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Thomas Muir LL.D.

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A PROPERTY OF AXISYMMETRIC DETERMINANTS,
CONNECTED WITH THE SIMULTANEOUS VANISHING
OF THE SURFACE AND VOLUME OF A
TETRAHEDRON.

By THOMAS MUIR, LL.D.

1. In a striking and characteristic paper,* published by Sylvester in 1853, he dealt with a consequence of the simultaneous vanishing of the volume and surface of a tetrahedron. Denoting the squared areas of the faces of the tetrahedron by F, G, H, K , and the volume by V , he supposes one of the vertices to become a point in the opposite face, with the result that

$$\left. \begin{aligned} \sqrt{F} + \sqrt{G} + \sqrt{H} + \sqrt{K} &= 0 \\ \text{and } V &= 0 \end{aligned} \right\},$$

and he thence concludes that if F, G, H, K, V be expressed in terms of one and the same set of variables—say the edges a, b, c, f, g, h of the tetrahedron—the norm of $\sqrt{F} + \sqrt{G} + \sqrt{H} + \sqrt{K}$ must contain V as a rational factor. Further, the said norm N being

$$\begin{aligned} &(\sqrt{F} + \sqrt{G} + \sqrt{H} + \sqrt{K}) \\ &\cdot (-\sqrt{F} + \sqrt{G} + \sqrt{H} + \sqrt{K})(\sqrt{F} - \sqrt{G} + \sqrt{H} + \sqrt{K}) \\ &\quad (\sqrt{F} + \sqrt{G} - \sqrt{H} + \sqrt{K})(\sqrt{F} + \sqrt{G} + \sqrt{H} - \sqrt{K}) \\ &\cdot (-\sqrt{F} - \sqrt{G} + \sqrt{H} + \sqrt{K})(-\sqrt{F} + \sqrt{G} - \sqrt{H} + \sqrt{K}) \\ &\quad (-\sqrt{F} + \sqrt{G} + \sqrt{H} - \sqrt{K}), \end{aligned}$$

or

$$\Sigma F^4 - 4\Sigma F^3G + 6\Sigma F^2G^2 + 4\Sigma F^2GH - 40FGHK,$$

where

$$16F = 2a^2h^2 + 2h^2g^2 + 2g^2a^2 - a^4 - h^4 - g^4,$$

$$16G = 2b^2f^2 + 2f^2h^2 + 2h^2b^2 - b^4 - f^4 - h^4,$$

$$16H = 2c^2g^2 + 2g^2f^2 + 2f^2c^2 - c^4 - g^4 - f^4,$$

$$16K = 2b^2c^2 + 2c^2a^2 + 2a^2b^2 - a^4 - b^4 - c^4;$$

* SYLVESTER, J. J., "On the Relation between the Volume of a Tetrahedron. . . ."—*Cambridge and Dub. Math. Journ.*, viii., pp. 171–178; or *Collected Math. Papers*, i., pp. 404–410.

it follows that

$$\Delta = \{(11) \sqrt{[11]} + (12) \sqrt{[22]} + \dots + (1n) \sqrt{[nn]}\} \cdot \sqrt{[11]},$$

$$= \dots\dots\dots$$

and therefore that

$$(11) \sqrt{[11]} + (12) \sqrt{[22]} + \dots + (1n) \sqrt{[nn]} = 0,$$

it being impossible that the vanishing of Δ could necessitate the vanishing of $[11]$. We thus reach the following general theorem: *If $[rr]$ be the cofactor of the element (rr) in an axisymmetric determinant Δ of the n th order, then the norm of*

$$(\sqrt{[11]}, \sqrt{[22]}, \dots, \sqrt{[nn]}) \text{ (any row of } \Delta \text{)}$$

is divisible by Δ .

(I.)

Sylvester's result is the particular case of this where the diagonal elements of Δ are zeros, and the elements of the last row and of the last column are units.

3. In examining the character of the cofactor of Δ in the norm let us confine ourselves for the present to the cases where there are not more than three non-zero elements in the first row of Δ , and first let us see what happens when all the elements of the first row vanish except the last two. The norm then is

$$N \{(1, n-1) \sqrt{[n-1, n-1]} + (1, n) \sqrt{[n, n]}\}$$

and therefore is

$$(1, n-1)^2 [n-1, n-1] - (1, n)^2 [n, n].$$

But in any determinant which has $(1, r)$ and $(r, 1)$ equal to zero for all values of r except $n-1$ and n , the cofactor of $(n-1, n-1)$ is

$$- (1, n) (n, 1) \cdot \begin{vmatrix} (2, 2) & (3, 3) & \dots & (n-2, n-2) \end{vmatrix},$$

and the cofactor of (n, n) is

$$- (1, n-1) (n-1, 1) \cdot \begin{vmatrix} (2, 2) & (3, 3) & \dots & (n-2, n-2) \end{vmatrix};$$

so that the ratio of the two cofactors is

$$(1, n) (n, 1) : (1, n-1) (n-1, 1). \quad (\text{II.})$$

This ratio in the case of an axisymmetric determinant is $(1, n)^2 : (1, n-1)^2$, which is the same as saying that the norm in

question vanishes. We thus have the theorem: *In any axisymmetric determinant Δ which has $(1, r)$ equal to zero for all values of r except $n-1$ and n , the cofactor of Δ in the norm of*

$$(1, n-1) \sqrt{[n-1, n-1]} + (1, n) \sqrt{[n, n]}$$

is 0,—in other words, the norm vanishes. (III.)

