It should be remarked that Dr. Graves gave the theorem as of spherical conics. So also did Chasles (1843). The theory of Chasles is very complete on arcs of conics whose difference is rectifiable.

On Operative Symbols in the Differential Calculus.

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1. The following paper contains several results in the theory of operative symbols as employed in the differential calculus. To claim them all as new would be, in these days, a somewhat hazardous pretension; the author is, besides, fully sensible of the fragmentary and unfinished shape in which a few detached results are here presented, which evidently form but indications of a great extension of the infinitesimal calculus.

2. The present paper is confined to the consideration of symbols of the form $f(x, D)$; where $D = \frac{d}{dx}$. If η be any such symbol, it is well to remember that it is commutative with any function of itself; in fact, that

$$\phi(\eta)\psi(\eta) \equiv \psi(\eta)\phi(\eta).$$

In treating of these symbols, an operand will be usually understood when not expressed. A symbol may be *isolated*; i.e., considered independently of any operand; it is important to remember that this is different from the same symbol operating on unity. To illustrate this distinction, and the notation employed, let us consider the symbol xD , as operated on by D^{-1} , in the following cases:—

1. $D^{-1} xD \equiv x - D^{-1},$
2. $D^{-1} |xD| \equiv \frac{1}{2}x^2D + \phi(D),$
3. $D^{-1} xD . 1 = C,$
4. $D^{-1} |xD| . 1 = C.$

In (1) and (2), an operand is understood; but in (2) the operation D^{-1} is limited by the bars $||$ to the symbol xD .

3. In dealing with these symbols, great care is required not to misapply, through habit, the ordinary rules of algebra and of differentiation, &c. For instance, if we require to differentiate the symbol

$$u = \overline{xD},$$

with regard to x , it is incorrect to give

$$\frac{du}{dx} = 2\overline{xD} \mid D,$$

for we must remember that $u = xDxD$. We have here to go back to the first principles of the differential calculus, and consider the increment of u produced by a small increment of x ; and we find

$$\frac{du}{dx} = DxD + xDD.$$

As we have the identity $Dx \equiv 1 + xD$,

this becomes $\frac{du}{dx} = 2\overline{xD} \mid D + D.$

To give rules for the differentiation of such symbols in general, is more than the present state of knowledge admits of; we shall have occasion to treat a few isolated cases in what follows.

4. The two following formulas are of frequent application. If η be any such operator,

$$D \mid \eta \mid = \frac{d\eta}{dx} \equiv D\eta - \eta D \dots\dots\dots(1),$$

$$\frac{d\eta}{dD} \equiv \eta x - x\eta \dots\dots\dots(2).$$

The first follows easily by considering that

$$D\eta X = \frac{d\eta}{dx} X + \eta \frac{dX}{dx} \dots\dots\dots(3);$$

the second from $\eta xX = f(x, D)xX = f(x, D_0 + D_1)xX$,

where D_0 is null as regards X , and D_1 as regards x , so that

$$\eta xX = x\eta X + \frac{d\eta}{dD} X.$$

As an example of how these formulas may sometimes be made use of, let us propose to differentiate

$$u - \overline{xD} \mid '.$$

$$\begin{aligned} \frac{du}{dx} &= D\overline{xD} \mid ' - \overline{xD} \mid ' D \\ &= \overline{Dx} \mid ' D - \overline{xD} \mid ' D \\ &= (\overline{1 + xD} \mid ' - \overline{xD} \mid ') D. \end{aligned}$$

If we put $xD = \zeta$, this may be written

$$\frac{du}{dx} = \Delta \zeta' D,$$

where Δ operates only on ζ ; therefore

$$\frac{d}{dx} \overline{x D} = \left(r \cdot \overline{x D} |^{r-1} + \frac{r(r-1)}{1 \cdot 2} \overline{x D} |^{r-2} + \&c. \right) D \dots\dots\dots(4),$$

also $\frac{d}{dx} f(xD) \equiv \Delta f(\zeta) D \equiv f'(xD) D + \frac{1}{1 \cdot 2} f''(xD) D + \&c. \dots\dots(5),$

thus $\frac{d}{dx} e^{xD} \equiv D | e^{xD} | \equiv (e-1) e^{xD} D \dots\dots\dots(6).$

