$$

Now, so far we have only restricted X'' to lie between X and $X+\frac{1}{2}$. Thus r, or $[X''\beta(X)/X]^{\sigma'}$, may vary between

$$[\beta(X)]^{\sigma'} \quad \text{and} \quad [1+1/(2X)]^{\sigma'}[\beta(X)]^{\sigma'}.$$

Moreover a_m and a_{m+2} do not lie between these limits, though a_{m+1} may possibly do so. These results follow at once from (10).

* The results (7) and (8) were suggested by the form of the asymptotic expansion of Barnes's function $P_\rho(z)$.

Hence it is possible to choose X'', within the limits already prescribed, such that

$$|r-a_{m+p}| > \tfrac{1}{2}\left\{[\beta(X)]^{\sigma'}\left(1+\frac{1}{2X}\right)^{\sigma'}-[\beta(X)]^{\sigma'}\right\} \quad (p=0,\,1,\,2)$$

$$> \frac{\sigma'}{4X}[\beta(X)]^{\sigma'} \quad (\text{since } \sigma' > 1). \tag{11}$$

Suppose that this is done. Then, since $|a_m/z| < 1$, we have

$$|T_3| = \left|\log\left|1+\frac{a_m}{z}\right|\right| \leqslant -\log\left(1-\frac{a_m}{r}\right)$$

$$\leqslant \log\left(\frac{r}{r-a_m}\right)$$

$$< \log\left[\left(\frac{X''}{X}\right)^{\sigma'}\frac{4X}{\sigma'}\right] \quad [\text{from (11), substituting for } r]$$

$$< \log\left[\frac{4(m+1)}{\sigma'}\left(1+\frac{1}{2m}\right)^{\sigma'}\right]$$

$$= m\epsilon(m).$$

Hence, from (4), $\qquad |T_3|/P = \epsilon(m).$ (12)

Next consider $\qquad |T_1| = \left|\log\left|1+\dfrac{z}{a_{m+1}}\right|\right|.$

(i.) Suppose $\qquad \left|\dfrac{z}{a_{m+1}}\right| \leqslant \tfrac{1}{2}.$

Then $\qquad |T_1| \leqslant \log\left(\dfrac{1}{1-\frac{1}{2}}\right) \leqslant \log 2$

$$= m\,.\,\epsilon(m).$$

(ii.) Suppose $\qquad \tfrac{1}{2} < \left|\dfrac{z}{a_{m+1}}\right| < 1.$

Then $\qquad |T_1| \leqslant -\log\left(1-\dfrac{r}{a_{m+1}}\right)$

$$\leqslant \log\left(\frac{a_{m+1}}{a_{m+1}-r}\right) < \log\left(\frac{2r}{a_{m+1}-r}\right)$$

$$< \log\left[2\left(\frac{X''}{X}\right)^{\sigma'}\frac{4X}{\sigma'}\right] \quad [\text{by (11)}]$$

$$= m\,.\,\epsilon(m),$$

as in the case of $|T_3|$. Hence in cases (i.) and (ii.),

$$|T_1|/P = \epsilon(m). \tag{13$_1$}$$

(iii.) The relation $|z/a_{m+1}| = 1$ is impossible on account of (11). Suppose, then, that $|z/a_{m+1}| > 1$. Then

$$|T_1| \leqslant \log\left|\frac{z}{a_{m+1}}\right| + \left|\log\left|1+\frac{a_{m+1}}{z}\right|\right|.$$

By means of (11) we prove, as in the case of $|T_3|$, that

$$\left|\log\left|1+\frac{a_{m+1}}{z}\right|\right| \Big/ P = \epsilon(m).$$

Also
$$\log\left|\frac{z}{a_{m+1}}\right| \Big/ P = \frac{\log\left|\dfrac{z}{a_{m+1}}\right|}{\sum\limits_{s=1}^{m} \log\left|\dfrac{z}{a_s}\right|} < \frac{1}{m} = \epsilon(m).$$