4. If all the elements of the first row vanish except the first and last, the norm then is

$$(1, 1)^2 [1, 1] - (1, n)^2 [n, n],$$

which

$$\begin{aligned} &= (1, 1)^2 [(2, 2)(3, 3) \dots (n, n)] - (1, n)^2 (1, 1) [(2, 2)(3, 3) \dots (n-1, n-1)] \\ &= (1, 1) \{ (1, 1) [(2, 2)(3, 3) \dots (n, n)] - (1, n)^2 [(2, 2)(3, 3) \dots (n-1, n-1)] \} \\ &= (1, 1) \Delta. \end{aligned}$$

Hence *in any axisymmetric determinant Δ which has $(1, r)$ equal to zero for all values of r except 1 and n the cofactor of Δ in the norm of $(1, 1) \sqrt{[1, 1]} + (1, n) \sqrt{[n, n]}$ is the element $(1, 1)$.* (IV.)

5. Passing now to those cases in which Δ has three non-zero elements in the first row, we naturally begin by considering the case where

$$\Delta = \begin{vmatrix} a & b & c \\ b & d & e \\ c & e & f \end{vmatrix}.$$

Here the fact already known from §2 that the norm of $a \sqrt{A} + b \sqrt{D} + c \sqrt{F}$ —that is,

$$N(a \sqrt{df - e^2} + b \sqrt{af - c^2} + c \sqrt{ad - b^2})$$

—is divisible by Δ is itself an aid in finding the cofactor. For Δ being altered and the norm not being altered by changing the sign of b or c or e or all of them, it follows that the norm must be divisible by

$$\begin{vmatrix} a & b & -c \\ b & d & e \\ -c & e & f \end{vmatrix}, \text{ or } \Delta' \text{ say.}$$

Since, however, the norm of $\sqrt{x} + \sqrt{y} + \sqrt{z}$ is of the second degree in x, y, z , our norm must be of the 8th degree in the elements of Δ :

consequently the whole of the cofactor has not been found. Trying a result of Cayley's, namely,

$$N(\sqrt{x} + \sqrt{y} + \sqrt{z}) = \begin{vmatrix} . & 1 & 1 & 1 \\ 1 & . & z & y \\ 1 & z & . & x \\ 1 & y & x & . \end{vmatrix},$$

and bearing in mind that for this determinant we may substitute

$$\begin{vmatrix} . & \rho & \sigma & \tau \\ \rho & . & z\bar{\tau}^1 & y\bar{\sigma}^1 \\ \sigma & z\bar{\tau}^1 & . & x\bar{\rho}^1 \\ \tau & y\bar{\sigma}^1 & x\bar{\rho}^1 & . \end{vmatrix}$$

we have

$$\begin{aligned} N(\sqrt{a^2A} + \sqrt{b^2D} + \sqrt{c^2F}) &= \begin{vmatrix} . & a^2 & b^2 & c^2 \\ a^2 & . & F & D \\ b^2 & F & . & A \\ c^2 & D & A & . \end{vmatrix} \\ &= \begin{vmatrix} . & a^2 & b^2 & c^2 \\ a^2 & . & ad-b^2 & af-c^2 \\ b^2 & ad-b^2 & . & df-e^2 \\ c^2 & af-c^2 & df-e^2 & . \end{vmatrix} = \begin{vmatrix} . & a^2 & b^2 & c^2 \\ a^2 & 2a^2 & ad & af \\ b^2 & ad & . & A \\ c^2 & af & A & . \end{vmatrix} \\ &= a^2 \begin{vmatrix} . & a & b^2 & c^2 \\ a & 2 & d & f \\ b^2 & d & . & A \\ c^2 & f & A & . \end{vmatrix} = a^2 \begin{vmatrix} 2 & a & d & f \\ a & . & b^2 & c^2 \\ d & b^2 & . & A \\ f & c^2 & A & . \end{vmatrix}. \end{aligned}$$