5. In a paper in the "Quarterly Journal of Mathematics," Oct., 1879, I have given the theorem

$$\phi \left(\frac{d}{dD} \right) f(D) \equiv f(D) \phi(x-x) \dots\dots\dots(7),$$

a series equivalent to which had previously been given by Prof. Donkin. In the second member, x is a letter as regards which D is null; it being replaced by x at the conclusion of the operations. This theorem can be generalized, and holds when $f(D)$ involves x as well as D ; so that, if η be any operator,

$$\phi \left(\frac{d}{dD} \right) \eta \equiv \eta \phi(x-x) \dots\dots\dots(8),$$

for, from (2),

$$\left(\frac{d}{dD} \right)^2 \eta = (\eta x - x \eta) x - x (\eta x - x \eta) = \eta x^2 - 2x \eta x + x^2 \eta;$$

therefore $\left(\frac{d}{dD} \right)^2 \eta \equiv \eta (x-x)^2,$

and this can easily be extended to any function ϕ .

From (8), $e^{\frac{d}{dD}} \eta \equiv e^{-x} \eta e^x,$

hence $f(x, D+1) \equiv e^{-x} f(x, D) e^x \dots\dots\dots(9),$

an extension of the well-known theorem as to $f(D)$; it is easily extended to $f(x, D+r)$.

For example,

$$a^{x(D+r)} F(x) = e^{-rx} a^{xD} e^{rx} F(x) = e^{-rx} e^{rax} F(ax),$$

therefore $a^{x(D+r)} F(x) = e^{rx(e-1)} F(ax) \dots\dots\dots(10),$

as it is known that the effect of the operator a^{xD} is to change x into ax in the subject.

6. An important principle in this theory is, that we may change D in an operator into $D_0 + D$; thus

$$\eta \equiv f(x, D) \equiv f(x, D_0 + D) \dots\dots\dots(11),$$

understanding that, in the second member, D is null as regards the operator, and D_0 null as regards the operand.

Thus we may put, in this sense, instead of the operator $\eta = \overline{x D}^3$,

$$\eta = \overline{x (D_0 + D)}^3,$$

therefore $\eta = \overline{x D_0}^3 + x D_0 x D + x D x D_0 + \overline{x D}^3$.

Here the first and third terms vanish, the second becomes $x D$, and the fourth $x^3 D^3$, so that

$$\overline{x D}^3 = x D x D = x D + x^3 D^3,$$

as is easily verified.

By expanding the second member of (11) according to powers of D , and suppressing the suffix 0; using f also for shortness instead of $f(x, D)$,

$$f(x, D) X \equiv f(x, D) \left| X + \frac{df}{dD} \right| DX + \frac{1}{1 \cdot 2} \frac{d^2 f}{dD^2} \left| D^2 X + \&c. \dots \dots (12), \right.$$

the bars limiting the effect of the D preceding them, the subject unity being understood before each bar.

To give an application of this theorem, let us consider (putting $\zeta = x D$), $u = f(x D) F(x) = f(\zeta) F(x)$.

As in Art. 4, we can easily show that

$$\frac{d}{dD} f(x D) = x \{ f(x D + 1) - f(x D) \} = x \Delta f(\zeta) \dots \dots \dots (13),$$

where Δ operates only on ζ . Hence

$$\begin{aligned} f(\zeta) F(x) &= \left\{ f(\zeta) \left| + x \Delta f(\zeta) \right| D + \frac{x^2}{1 \cdot 2} \Delta^2 f(\zeta) \left| D^2 + \dots \right. \right\} F(x) \\ &= e^{x \Delta} f(\zeta) | F(x) \dots \dots \dots (14), \end{aligned}$$

where x is null as to all the operators, Δ affects only ζ , D affects only $F(x)$, and ζ is null as to $F(x)$, as is indicated by the bar.