Hence
$$|T_1|/P = \epsilon(m), \tag{13$_2$}$$

and hence, in any case,
$$|T_1|/P = \epsilon(m). \tag{13}$$

Consider now
$$|T_2| = \log\left|1+\frac{z}{a_{m+2}}\right|.$$

(i.) If $\left|\dfrac{z}{a_{m+1}}\right| \leqslant \frac{1}{2}$,

$$|T_2| \leqslant \log 2 \quad \text{and} \quad |T_2|/P = \epsilon(m).$$

(ii.) If $\frac{1}{2} < \left|\dfrac{z}{a_{m+2}}\right| < 1$, then, as in the case (ii.) for $|T_1|$, we obtain, by means of (11),
$$|T_2|/P = \epsilon(m).$$

Hence, in any case,
$$|T_2|/P = \epsilon(m). \tag{14}$$

Finally, when r is large,

$$P = \log\left(\frac{r^m}{a_1 a_2 \dots a_m}\right) > \log\left(\frac{r^{p/\epsilon}}{a_1 a_2 \dots a_{p/\epsilon}}\right)$$
$$> \frac{p}{\epsilon}\log r - \log(a_1 a_2 \dots a_{p/\epsilon})$$
$$> \frac{p}{\epsilon}\log r\,[1+\epsilon(r)],$$

however small ϵ may be.

Hence $$\{\,|\log|C||+p\log r\}/P=\epsilon(r)=\epsilon(m). \tag{15}$$

From (12), (13), (14), and (15) we obtain (9).

The first part of our theorem, for $\rho\neq 0$, now follows easily.

Denote by $[f(z)]_M$, $[f(z)]_m$, the maximum and the minimum algebraic values of the (real) expression $f(z)$, for all points z on the circle $|z|=r$. Then we have

$$\log M(r)=P+[R+S]_M+[T_1+T_2+T_3+\log|C|+p\log r]_M,$$

$$\log m(r)=P+[R+S]_m+[T_1+T_2+T_3+\log|C|+p\log r]_m.$$

From (7) and (8) we have

$$[R+S]_M<\quad(m+2)\left(\pi\operatorname{cosec}\frac{\pi}{\sigma'}-\sigma'\right),$$

$$[R+S]_m>-(m+2)\left(\sigma'-\pi\cot\frac{\pi}{\sigma'}\right).$$

Hence, by means of (10), noticing that $\left(\pi\operatorname{cosec}\frac{\pi}{\sigma'}-\sigma'\right)$ and $\left(\sigma'-\pi\cot\frac{\pi}{\sigma'}\right)$ are positive, we have

$$\frac{\log m(r)}{\log M(r)}>\frac{1-\dfrac{m+2}{P}\left(\sigma'-\pi\cot\dfrac{\pi}{\sigma'}\right)+\epsilon(m)}{1+\dfrac{m+2}{P}\left(\pi\operatorname{cosec}\dfrac{\pi}{\sigma'}-\sigma'\right)+\epsilon(m)}$$

$$>\frac{1-\dfrac{1}{\sigma'}\left(\sigma'-\pi\cot\dfrac{\pi}{\sigma'}\right)+\epsilon(m)}{1+\dfrac{1}{\sigma'}\left(\pi\operatorname{cosec}\dfrac{\pi}{\sigma'}-\sigma'\right)+\epsilon(m)}$$

[on replacing P by its minimum given by (4)]

$$>\cos\frac{\pi}{\sigma'}+\epsilon(m). \tag{16}$$

We can make m as large as we please, and consequently r as large as we please, by choosing h sufficiently large. Choose h so that m is so large that $\epsilon(m)<\epsilon_1$. Then

$$\frac{\log m(r)}{\log M(r)}>\cos\left(\frac{\pi}{\sigma'}\right)-\epsilon_1>\cos[2\pi(\rho+\epsilon_1)]-\epsilon_1.$$

Choose ϵ_1 so that

$$\cos[2\pi(\rho+\epsilon_1)]-\epsilon_1>\cos(2\pi\rho)-\epsilon.$$

(This can always be done, since $\rho < \frac{1}{2}$.) Then

$$\frac{\log m(r)}{\log M(r)} > \cos(2\pi\rho) - \epsilon,$$

and
$$m(r) > [M(r)]^{\cos(2\pi\rho)-\epsilon}. \tag{17}$$

We have therefore proved that when ϵ is assigned we can find a value of r, as large as we please, such that the above inequality holds.