It is thus seen that the missing cofactor is a^2 , and that we have reached at one and the same time two theorems, namely, *In any three-line axisymmetric determinant Δ the cofactor of Δ in the norm of (11) $\sqrt{[11]}$ + (12) $\sqrt{[22]}$ + (13) $\sqrt{[33]}$ is*

$$(11)^2 \begin{vmatrix} (11) & (12) & (13) \\ (12) & (22) & (23) \\ -(13) & (23) & (33) \end{vmatrix} \quad (V.)$$

and

$$\begin{vmatrix} 2 & a & d & f \\ a & . & b^2 & c^2 \\ d & b^2 & . & A \\ f & c^2 & A & . \end{vmatrix} = \begin{vmatrix} a & b & c \\ b & d & e \\ c & e & f \end{vmatrix} \cdot \begin{vmatrix} a & b & -c \\ b & d & e \\ -c & e & f \end{vmatrix} \quad (VI.)$$

6. Since the right-hand member of (VI.) is not altered by performing the cyclical substitutions

$$\begin{pmatrix} a & d & f \\ d & f & a \end{pmatrix}, \quad \begin{pmatrix} e & c & b \\ c & b & e \end{pmatrix},$$

neither will the left-hand member. We thus have

$$\begin{vmatrix} 2 & a & d & f \\ a & . & b^2 & c^2 \\ d & b^2 & . & A \\ f & c^2 & A & . \end{vmatrix} = \begin{vmatrix} 2 & d & f & a \\ d & . & e^2 & b^2 \\ f & e^2 & . & D \\ a & b^2 & D & . \end{vmatrix} = \begin{vmatrix} 2 & f & a & d \\ f & . & c^2 & e^2 \\ a & c^2 & . & F \\ d & e^2 & F & . \end{vmatrix} \quad (\text{VII.})$$

This result could also have been got by considering the remaining norms

$$\begin{aligned} &N(\sqrt{A}, \sqrt{D}, \sqrt{F})(b, d, e) \\ &N(\sqrt{A}, \sqrt{D}, \sqrt{F})(c, e, f) \end{aligned}$$

—a fact which is only natural when we note that these expressions are obtained from the corresponding expression of §5 by performing on it the cyclical substitutions.

7. When zero elements exist in the first row beside the three non-zero elements a distinction is necessary, as in §§3, 4, between the case where the first place is filled by a zero element and the case where it is not so filled. The latter case being the more closely allied to that of §5 falls to be considered first.

For convenience in writing let us limit the number of the said zeros, Δ being

$$\begin{vmatrix} a & . & . & . & e & f \\ . & g & h & i & j & k \\ . & h & l & m & n & o \\ . & i & m & p & q & r \\ e & j & n & q & s & t \\ f & k & o & r & t & u \end{vmatrix}.$$

Then, as before, we have

$$N(a\sqrt{A} + e\sqrt{S} + f\sqrt{U}) = \begin{vmatrix} . & a^2 & e^2 & f^2 \\ a^2 & . & U & S \\ e^2 & U & . & A \\ f^2 & S & A & . \end{vmatrix}.$$

But if we use $\begin{vmatrix} r & s \\ r' & s' \end{vmatrix} \dots$ to denote the minor of Δ got by deleting the r th, s th, ... rows and the r' th, s' th, ... columns, it is readily seen that

$$S = a \begin{vmatrix} 15 \\ 15 \end{vmatrix} - f^2 \begin{vmatrix} 156 \\ 156 \end{vmatrix},$$

and

$$U = a \begin{vmatrix} 16 \\ 16 \end{vmatrix} - e^2 \begin{vmatrix} 156 \\ 156 \end{vmatrix}.$$

Consequently, by performing the operations

$$\begin{aligned} \text{row}_2 + \begin{vmatrix} 156 \\ 156 \end{vmatrix} \text{row}_1, & \quad \text{col}_2 + \begin{vmatrix} 156 \\ 156 \end{vmatrix} \text{col}_1, \\ \text{row}_2 \div a, & \quad \text{col}_2 \div a, \end{aligned}$$

we obtain

$$N(a\sqrt{A} + e\sqrt{S} + f\sqrt{U}) = a^2 \cdot \begin{vmatrix} . & a & e^2 & f^2 \\ a & 2 \begin{vmatrix} 156 \\ 156 \end{vmatrix} & \begin{vmatrix} 15 \\ 15 \end{vmatrix} & \begin{vmatrix} 16 \\ 16 \end{vmatrix} \\ e^2 & \begin{vmatrix} 15 \\ 15 \end{vmatrix} & . & A \\ f^2 & \begin{vmatrix} 16 \\ 16 \end{vmatrix} & A & . \end{vmatrix}.$$

Further, having the knowledge from §2 that the four-line determinant here is divisible by Δ , and seeing that it would remain unaltered although the sign of f were changed, we conclude that it is also divisible by the determinant resulting from Δ by substituting $-f$ for f . We thus have the theorem: *If any axisymmetric determinant Δ which has $(1r)$ equal to zero for all values of r except $1, n-1, n$, the cofactor of Δ in the norm of*

$$(1, 1) \sqrt{[1, 1]} + (1, n-1) \sqrt{[n-1, n-1]} + (1, n) \sqrt{[n, n]}$$

is $a^2\Delta'$, where Δ' is the determinant differing from Δ merely in the sign of $(1, n)$. (VIII.)