Therefore $u = f(\zeta) F(x) = F(x + x \Delta) f(\zeta) |$,

or $u = F[\dot{x}(1 + \Delta)] f(\zeta) | = F(\dot{x} e^{\Delta}) f(\zeta) \cdot 1 \dots \dots \dots (15).$

Thus, if we have to expand $u = f(x D) e^x$ according to powers of x ,

$$\begin{aligned} u &= \exp. (\dot{x} e^{\Delta}) f(\zeta) \cdot 1 \\ &= f(\zeta) + x f(\zeta + 1) + \frac{x^2}{1 \cdot 2} f(\zeta + 2) + \dots, \end{aligned}$$

the right member operating on 1. But

$$f(x D + r) \cdot 1 = f(r),$$

therefore $f(xD) e^x = f(0) + xf(1) + \frac{x^2}{1.2} f(2) + \&c. \dots\dots\dots(16),$

easily verified by first expanding e^x . Again, if

$$u = f(xD) (\log x)^r,$$

$$u = \left(\log x + \frac{d}{d\zeta} \right)^r f(\zeta) . 1,$$

therefore

$$u = \log^r x f(0) + r \log^{r-1} x . f'(0) + \frac{r(r-1)}{1.2} \log^{r-2} x . f''(0) + \&c. \dots\dots(17).$$

Of course the chief difficulty in applying such a formula as (12) is the absence of rules for differentiating symbols with regard to the x or D which they contain.

7. LAGRANGE'S THEOREM.—We may here give a proof of Lagrange's theorem by this method; let us propose to find the value of

$$u = e^{D \cdot \phi} F(x),$$

where, for shortness, ϕ is put for $\phi(x)$; a point is placed between D and ϕ (though dropped for shortness in what follows) to remind us that $D\phi$ is not here an indivisible symbol, but that in fact we define

$$e^{D \cdot \phi} \equiv 1 + D\phi + \frac{D^2 \phi^2}{1.2} + \frac{D^3 \phi^3}{1.2.3} + \&c. \dots\dots\dots(18).$$

As in (11), we may write here $D_0 + D$ for D , and we easily see, from (18), that

$$u = e^{(D_0 + D) \cdot \phi} F(x) = e^{D_0 \cdot \phi} e^{D \cdot \phi} F(x),$$

where D only affects $F(x)$, being null as regards $\phi(x)$; hence

$$u = e^{D_0 \cdot \phi} F[\dot{x} + \phi(x)] = e^{D \cdot \phi} F[\dot{x} + \phi(x)],$$

dropping the suffix 0, and marking x with a point to denote that D is null as regards it.

Thus
$$u \equiv e^{D \cdot \phi} F(x) \equiv e^{D \cdot \phi} F[\dot{x} + \phi(x)],$$

so that we may replace x by $\dot{x} + \phi(x)$ without change; but, if so, we may repeat the process, and put

$$u = e^{D \cdot \phi} F[\dot{x} + \phi(\dot{x} + \phi x)],$$

and so on *ad infinitum*. Hence, putting

$$\begin{aligned} z &= \dot{x} + \phi \{ \dot{x} + \phi(\dot{x} + \dots) \} = \dot{x} + \phi(z), \\ u &= e^{D \cdot \phi} F(z) = F(z) e^{D \cdot \phi} . 1 \dots\dots\dots(19), \end{aligned}$$

as D is null as to z . To determine $e^{D \cdot \phi} . 1$, it is easy to see, from (18), that

$$e^{D \cdot \phi} . 1 = 1 + e^{D \cdot \phi} \phi'(x) = 1 + \phi'(z) e^{D \cdot \phi} . 1,$$

therefore
$$e^{D\phi} \cdot 1 = \frac{1}{1-\phi'(x)} = \frac{dz}{dx}.$$

Thus we have proved $u = e^{D\phi} F(x) = F(x) \frac{dz}{dx} \dots\dots\dots(20),$

where $z = x + \phi(x) \dots\dots\dots(21);$

put F' for F , and we have

$$\frac{d}{dx} F(z) = e^{D\phi} F'(x),$$

or
$$F'(z) = F'(x) + \phi \cdot F''(x) + \frac{1}{1 \cdot 2} D\phi^2 \cdot F''(x) + \dots$$

which is Lagrange's theorem.