Now, in the above analysis, the number r was determined quite independently of the arguments of a_1, a_2, ..., although it may depend on the constant C.

We may therefore determine our sequence r_1, r_2, ... as follows.

Determine, by the above methods, a number r_n, such that

$$r_n > r_{n-1} + 1,$$

and such that (17) holds for $r = r_n$ when $\epsilon = 2^{-n}$.

Then $\underset{n=\infty}{\text{Lt}}\, r_n = \infty$ and $m(r_s) > [M(r_s)]^{\cos(2\pi\rho)-\epsilon}$

when $s >$ some finite μ.

Again, P is independent of the arguments of z and of the zeros, while the limits which we found for $[R+S]_M$ and $[R+S]_m$ are independent of the arguments of the zeros. Moreover, the relation (9) always holds.

Then it follows, by considerations similar to those which gave us the result (16), that if

$$F_1(z) = Cz^p . \prod_{s=1}^{\infty} \left(1 + \frac{z}{a_s'}\right)$$

be a function with the same constant C, and with the same sequence of moduli of zeros as that of $F(z)$, then $\frac{\log m_1(r)}{\log M(r)}$ and $\frac{\log m(r)}{\log M_1(r)}$ are greater than $\cos\frac{\pi}{\sigma'} + \epsilon(m)$.

It follows, then, that

$$\left.\begin{aligned} m_1(r_s) &> [M(r_s)]^{\cos(2\pi\rho)-\epsilon} \\ \text{and} \qquad m(r_s) &> [M_1(r_s)]^{\cos(2\pi\rho)-\epsilon} \end{aligned}\right\}, \tag{18}$$

when $s > \mu$.

Finally, if the C and the p which occur in the product-forms for $F_1(z)$ be replaced by C' and p', it is easily seen that the inequalities (18) still hold when $s >$ some finite μ', which is possibly different from μ.

Thus we have proved our theorem for the case when $\rho \neq 0$.

Although the above analysis breaks down when $\rho = 0$, it is easy to see that the theorem remains valid for the limiting case, the main inequality taking the form*

$$m(r) > [M(r)]^{1-\epsilon}.$$

We have proved implicitly that the theorem stated at the beginning of the article holds when $F(z)$, instead of being of order ρ (ρ not zero), is of order less than ρ. Now a function of zero order is of order less than ϵ, and we can therefore determine a sequence of circles of radii $r_1, r_2, \ldots$, such that

$$m(r_s) > [M(r_s)]^{\cos(2\pi\epsilon)-\epsilon}, \quad \text{when} \quad s > \mu,$$

with the corresponding relations between $m_1(r_s)$, $M_1(r_s)$, ...

The theorem for the limiting case then follows immediately.

5. When $\rho < \frac{1}{4}$, we have $\cos 2\pi\rho > 0$.

It follows that, if $F(z)$ be an integral function of order less than $\frac{1}{4}$, then, on circles of radii as large as we please,

$$m(r) \gtrdot [M(r)]^c,$$

where

$$c > 0.$$

Now, if any number p be assigned,

$$\operatorname*{Lt}_{r=\infty} r^{-p/c} M(r) = \infty.$$

Hence $r^{-p}m(r)$ has an upper limit infinity, and as z tends to infinity along any radius vector through the origin, $|z^{-p}F(z)|$ must have infinity for its upper limit.

6. The theorem of § 4 is only provisional. I am convinced that it remains true when the condition $0 \leqslant \rho < \frac{1}{2}$ is replaced by $0 \leqslant \rho < 1$, and when the index $\cos(2\pi\rho)$ is replaced by $\cos(\pi\rho)$.

But, although it would seem that the expression which I have called P must play a prominent part in the proof, I believe that to establish the result an entirely new point of view is required.

The line of proof of this paper is based essentially on the Lemma. It might be thought that by further refinement the Lemma might be extended into a form which should enable us to deal with the case $\frac{1}{2} \leqslant \rho < 1$. This, however, is not the case: if, with the notation of the statement of the Lemma, $\beta(x)$ tends to infinity like x^{2-k}, no more valuable result is true than that X' is of order X^{1+k}, and this result is clearly useless for our purpose.

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* I have already given the theorem for this limiting case, *Proc. London Math. Soc.*, Ser. 2, Vol. 5, p. 365.