Also, if Δ be any axisymmetric determinant which has $(1, r)$ equal to zero for all values of r except $1, n-1, n$, and Δ' be the determinant differing from Δ merely in the sign of $(1, n)$

$$\Delta\Delta' = \begin{vmatrix} 2 \begin{vmatrix} 1, n, n-1 \\ 1, n, n-1 \end{vmatrix} & (1, 1) & \begin{vmatrix} 1, n \\ 1, n \end{vmatrix} & \begin{vmatrix} 1, n-1 \\ 1, n-1 \end{vmatrix} \\ (1, 1) & . & (1, n-1)^2 & (1, n)^2 \\ \begin{vmatrix} 1, n \\ 1, n \end{vmatrix} & (1, n-1)^2 & . & 1 \\ \begin{vmatrix} 1, n-1 \\ 1, n-1 \end{vmatrix} & (1, n)^2 & 1 & . \end{vmatrix} \quad (\text{IX.})$$

It would be interesting to see this last theorem proved directly, that is to say, to see the right-hand side evolved from the left-hand side by mere application of the elementary properties of determinants. In default of such a proof the theorem may be verified as follows, the return to the special Δ with which we started being made to save space in writing:—

$$\Delta = aA - e^2 \begin{vmatrix} 15 \\ 15 \end{vmatrix} - f^2 \begin{vmatrix} 16 \\ 16 \end{vmatrix} + 2ef \begin{vmatrix} 15 \\ 16 \end{vmatrix},$$

$$\begin{aligned}
\text{and } \Delta' &= aA - c^2 \begin{vmatrix} 15 \\ 15 \end{vmatrix} - f^2 \begin{vmatrix} 16 \\ 16 \end{vmatrix} - 2ef \begin{vmatrix} 15 \\ 16 \end{vmatrix}, \\
\therefore \Delta\Delta' &= \left\{ aA - c^2 \begin{vmatrix} 15 \\ 15 \end{vmatrix} - f^2 \begin{vmatrix} 16 \\ 16 \end{vmatrix} \right\}^2 - 4e^2 f^2 \begin{vmatrix} 15 \\ 16 \end{vmatrix}^2, \\
&= a^2 A^2 + c^4 \begin{vmatrix} 15 \\ 15 \end{vmatrix}^2 + f^4 \begin{vmatrix} 16 \\ 16 \end{vmatrix}^2 - 2aAc^2 \begin{vmatrix} 15 \\ 15 \end{vmatrix} \begin{vmatrix} 16 \\ 16 \end{vmatrix} - 2aAf^2 \begin{vmatrix} 15 \\ 16 \end{vmatrix} \begin{vmatrix} 16 \\ 16 \end{vmatrix} \\
&\quad + 2c^2 f^2 \begin{vmatrix} 15 \\ 15 \end{vmatrix} \begin{vmatrix} 16 \\ 16 \end{vmatrix} - 4e^2 f^2 \begin{vmatrix} 15 \\ 16 \end{vmatrix}^2.
\end{aligned}$$

On the other hand, the determinant on the right side of (IX.), if we express it in a series of terms proceeding according to binary products of the elements of the first row and first column, is

$$\begin{aligned}
4e^2 f^2 A \begin{vmatrix} 156 \\ 156 \end{vmatrix} + a^2 A^2 + \begin{vmatrix} 16 \\ 16 \end{vmatrix}^2 f^4 + \begin{vmatrix} 15 \\ 15 \end{vmatrix}^2 c^4 \\
- 2a \begin{vmatrix} 16 \\ 16 \end{vmatrix} A f^2 - 2a \begin{vmatrix} 15 \\ 15 \end{vmatrix} A c^2 - 2 \begin{vmatrix} 16 \\ 16 \end{vmatrix} \begin{vmatrix} 15 \\ 15 \end{vmatrix} e^2 f^2.
\end{aligned}$$

Now these two expressions, having five terms in common, will manifestly be equal if we can show that

$$A \begin{vmatrix} 156 \\ 156 \end{vmatrix} = \begin{vmatrix} 15 \\ 15 \end{vmatrix} \begin{vmatrix} 16 \\ 16 \end{vmatrix} - \begin{vmatrix} 15 \\ 16 \end{vmatrix}^2;$$

and this is unnecessary because of the well-known theorem regarding a minor of the adjugate.