8. Since $\overline{x^2 D}^n x^r = r(r+1) \dots (r+n-1) x^{r+n},$

it is easy to shew by expanding the exponential that

$$e^{ax^2 D} x^r = \left(\frac{x}{1-ax} \right)^r \dots\dots\dots(22),^*$$

and thence that $e^{ax^2 D} f(x) \equiv f\left(\frac{x}{1-ax} \right) \dots\dots\dots(23)$

and, in a similar way, it can be proved that in general

$$\exp. (ax^{r+1} D) f(x) \equiv f \left\{ \frac{x}{(1-ax^r)^{\frac{1}{r}}} \right\} \dots\dots\dots(24).$$

This may be shewn to include the known cases, when $r = 0,$

$$e^{ax^2 D} f(x) = f(e^{ax} x),$$

and Taylor's theorem, when $r = -1.$ We have also

$$\exp. (ax^{-1} D) f(x) \equiv f(\sqrt{x^2+2a}) \dots\dots\dots(25);$$

thus the expansion of this by powers of a is

$$f(\sqrt{x^2+2a}) = f(x) + a \overline{x^{-1} D} f(x) + \frac{a^2}{1 \cdot 2} \overline{x^{-1} D}^2 f(x) + \dots$$

By considering $f(x)$ as consisting of a series of powers of e^x , it is not difficult to shew also that

$$\exp. (e^{ax} D) f(x) \equiv f \left\{ x - \frac{1}{a} \log(1 - ae^{ax}) \right\} \dots\dots\dots(26).$$

9. The above however are only cases of a general theorem, somewhat resembling Lagrange's, which was first arrived at by induction from such cases as the above, but which, once stated, can be easily proved as follows.

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* N.B.— $x^2 D$ is here an indivisible symbol, its square being $x^2 D x^2 D$, not $x^4 D^2$, &c. The same remark applies to $e^{x^2 D}$, e^{D^2} , &c., in what follows; so that the series (18) now ceases to be employed.

If z, x are related by the equation

$$\psi(z) = 1 + \psi(x) \dots\dots\dots(27),$$

it is required to express any function $F(z)$ of z in terms of x .

The result is very simple, viz., putting for convenience of notation

$$\phi = \phi(x) = \frac{1}{\psi'(x)}, \quad \text{i.e., } \psi(x) = D^{-1}\phi^{-1} \dots\dots\dots(28),$$

we shall have

$$F(z) \equiv e^{\phi D} F(x) \dots\dots\dots(29).$$

The simplest proof is probably to observe first that (28)

$$\phi D \{\psi(x)\}^n = n \{\psi(x)\}^{n-1} = n\psi^{n-1},$$

putting ψ for $\psi(x)$; hence $\phi Df(\psi) = f'(\psi)$,

$$\text{and hence} \quad F(\phi D)f\{\psi(x)\} \equiv \int_x^{\psi} F(D)f(x) \dots\dots\dots(30),$$

Sarrus' sign of substitution being used to show that $\psi(x)$ is finally put for x . A case of (30) is

$$e^{\phi D} f(\psi) = \int_x^{\psi} e^{\psi D} f(x) = f(\psi+1),$$

that is, by (27),

$$e^{\phi D} f(\psi x) = f(\psi z),$$

or

$$e^{\phi D} F(x) = F(z),$$

a general theorem which includes the cases in Art. 9, and, of course, many others.

10. To give one or two more examples of the application of (29); let it be proposed—(1) to find an operator which for all functions of x , shall change x into $ax+b$.

In (27) we have to determine the function ψ from the equation

$$\psi(ax+b) \equiv 1 + \psi(x);$$

$$\text{this is satisfied by} \quad \psi(x) = \frac{1}{\log a} \log \left(x - \frac{b}{1-a} \right);$$

$$\text{hence} \quad a^{(s-\frac{b}{1-a})D} F(x) \equiv F(ax+b) \dots\dots\dots(31),$$

$$\text{From this we have} \quad a^{sD} e^{bD} \equiv a^{(s-\frac{b}{1-a})D}.$$

$$(2.) \quad \exp.(\overline{\log x \cdot D}) F(x) = F(z) \dots\dots\dots(32),$$

$$\text{where } z \text{ is related to } x \text{ by } li(z) = 1 + li(x) \dots\dots\dots(33),$$

li denoting the logarithm-integral,

$$li(x) = \int_0^x \frac{dx}{\log x}.$$

Hence, if z is related to x , as in (33), its value in terms of x is

$$z = x + \overline{\log x \cdot D} \left| x + \frac{1}{1 \cdot 2} \overline{\log x \cdot D} \right|^2 x + \&c.$$