7.* The extended form of the theorem, if it be true, is chiefly interesting as showing that when $\rho < \frac{1}{2}$, $m(r) > [M(r)]^c$, where c is positive. For it is easy to deduce, by a well-known device, from the theorem established in § 4, a theorem applicable to all functions of finite order, and analogous to the result which we have obtained for the case $\frac{1}{4} < \rho < \frac{1}{2}$. This theorem is as follows.†

Let $F(z)$ be any integral function of finite (apparent) order ρ. Then there exists a sequence of circles of radii $r_1, r_2, \ldots$ tending to infinity, and depending only on the moduli of the zeros of $F(z)$, such that

$$m(r_s) > [M(r_s)]^{-c(\rho)},$$

where $c(\rho)$ is a constant depending only on ρ.

Let λ be the least integer greater than 2ρ, and let ω be a primitive λ-th root of unity.‡

Let $$F(z)\,F(\omega z)\,\ldots\,F(\omega^{\lambda-1}z) = G(z),$$

and let $$\zeta = z^\lambda, \qquad |\zeta| = t.$$

Now $G(z)$ is clearly an integral function of z^λ or ζ, $H(\zeta)$ suppose.

Since $$|H(\zeta)| < [M(r)]^\lambda < \exp(\lambda r^{\rho+\epsilon}) < \exp(\lambda t^{(\rho+\epsilon)/\lambda}),$$

for large values of r or t, $H(\zeta)$ is of order in ζ not greater than ρ/λ.§

Let $\mathcal{M}(t)$ be the maximum modulus of $H(\zeta)$ on the circle $|\zeta| = t$.

Since $\rho/\lambda < \frac{1}{2}$, there exists a sequence of circles $|\zeta| = t$, of radii $r_1^\lambda, r_2^\lambda, \ldots$, tending to infinity, and depending only on the *moduli* of the zeros of $H(\zeta)$, *i.e.*, depending only on the *moduli* of the zeros of $F(z)$, such that, on each circle

$$\begin{aligned} |H(\zeta)| &> [\mathcal{M}(r_s^\lambda)]^{\cos(2\pi\rho/\lambda) - [1-\cos(2\pi\rho/\lambda)]} \\ &> [\mathcal{M}(r^\lambda)]^{-1} \\ &> [\{M(r)\}^\lambda]^{-1}. \end{aligned}$$

In particular, if z_1 be the point of the circle $|z| = r_s$ for which $|F(z_1)| = m(r_s)$, we obtain

$$m(r_s)\,[M(r_s)]^{\lambda-1} \geqslant |G(z_1)| > [M(r_s)]^{-\lambda}.$$

* The remainder of the paper was added December 24th.

† I must here withdraw the statement which I put forward in a previous paper (*Proc. London Math. Soc.*, *loc. cit.*, § 5), that no theorem such as the above could exist.

‡ The method here employed is substantially that used in extending Hadamard's theorem that $m(r) > \exp(-r^{\rho+\epsilon})$, from the case when $\rho < 1$, to the case when ρ is general.

§ $H(\zeta)$ is actually of order ρ/λ, when ρ is not an integer.

Hence $m(r_s) > [M(r_s)]^{-(2\lambda-1)}$, and if we take

$$c(\rho) = 2\lambda - 1,$$

it is seen that the theorem is true.

8. It is interesting to notice that, if $F(z)$ and $F_1(z)$ be two integral functions of the same apparent order ρ, with the same sequence of moduli of zeros, it is *not* necessarily true that we can find a sequence of circles of radii tending to infinity, such that

$$m_1(r_s) > [M(r_s)]^{-c}.$$

We have proved that this inequality holds when $\rho < \frac{1}{2}$, and I believe that it holds when ρ is not an integer; but a simple example, due to M. Borel, shows that when, *e.g.*, $\rho = 1$, the inequality need not hold.

If $$F(z) = \sin \pi z, \qquad F_1(z) = z^{-1}[\Gamma(z)]^{-2},$$

we have

$$M(r) = \exp[\{1+\epsilon(r)\}\pi r], \qquad m_1(r) = \exp[-\{1+\epsilon(r)\}2r \log r],$$

and we cannot have $$m_1(r) > [M(r)]^{-c}.$$