8. Before proceeding farther it is necessary to recall the series of identities*

$$\begin{aligned}
|a_{11}a_{22}\dots a_{nn}| &= \begin{vmatrix} |a_{11}a_{22}| & |a_{11}a_{23}| & \dots & |a_{11}a_{2n}| \\ |a_{11}a_{32}| & |a_{11}a_{33}| & \dots & |a_{11}a_{3n}| \\ \dots & \dots & \dots & \dots \\ |a_{11}a_{n2}| & |a_{11}a_{n3}| & \dots & |a_{11}a_{nn}| \end{vmatrix} \div |a_{11}|^{n-2}, \\
&= \begin{vmatrix} |a_{11}a_{22}a_{33}| & |a_{11}a_{22}a_{34}| & \dots & |a_{11}a_{22}a_{3n}| \\ |a_{11}a_{22}a_{43}| & |a_{11}a_{22}a_{44}| & \dots & |a_{11}a_{22}a_{4n}| \\ \dots & \dots & \dots & \dots \\ |a_{11}a_{22}a_{n3}| & |a_{11}a_{22}a_{n4}| & \dots & |a_{11}a_{22}a_{nn}| \end{vmatrix} \div |a_{11}a_{22}|^{n-3}, \\
&= \dots
\end{aligned}$$

and to point out the effect of putting

$$\begin{aligned}
0 &= a_{nn} = a_{1n} \left\{ \begin{array}{l} \text{in the first,} \\ = a_{n1} \end{array} \right\} \\
0 &= a_{nn} = a_{1n} = a_{2n} \left\{ \begin{array}{l} \text{in the second,} \\ = a_{n1} = a_{n2} \end{array} \right\}
\end{aligned}$$

* *Trans. R. Soc., Edinburgh*, xxix., pp. 47-54.

and so on. In the first place all the elements of the last row and of the last column of each compound determinant become resolvable into two factors, one of which is the minor common to all the elements of the said determinant. The division indicated can thus in part be performed, namely, in the first identity we can divide by a_i^2 , in the second by $|a_{i1}a_{22}|^2$, and so on, with the result that the row and column in question become, as far as may be, a repetition of the corresponding row and column on the left-hand side. Thus, in the case where $n = 6$ we have

$$\begin{vmatrix} a_1 & a_2 & a_3 & a_4 & a_5 & . \\ b_1 & b_2 & b_3 & b_4 & b_5 & b_6 \\ c_1 & c_2 & c_3 & c_4 & c_5 & c_6 \\ d_1 & d_2 & d_3 & d_4 & d_5 & d_6 \\ e_1 & e_2 & e_3 & e_4 & e_5 & e_6 \\ . & f_2 & f_3 & f_4 & f_5 & . \end{vmatrix} = \begin{vmatrix} |a_1b_2| & |a_1b_3| & |a_1b_4| & |a_1b_5| & b_6 \\ |a_1c_2| & |a_1c_3| & |a_1c_4| & |a_1c_5| & c_6 \\ |a_1d_2| & |a_1d_3| & |a_1d_4| & |a_1d_5| & d_6 \\ |a_1e_2| & |a_1e_3| & |a_1e_4| & |a_1e_5| & e_6 \\ f_2 & f_3 & f_4 & f_5 & . \end{vmatrix} \div a_i^2,$$

$$\begin{vmatrix} a_1 & a_2 & a_3 & a_4 & a_5 & . \\ b_1 & b_2 & b_3 & b_4 & b_5 & . \\ c_1 & c_2 & c_3 & c_4 & c_5 & c_6 \\ d_1 & d_2 & d_3 & d_4 & d_5 & d_6 \\ e_1 & e_2 & e_3 & e_4 & e_5 & e_6 \\ . & . & f_3 & f_4 & f_5 & . \end{vmatrix} = \begin{vmatrix} |a_1b_2c_3| & |a_1b_2c_4| & |a_1b_2c_5| & c_6 \\ |a_1b_2d_3| & |a_1b_2d_4| & |a_1b_2d_5| & d_6 \\ |a_1b_2e_3| & |a_1b_2e_4| & |a_1b_2e_5| & e_6 \\ f_3 & f_4 & f_5 & . \end{vmatrix} \div |a_1b_2|,$$

$$\begin{vmatrix} a_1 & a_2 & a_3 & a_4 & a_5 & . \\ b_1 & b_2 & b_3 & b_4 & b_5 & . \\ c_1 & c_2 & c_3 & c_4 & c_5 & . \\ d_1 & d_2 & d_3 & d_4 & d_5 & d_6 \\ e_1 & e_2 & e_3 & e_4 & e_5 & e_6 \\ . & . & . & f_4 & f_5 & . \end{vmatrix} = \begin{vmatrix} |a_1b_2c_3d_4| & |a_1b_2c_3d_5| & d_6 \\ |a_1b_2c_3e_4| & |a_1b_2c_3e_5| & e_6 \\ f_4 & f_5 & . \end{vmatrix} \quad (X.)$$

9. Let us consider now the case where none of the three non-zero elements is in the place 11, Δ being

$$\begin{vmatrix} . & . & d & e & f \\ . & g & h & i & j & k \\ . & h & l & m & n & o \\ d & i & m & p & q & r \\ e & j & n & q & s & t \\ f & k & o & r & t & u \end{vmatrix}.$$