(3.) To find an operator which shall change x to x^r , in any function. The solution of the functional equation

$$\psi(x^r) \equiv 1 + \psi(x)$$

is

$$\psi(x) = \frac{\log \log x}{\log r} + C.$$

Hence (29)

$$r^{x \log x \cdot D} F(x) \equiv F(x^r) \dots\dots\dots (34),$$

$$\exp. (x \log x \cdot D) F(x) = F(x^e) \dots\dots\dots (35).$$

(4.) To find a function X such that, whatever be the function F ,

$$F(e^x D) X \equiv F(x) X.$$

In (28) put $\phi(x) = e^x$, therefore $\psi(x) = -e^{-x}$; by (30),

$$F(e^x D) f(-e^{-x}) = \int_x^{-e^{-x}} F(D) f(x).$$

Now, if $f(x) = e^x$, this becomes $F(x) f(-e^{-x})$; hence the required function is

$$X = \exp. (-ae^{-x}).$$

11. If we put for shortness ϕ for $\phi(x)$, ψ for $\psi(x)$, these being any two functions, and $u = D^{-1} \frac{\psi}{\phi}$, or $\int \frac{\psi(x)}{\phi(x)} dx$,

$$\begin{aligned} \text{then for any operand } \phi D e^u &= \phi \left(e^u D + e^u \frac{\psi}{\phi} \right) \\ &= e^u (\phi D + \psi), \end{aligned}$$

and, by repeating the operation,

$$f(\phi D) e^u = e^u f(\phi D + \psi);$$

so that in general, whatever be the operand,

$$f(\phi D + \psi) \equiv \exp. \left(-D^{-1} \frac{\psi}{\phi} \right) f(\phi D) \exp. \left(D^{-1} \frac{\psi}{\phi} \right)^* \dots\dots (36).$$

Let $\psi(x) = \phi'(x)$, $f(\phi D + \phi') = \phi^{-1} f(\phi D) \phi$;

but

$$\phi D + \phi' \equiv D\phi,$$

hence

$$\phi f(D\phi) = f(\phi D) \phi \dots\dots\dots (37);$$

this may be applied to the results in Arts. 8, 9. Thus, from (23),

$$e^{Dx} f(x) = x^{-1} e^{x^2 D} x^2 f(x) = \frac{1}{(1-x)^2} f\left(\frac{x}{1-x}\right).$$

* N.B.—In the exponentials, D^{-1} only affects the function $\frac{\psi}{\phi}$.

Again, from (29),

$$e^{D\psi} F(x) = \frac{1}{\phi(x)} e^{D\phi(x)} F(x) = \frac{\phi(z)}{\phi(x)} F(z).$$

But $\psi(z) = 1 + \psi(x)$, where $\psi(x) = D^{-1}\phi^{-1}$, therefore $\psi'(x) = \frac{1}{\phi}$,

therefore
$$\frac{dz}{dx} = \frac{\phi(z)}{\phi(x)},$$

therefore
$$e^{D\psi} F(x) = \frac{dz}{dx} F(z) \dots\dots\dots(38).$$

This would have followed at once also from the second formula in

$$f(D\phi) \equiv \phi^{-1} f(\phi D) \phi \equiv Df(\phi D) D^{-1} \dots\dots\dots(39),$$

which can be shewn by expansion; as well as by (36), as regards the first result. The second also follows from the first, by the remark in Art. 2.

12. If, in (36), we put $\phi(x) \alpha'(x)$ instead of $\psi(x)$, $\alpha(x)$ being any function; calling it α , and $\phi(x)$, ϕ ,

$$f\{\phi(D + \alpha')\} = e^{-\alpha} f(\phi D) e^{\alpha} \dots\dots\dots(40).$$

Thus, in the theorem in (29), let D receive as an increment any function $\alpha'(x)$, $\exp.\{\phi[D + \alpha'(x)]\} F(x) = e^{-\alpha(x)} e^{\alpha(x)} F(x) \dots\dots\dots(41),$

α being related to x by equation (27).

Suppose, for example, $\phi(x) = 1$, then $\alpha = x + 1$,

$$e^{D + \alpha'(x)} F(x) = e^{-\alpha(x)} e^{\alpha(x+1)} F(x+1),$$

$\alpha(x)$ being any function; this is easily deducible from known theorems.

Many results might be made to follow from (41); e.g., to find the value of

$$U = \exp.(x^2 D + x^2) F(x).$$

Here $\alpha'(x) = x$, $\alpha(x) = \frac{1}{2}x^2$; hence (23)

$$U = e^{-\frac{1}{2}x^2} e^{x^2 D} e^{\frac{1}{2}x^2} F(x) = \exp.(-\frac{1}{2}x^2) \cdot \exp.\left(\frac{\frac{1}{2}x^2}{(1-x)^2}\right) F\left(\frac{x}{1-x}\right) \dots\dots(42).$$

13. If, in (36), we put $\psi(x) = 1$, and for f the exponential, and

$$D^{-1}\phi^{-1} = \beta(x) = \beta,$$

$$e^{1+\phi D} \equiv e e^{\phi D} \equiv e^{-\beta} e^{\phi D} e^{\beta}.$$

Hence

$$e^{\phi D} e^{\beta} = e^{1+\beta} e^{\phi D},$$

therefore

$$e^{\phi D} e^{\beta} = e^{\beta + \phi D},$$

therefore

$$e^{\phi D} f(e^{\beta}) = f(e^{\beta + \phi D}).$$

If we now put

$$1 + \beta(x) = \beta(z),$$

and take unity as operand, we shall easily find

$$e^{\phi D} F\{\beta(x)\} = F\{\beta(z)\},$$

therefore $e^D F(x) = F(z),$

thus proving the theorem in (29) by a different process.

14. If we put $\zeta = D\phi$, where $\phi = \phi(x)$, and if u, v are functions of x ,

$$\begin{aligned}\zeta uv &= vD\phi u + u\phi Dv \\ &= v\zeta u + uD^{-1}\zeta Dv,\end{aligned}$$

therefore $\zeta uv = (\zeta' + D_1^{-1}\zeta_1 D_1) uv \dots\dots\dots(43),$

provided ζ' is null as to v , and D_1, ζ_1 null as to u .

By repeating this operation on uv , it is easy to see that

$$\zeta^n uv = (\zeta' + D_1^{-1}\zeta_1 D)^n uv, \text{ or}$$

$$\overline{D\phi}^n uv = D_1^{-1} \left\{ v' \zeta^n u + n \zeta v' \cdot \zeta^{n-1} u + \frac{n(n-1)}{1 \cdot 2} \zeta^2 v' \cdot \zeta^{n-2} u + \dots \right\} \dots\dots(44),$$

where $v' = Dv$; D_1^{-1} refers only to v' , and the points indicate multiplication.

It is also evident, from (43), that

$$e^{\zeta} uv = e^{\zeta} u \cdot D^{-1} e^{\zeta} Dv.$$

Another form may be given to (44), by putting

$$\eta = \phi D,$$

η only operating on v , ζ only on u ; we shall then have

$$D\phi uv = (\zeta + \eta) uv,$$

$$\text{and } \overline{D\phi}^n uv = v \zeta^n u + n \eta v \cdot \zeta^{n-1} u + \frac{n(n-1)}{1 \cdot 2} \eta^2 v' \cdot \zeta^{n-2} u + \dots \dots\dots(45).$$

As an example, put $v = \exp. (D^{-1}\phi^{-1})$, then

$$\overline{D\phi}^n vu = v (1 + D\phi)^n u,$$

a case of (36).

15. Mr. Walker has given ("Proceedings, London Mathematical Society," April, 1880) a very remarkable formula for $D^n \phi^r uv$, where $\phi = \phi(x)$, and u, v are functions of x , viz.,

$$D^n \phi^r uv = v D^n \phi^r u + n v' \phi \cdot D^{n-1} \phi^{r-1} u + \frac{n(n-1)}{1 \cdot 2} D \phi^2 v' \cdot D^{n-2} \phi^{r-2} u + \&c.$$

This may be put in the form

$$\begin{aligned}D^n \phi^r uv &= D_1^{-1} \left\{ v' \cdot D^n \phi^r u + n D \phi v' \cdot D^{n-1} \phi^{r-1} u \right. \\ &\quad \left. + \frac{n(n-1)}{1 \cdot 2} D^2 \phi^2 v' \cdot D^{n-2} \phi^{r-2} u + \&c. \right\} \dots\dots(46),\end{aligned}$$

being, as is very remarkable, exactly the same series as (44), except that $D^r \phi^r$ everywhere takes the place of $\overline{D\phi}^r$.