Here, as before, we have

$$N(d\sqrt{P} + e\sqrt{S} + f\sqrt{U}) = \begin{vmatrix} . & d^2 & e^2 & f^2 \\ d^2 & . & U & S \\ e^2 & U & . & P \\ f^2 & S & P & . \end{vmatrix}$$

where

$$U = \begin{vmatrix} . & . & . & d & e \\ . & g & h & i & j \\ . & h & l & m & n \\ d & i & m & p & q \\ e & j & n & q & s \end{vmatrix}$$

$$\begin{aligned}
&= -d^2 \begin{vmatrix} 146 \\ 146 \end{vmatrix} - e^2 \begin{vmatrix} 156 \\ 156 \end{vmatrix} + 2de \begin{vmatrix} 156 \\ 146 \end{vmatrix}, \\
S &= -d^2 \begin{vmatrix} 145 \\ 145 \end{vmatrix} - f^2 \begin{vmatrix} 156 \\ 156 \end{vmatrix} + 2df \begin{vmatrix} 146 \\ 145 \end{vmatrix}, \\
P &= -e^2 \begin{vmatrix} 145 \\ 145 \end{vmatrix} - f^2 \begin{vmatrix} 146 \\ 146 \end{vmatrix} + 2ef \begin{vmatrix} 146 \\ 145 \end{vmatrix}.
\end{aligned}$$

If, therefore, we perform on the determinant the operations

$$\begin{aligned}
\text{row}_2 + \begin{vmatrix} 156 \\ 156 \end{vmatrix} \text{row}_1, & \quad \text{row}_3 + \begin{vmatrix} 146 \\ 146 \end{vmatrix} \text{row}_1, \\
\text{col}_2 + \begin{vmatrix} 156 \\ 156 \end{vmatrix} \text{col}_1, & \quad \text{col}_3 + \begin{vmatrix} 146 \\ 146 \end{vmatrix} \text{col}_1,
\end{aligned}$$

there is obtained

$$\begin{aligned}
N &= \begin{vmatrix} d^2 & e^2 & f^2 \\ d^2 & 2d^2 \begin{vmatrix} 156 \\ 156 \end{vmatrix} & 2de \begin{vmatrix} 156 \\ 146 \end{vmatrix} & 2df \begin{vmatrix} 156 \\ 145 \end{vmatrix} \\ e^2 & 2de \begin{vmatrix} 156 \\ 146 \end{vmatrix} & 2e^2 \begin{vmatrix} 146 \\ 146 \end{vmatrix} & 2ef \begin{vmatrix} 146 \\ 145 \end{vmatrix} \\ f^2 & 2df \begin{vmatrix} 156 \\ 145 \end{vmatrix} & 2ef \begin{vmatrix} 146 \\ 145 \end{vmatrix} & 2f^2 \begin{vmatrix} 145 \\ 145 \end{vmatrix} \end{vmatrix}, \\
&= 4d^2e^2f^2 \begin{vmatrix} d & e & f \\ d & \begin{vmatrix} 156 \\ 156 \end{vmatrix} & \begin{vmatrix} 156 \\ 146 \end{vmatrix} & \begin{vmatrix} 156 \\ 145 \end{vmatrix} \\ e & \begin{vmatrix} 146 \\ 146 \end{vmatrix} & \begin{vmatrix} 146 \\ 146 \end{vmatrix} & \begin{vmatrix} 146 \\ 145 \end{vmatrix} \\ f & \begin{vmatrix} 145 \\ 145 \end{vmatrix} & \begin{vmatrix} 145 \\ 146 \end{vmatrix} & \begin{vmatrix} 145 \\ 145 \end{vmatrix} \end{vmatrix}, \\
&= 4d^2e^2f^2 \cdot \begin{vmatrix} g & h \\ h & l \end{vmatrix} \cdot \Delta \quad \text{by §8.}
\end{aligned}$$

We thus have the theorem: *In any axisymmetric determinant Δ which has $(1, r)$ equal to zero for all values of r except $n-2, n-1, n$, the cofactor of Δ in the norm of*

$$(1, n-2) \sqrt{[n-2, n-2]} + (1, n-1) \sqrt{[n-1, n-1]} + (1, n) \sqrt{[n, n]}$$

is

$$4(1, n-2)^2(1, n-1)^2(1, n)^2 \cdot \begin{vmatrix} 2 & 3 & \dots & n-3 \\ 1 & 3 & \dots & n-3 \end{vmatrix} \quad (\text{XI.})$$

where $\begin{vmatrix} 2 & 3 & \dots & n-3 \\ 2 & 3 & \dots & n-3 \end{vmatrix}$ is the minor of Δ occupying the rows and columns whose numbers are 2, 3, ..., $n-3$.

10. It is easy to see from §3 why the norm in the preceding case should vanish when any one of the three elements $(1, n-2), (1, n-1),$

(1, n) vanishes: why it should vanish when $\begin{vmatrix} 2 & 3 & \dots & n-3 \\ 2 & 3 & \dots & n-3 \end{vmatrix}$ vanishes is not so apparent, and the reason when found is of considerable interest.