To establish this result by the method of symbols, let U be any operator, being a function of x , D , and let us define the symbolic operator ζ to be

$$\zeta U \equiv DU\phi \dots\dots\dots (47).$$

Hence

$$\zeta' U \equiv D'U\phi'.$$

Let v be any function of x , then

$$DUv\phi = UDv\phi + DU|\phi v,$$

therefore

$$\zeta Uv = U\zeta v + \{D^{-1}\zeta\dot{D}U\}v,$$

where $\dot{D}$ is marked so, to indicate that the operation $\dot{D}$ is performed before the ζ which precedes.

$$\text{This may be put } \zeta Uv = (\zeta_1 + D_1^{-1}\zeta_1\dot{D}_1)Uv,$$

the suffix $_1$ denoting operations on U only, $_1$ on v only, the former being thus commutative with the latter. We shall have thus

$$\zeta^2 Uv = (\zeta_1^2 + 2\zeta_1 D_1^{-1}\zeta_1\dot{D}_1 + D_1^{-1}\zeta_1^2\dot{D}_1)Uv,$$

since

$$D^{-1}\zeta\dot{D}D^{-1}\zeta\dot{D}U = D^{-1}\zeta\zeta\dot{D}U|.$$

Hence, if for shortness we put $DU = U'$,

$$\zeta^n Uv = D^n Uv\phi^n = D_1^{-1}\left\{U'D^n\phi^n v + nDU'\phi|D_1^{-1}\phi^{n-1}v + \frac{n(n-1)}{1.2}D^2U'\phi^2|D_1^{-2}\phi^{n-2}v + \&c.\right\} \dots\dots (48),$$

which includes Mr. Walker's theorem (46), when $U = u$, a function of x (the letters u, v being interchanged). The bars in (48) limit the operation of the $D, D^2, \&c.$, which precede them, also of D_1^{-1} at the commencement, which is null as regards v .

16. As we have from last article, if u, v are functions of x ,

$$f(\zeta)uv \equiv f(\zeta_1 + D_1^{-1}\zeta_1\dot{D}_1)uv,$$

$$e^{\zeta}uv = \exp.(D^{-1}\zeta D)u.e^{\zeta}v.$$

Suppose $v = 1$,

$$e^{\zeta}u = e^{\zeta}1.\exp.(D^{-1}\zeta D)u \dots\dots\dots (49),$$

or

$$e^{\zeta}u = e^{\zeta}1.D^{-1}e^{\zeta}Du.$$

Let $u = x$,

$$e^{\zeta}x = e^{\zeta}1.D^{-1}e^{\zeta}1;$$

if we put $z = D^{-1}e^{\zeta}1$,

$$D^{-1}e^{\zeta}x = \frac{1}{2}z^2;$$

therefore

$$D^{-1}e^{\zeta}Dx^2 = z^2,$$

and in like manner

$$D^{-1}e^{\zeta}Dx' = x';$$

therefore

$$D^{-1}e^{\zeta}DF(x) = F(z) \dots\dots\dots (50),$$

where
$$e_{\zeta} = 1 + D\phi + \frac{D^2\phi^2}{1 \cdot 2} + \&c. \dots\dots\dots(51).$$

This is Lagrange's theorem proved by a different process ; for

$$z = x + \phi(z),$$

because, from (50), $D^{-1}e^{\zeta}D\phi(x) = \phi(z),$

and it is easy to see, (51), that

$$e^{\zeta}1 = 1 + e^{\zeta}D\phi(x),$$

therefore
$$z = D^{-1}e^{\zeta}1 = x + D^{-1}e^{\zeta}D\phi(x).$$

17. Mr. Hargreave has remarked (*Phil. Trans.*, 1848, see Boole's "Differential Equations," p. 455) that any relation between the symbols x and D will hold if we substitute D for x , and $-x$ for D . Thus Taylor's theorem gives us the relation

$$e^{\lambda D}f(x) \equiv f(x+h)e^{\lambda D},$$

any subject being understood. By the above substitution, we deduce the known relation $f(D+h) \equiv e^{-\lambda x}f(D)e^{\lambda x}.$