As a preliminary let us consider an axisymmetric determinant of the form

$$\begin{vmatrix} \cdot & \cdot & \cdot & \cdot & b & c \\ \cdot & d & e & f & h & i \\ \cdot & e & j & k & m & n \\ \cdot & f & k & o & q & r \\ b & h & m & q & v & w \\ c & i & n & r & w & x \end{vmatrix}$$

on the supposition that the minor

$$\begin{vmatrix} d & e & f \\ e & j & k \\ f & k & o \end{vmatrix} = 0.$$

It is at once evident that the determinant is equal to

$$\begin{aligned} & -b^2 \begin{vmatrix} d & e & f & i \\ e & j & k & n \\ f & k & o & r \\ i & n & r & x \end{vmatrix} + 2bc \begin{vmatrix} d & e & f & i \\ e & j & k & n \\ f & k & o & r \\ h & m & q & w \end{vmatrix} - c^2 \begin{vmatrix} d & e & f & h \\ e & j & k & m \\ f & k & o & q \\ h & m & q & v \end{vmatrix} \\ & = -b^2 \xi + 2bc\eta - c^2\zeta, \text{ say,} \end{aligned}$$

where ξ , η , ζ may be looked on as the principal minors corresponding to the elements v , w , x of

$$\begin{vmatrix} d & e & f & h & i \\ e & j & k & m & n \\ f & k & o & q & r \\ h & m & q & v & w \\ i & n & r & w & x \end{vmatrix},$$

and where therefore

$$\begin{vmatrix} \xi & \eta \\ \eta & \zeta \end{vmatrix} = \begin{vmatrix} d & e & f & h & i \\ e & j & k & m & n \\ f & k & o & q & r \\ h & m & q & v & w \\ i & n & r & w & x \end{vmatrix} \cdot \begin{vmatrix} d & e & f \\ e & j & k \\ f & k & o \end{vmatrix} = 0.$$

From this it follows that

$$-b^2\xi + 2bc\eta - c^2\zeta = -(b\xi^{\frac{1}{2}} - c\xi^{\frac{1}{2}})^2;$$

so that we have the theorem: *Any axisymmetric determinant which has (1, r) equal to zero for all values of r except $n-1$ and n is expressible as a square if the minor $\begin{vmatrix} 2 & 3 & \dots & n-2 \\ 2 & 3 & \dots & n-2 \end{vmatrix}$ vanishes.* (XII.)

11. In order to find the actual quadrate expression in (XII.) we note that ξ , ζ being both axisymmetric determinants with a vanishing principal minor are themselves expressible as squares, namely,

$$\begin{aligned} -\xi &= \left\{ r \begin{vmatrix} d & e \\ e & j \end{vmatrix}^{\frac{1}{2}} - n \begin{vmatrix} d & f \\ f & o \end{vmatrix}^{\frac{1}{2}} + i \begin{vmatrix} j & k \\ k & o \end{vmatrix}^{\frac{1}{2}} \right\}^2, \\ -\zeta &= \left\{ q \begin{vmatrix} d & e \\ e & j \end{vmatrix}^{\frac{1}{2}} - m \begin{vmatrix} d & f \\ f & o \end{vmatrix}^{\frac{1}{2}} + h \begin{vmatrix} j & k \\ k & o \end{vmatrix}^{\frac{1}{2}} \right\}^2. \end{aligned}$$

We thus have the determinant with which we started, and which has been shown equal to $-(b\xi^{\frac{1}{2}} - c\zeta^{\frac{1}{2}})^2$,

$$\begin{aligned} &= - \left[b \sqrt{-1} \left\{ r \begin{vmatrix} d & e \\ e & j \end{vmatrix}^{\frac{1}{2}} - n \begin{vmatrix} d & f \\ f & o \end{vmatrix}^{\frac{1}{2}} + i \begin{vmatrix} j & k \\ k & o \end{vmatrix}^{\frac{1}{2}} \right\} \right. \\ &\quad \left. - c \sqrt{-1} \left\{ q \begin{vmatrix} d & e \\ e & j \end{vmatrix}^{\frac{1}{2}} - m \begin{vmatrix} d & f \\ f & o \end{vmatrix}^{\frac{1}{2}} + h \begin{vmatrix} j & k \\ k & o \end{vmatrix}^{\frac{1}{2}} \right\} \right]^2, \\ &= \left[\begin{vmatrix} b & c \\ q & r \end{vmatrix} \begin{vmatrix} d & e \\ e & j \end{vmatrix}^{\frac{1}{2}} - \begin{vmatrix} b & c \\ m & n \end{vmatrix} \begin{vmatrix} d & f \\ f & o \end{vmatrix}^{\frac{1}{2}} + \begin{vmatrix} b & c \\ h & i \end{vmatrix} \begin{vmatrix} j & k \\ k & o \end{vmatrix}^{\frac{1}{2}} \right]^2, \end{aligned}$$

as required.