Again, Lagrange's theorem may be stated

$$e^{D \cdot \phi(x)}F(x) \equiv F(z)e^{D \cdot \phi(x)} \dots\dots\dots(52),$$

where we define $e^{D \cdot \phi(x)} \equiv 1 + D\phi x + \frac{1}{1 \cdot 2} D^2(\phi x)^2 + \&c. \dots\dots\dots(53),$

and
$$z = x + \phi(z).$$

We deduce
$$e^{-x \cdot \phi(D)}F(D) \equiv F(\zeta)e^{-x \cdot \phi(D)} \dots\dots\dots(54),$$

where
$$\zeta \equiv D + \phi(\zeta) \dots\dots\dots(55).$$

It may easily be shewn that the reciprocal or inverse of the operator $e^{D \cdot \phi x}$, in (53), is not $e^{-D \cdot \phi x}$, but $e^{-D \cdot \phi^2}$. Hence also

$$e^{-x \cdot \phi D}e^{D \cdot \phi^2} \equiv 1,$$

so that (54) gives
$$F(\zeta) \equiv e^{-x \cdot \phi D}F(D)e^{x \cdot \phi^2} \dots\dots\dots(56).$$

Suppose, e.g.,
$$\phi(D) = hD,$$

then
$$\zeta = \frac{D}{1-h},$$

therefore
$$F\left(\frac{D}{1-h}\right) \equiv e^{-\lambda x \cdot D}F(D)e^{\frac{\lambda x}{1-h}D}$$

Again, if $\phi(D) = D^2$; by taking in (56), $F(D-D^2)$ for $F(D),$

$$e^{-x \cdot D^2}F(D-D^2) \equiv F(D)e^{-x \cdot D^2} \dots\dots\dots(57),$$

which is easy to verify when $e^{\lambda x}$ is the subject.

If we take $\phi(D) = D - D^2$; then, from (55), $\zeta^2 = D$,

$$F(D^2) \equiv e^{\pi \cdot (D^2 - D)} F(D) e^{\pi \cdot (D^2 - D)} \dots\dots\dots (58).$$

18. Many of the results in this paper may be transformed in like manner. Thus from formula (23) we derive

$$F\left(\frac{D}{1-D}\right) \equiv e^{-D^2} F(D) e^{D^2},$$

or
$$*F\left(\frac{D}{1-D}\right) \equiv e^{-\pi D^2} F(D) e^{\pi D^2} \dots\dots\dots (59),$$

where, we must remember, D^2x is treated as one symbol, and the definition (53), relating to Lagrange's theorem and its applications, no longer holds. In (59) let the subject be

$$e^{-\pi D^2} e^{\lambda x},$$

$$F\left(\frac{D}{1-D}\right) e^{-\pi D^2} e^{\lambda x} = F(h) e^{-\pi D^2} e^{\lambda x}.$$

This is easy to verify, because

$$e^{\lambda x D^2} e^{\lambda x} = \exp. \left(\frac{hx}{1-hk} \right) \dots\dots\dots (60).$$

It is perhaps worth noticing one method of deducing this theorem :

by putting $u = e^{\lambda x D^2} e^{\lambda x}$,

we find, on differentiating, $\frac{du}{dk} = h^2 \frac{du}{dh}$,

therefore $u = \psi \left(\frac{h}{1-hk} \right)$;

but when $k = 0$, $u = \psi(h) = e^{\lambda x}$.

Since $\exp. (D\alpha D^{-1}) \equiv D \exp. (\alpha) D^{-1}$,

whatever be α , we deduce from (60)

$$e^{\lambda D^2 D} e^{\lambda x} = D e^{\lambda x D^2} h^{-1} e^{\lambda x}$$

$$= \frac{1}{1-hk} \exp. \frac{hx}{1-hk} \dots\dots\dots (61).$$

Again,
$$e^{\lambda D^2 x} e^{\lambda x} = \frac{1}{(1-hk)^2} \exp. \frac{hx}{1-hk} \dots\dots\dots (62).$$

19. Since, by formula (34),

$$r^{(\log x) D} F(x) \equiv F(x^r) r^{(\log x) D}$$

we deduce
$$F(D^r) \equiv r^{-(\log D)^2} F(D) r^{(\log D)^2} \dots\dots\dots (63).$$

The full significance of this and other similar formulas, no doubt, can hardly be appreciated, till our present limited knowledge of the theory of these symbols be much extended.

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* Because $e^{D^2 x} = D^2 e^{x D^2} D^{-2}$; also for $e^{-D^2 x}$.