It may be noted that the form of the result suggests the following alternative proof: By performing the operations

$$c \text{ row}_5 - b \text{ row}_6, \quad c \text{ col}_5 - b \text{ col}_6$$

we have the given determinant

$$\begin{aligned} &= \begin{vmatrix} \cdot & \dot{d} & \dot{e} & \dot{f} & ch - bi & c \\ \cdot & e & j & k & cm - bn & i \\ \cdot & f & k & o & cq - br & n \\ \cdot & ch - bi & cm - bn & cq - br & c^2v - 2bcw + bx^2 & r \\ c & i & n & r & cw - bx & x \end{vmatrix} \div c^2, \\ &= - \begin{vmatrix} d & e & f & ch - bi \\ e & j & k & cm - bn \\ f & k & o & cq - br \\ ch - bi & cm - bn & cq - br & c^2v - 2bcw + bx^2 \end{vmatrix}, \\ &= \left[(ch - bi) \begin{vmatrix} j & k \\ k & o \end{vmatrix}^{\frac{1}{2}} - (cm - bn) \begin{vmatrix} d & f \\ f & o \end{vmatrix}^{\frac{1}{2}} + (cq - br) \begin{vmatrix} d & e \\ e & j \end{vmatrix}^{\frac{1}{2}} \right]^2 \quad \text{(XIII.)} \end{aligned}$$

by the theorem just used in regard to ξ and ζ .

12. Returning now to the norm

$$N(a\sqrt{S} + b\sqrt{V} + c\sqrt{X})$$

connected with the determinant

$$\begin{vmatrix} \cdot & \cdot & \cdot & \cdot & a & b & c \\ \cdot & d & e & f & g & h & i \\ \cdot & e & j & k & l & m & n \\ \cdot & f & k & o & p & q & r \\ a & g & l & p & s & t & u \\ b & h & m & q & t & v & w \\ c & i & n & r & u & w & x \end{vmatrix},$$

let us see what happens when the minor $\begin{vmatrix} 2 & 3 & 4 \\ 2 & 3 & 4 \end{vmatrix}$ vanishes. From §10 we learn that on that supposition S, V, X are expressible as squares, and consequently can have their roots extracted. We thus have from §11

$$\begin{aligned} & a\sqrt{S} + b\sqrt{V} + c\sqrt{X} \\ = & a \left\{ \begin{vmatrix} b & c \\ q & r \end{vmatrix} \cdot \begin{vmatrix} d & e \\ e & j \end{vmatrix}^{\frac{1}{2}} - \begin{vmatrix} b & c \\ m & n \end{vmatrix} \cdot \begin{vmatrix} d & f \\ f & o \end{vmatrix}^{\frac{1}{2}} + \begin{vmatrix} b & c \\ h & i \end{vmatrix} \cdot \begin{vmatrix} d & k \\ k & o \end{vmatrix}^{\frac{1}{2}} \right\} \\ & - b \left\{ \begin{vmatrix} a & c \\ p & r \end{vmatrix} \cdot \begin{vmatrix} d & e \\ e & j \end{vmatrix}^{\frac{1}{2}} - \begin{vmatrix} a & c \\ l & n \end{vmatrix} \cdot \begin{vmatrix} d & f \\ f & o \end{vmatrix}^{\frac{1}{2}} + \begin{vmatrix} a & c \\ g & i \end{vmatrix} \cdot \begin{vmatrix} d & k \\ k & o \end{vmatrix}^{\frac{1}{2}} \right\} \\ & + c \left\{ \begin{vmatrix} a & b \\ p & q \end{vmatrix} \cdot \begin{vmatrix} d & e \\ e & j \end{vmatrix}^{\frac{1}{2}} - \begin{vmatrix} a & b \\ l & m \end{vmatrix} \cdot \begin{vmatrix} d & f \\ f & o \end{vmatrix}^{\frac{1}{2}} + \begin{vmatrix} a & b \\ g & h \end{vmatrix} \cdot \begin{vmatrix} d & k \\ k & o \end{vmatrix}^{\frac{1}{2}} \right\}, \\ = & \begin{vmatrix} a & b & c \\ a & b & c \\ p & q & r \end{vmatrix} \cdot \begin{vmatrix} d & e \\ e & j \end{vmatrix}^{\frac{1}{2}} - \begin{vmatrix} a & b & c \\ a & b & c \\ l & m & n \end{vmatrix} \cdot \begin{vmatrix} d & f \\ f & o \end{vmatrix}^{\frac{1}{2}} + \begin{vmatrix} a & b & c \\ a & b & c \\ g & h & i \end{vmatrix} \cdot \begin{vmatrix} d & k \\ k & o \end{vmatrix}^{\frac{1}{2}}, \\ = & 0. \end{aligned}$$

And this vanishing expression being one of the eight whose product constitutes the norm, the norm itself must vanish.

CAPETOWN, S.A.
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