$$= \frac{1}{\underline{7}} \begin{vmatrix} 1, 2, 0, 0, 0, 0 \\ 0, 1, 3, 0, 0, 0 \\ 0, 0, 0, 4, 0, 0 \\ 0, 0, 1, 2, 5, 0 \\ 0, 0, 1, 4, 5, 6 \\ 0, 0, 1, 5, 10, 9 \end{vmatrix} = \frac{1}{\underline{7}} \begin{vmatrix} 1, 2, 0, 0, 0, 0 \\ 0, 1, 3, 0, 0, 0 \\ 0, 0, 0, 4, 0, 0 \\ 0, 0, 1, 2, 5, 0 \\ 0, 0, 0, 2, 0, 6 \\ 0, 0, 0, 1, 5, 3 \end{vmatrix}$$

$$= \frac{4 \cdot 5 \cdot 6}{\underline{7}} = \frac{1}{42}.$$

$$(2) \dots B_1 = \frac{1}{63 \cdot 8 \cdot 2} \begin{vmatrix} 1, 2, 0 \\ 1, 4, 4 \\ 1, 6, 20 \end{vmatrix} = \frac{56 - 2 \cdot 16}{63 \cdot 8 \cdot 2} = \frac{1}{42}.$$

$$(3) \dots B_1 = \frac{6}{64 \cdot 63} \begin{vmatrix} 1, 1, 0 \\ 1, 3, 1 \\ 1, 5, 10 \end{vmatrix} = \frac{(25-9)6}{64 \cdot 63} = \frac{1}{42}.$$

$$(4) \dots B_1 = \frac{4 \cdot 6}{\underline{7} (32-1)} \begin{vmatrix} 1, 3, 0 \\ 1, 10, 5 \\ 1, 21, 35 \end{vmatrix} = \frac{4 \cdot 6 (350-105-3 \cdot 30)}{31 \underline{7}}$$

$$= \frac{4 \cdot 6 \cdot 155}{31 \underline{7}} = \frac{1}{42}.$$

The formula (5) gives

$$\sec x = 1 + \frac{x^2}{\underline{2}} + \begin{vmatrix} 1, 1 \\ 1, 6 \end{vmatrix} \frac{x^4}{\underline{4}} + \begin{vmatrix} 1, 1, 0 \\ 1, 6, 1 \\ 1, 15, 15 \end{vmatrix} \frac{x^6}{\underline{6}} + \dots$$

Or
$$\sec x = 1 + \frac{x^2}{\underline{2}} + \frac{5x^4}{\underline{4}} + \frac{61x^6}{\underline{6}} + \dots$$

This result may be verified by ordinary division, or otherwise.

On Three-bar Motion in Plane Spaces. By S. ROBERTS, M.A.

[Read Nov. 11th, 1875.]

1. I propose to extend briefly to general three-bar motion a discussion which I have before given of some particular cases; and fortunately I find my task rendered easy by some recent discoveries in relation to the subject.

It is known that the locus of a point rigidly attached to a link connected immediately with two others moving in a plane about two

fixed points, is a sextic trircular curve having three nodes besides the triple circular points.

If we consider the general movement as that of a plane moving with two of its points on two fixed circles in the same level, there are nine constants at our disposal; namely, the three parameters of each of the circles and the three parameters connecting the describing point with the points conditioned to move in circles.

Practically, and with reference to what is more strictly regarded as mechanical three-bar motion, there are further limitations, since the bars are finite. Nevertheless, it will be convenient to consider generally the curves represented by the analytical equation. In the first instance, therefore, I give one form out of many of the equation of the locus.

2. Our curve is generated by the vertex C of a triangle ABC, moving with its base angles A and B on two circles, having respectively their centres at O, O', and the radii OA and O'B.

Take O as the origin of rectangular axes and let OO' be the axis of x . Let the co-ordinates of A, B, C respectively be (x', y') , (x'', y'') , and (x, y) .

Then, denoting as usual the sides and angles of the triangle by a, b, c, A, B, C , and by θ the angle which AC makes with the axis of x , we have

$$x' = x - b \cos \theta, \quad x'' = x - a \cos (C + \theta),$$

$$y' = y - b \sin \theta, \quad y'' = y - a \sin (C + \theta).$$

and writing r for OA, and s for O'B, and γ for OO', we have also

$$x'^2 + y'^2 = r^2, \quad (x'' - \gamma)^2 + y''^2 = s^2.$$

Determining therefore $\cos \theta$, $\sin \theta$ by means of

$$x \cos \theta + y \sin \theta = \frac{x^2 + y^2 + b^2 - r^2}{2b},$$

$$\begin{aligned} \{ (x - \gamma) \cos C + y \sin C \} \cos \theta - \{ (x - \gamma) \sin C - y \cos C \} \sin \theta \\ = \frac{(x - \gamma)^2 + y^2 + a^2 - s^2}{2a}; \end{aligned}$$

and putting k for $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$, we get for the equation of the locus

$$\begin{aligned} & \left\{ \sin A [(x - \gamma) \sin C - y \cos C] [x^2 + y^2 + b^2 - r^2] \right. \\ & \quad \left. + \sin B y [(x - \gamma)^2 + y^2 + a^2 - s^2] \right\}^2 \\ & + \left\{ \sin A [(x - \gamma) \cos C + y \sin C] [x^2 + y^2 + b^2 - r^2] \right. \\ & \quad \left. - \sin B x [(x - \gamma)^2 + y^2 + a^2 - s^2] \right\}^2 \\ & = 4k^2 \sin^2 A \sin^2 B \sin^2 C [x(x - \gamma) + y^2 - \gamma \cotan C y]^2 = 0 \dots (1). \end{aligned}$$

3. The equation is of the form

$$(L\phi + M\psi)^2 + (N\phi + P\psi)^2 = K(LP - MN)^2,$$

where ϕ , ψ , and $LP - MN$ represent circles; and it is immediately seen that the three finite nodes lie on the last circle.

It can be shown geometrically, and very simply, that OO' subtends an angle C , or $180^\circ - C$, at a node. For, suppose ABC , $A'B'C$ are the two positions of the traversing triangle which determine a node; then we have

$$\angle OCO' = \angle OCA - \angle O'CB + \angle ACB = C,$$

$$\text{or } \angle OCO' = \angle OCA + \angle O'CB - \angle ACB = 180^\circ - C.$$

Again, if ρ , ρ' are the distances of the node at C from O , O' respectively, we have

$$r^2 = \rho^2 + b^2 - 2\rho b \cos A'CO,$$

$$s^2 = \rho'^2 + a^2 - 2\rho a \cos A'CO,$$

$$\gamma^2 = \rho^2 + \rho'^2 - 2\rho\rho' \cos C;$$

giving, to determine the nodes,

$$\rho^2 \{ \sin C \cos (C-B) \rho^2 + \sin B (\gamma^2 + a^2 - s^2) + \sin A \cos C (b^2 - r^2) \}^2 - (\gamma^2 - \sin^2 C \rho^2) \{ \sin (C-B) \rho^2 + \sin A (b^2 - r^2) \}^2 = 0 \dots (2).$$

4. I remark, also, that the two centres O , O' are singular foci ("foyers singuliers" of M. Laguerre), that is to say, real intersections of the circular asymptotes. There is plainly another such focus, and it is natural to infer that it lies on the circle $LP - MN = 0$, as do the two foci O and O' .

In fact, by substituting X for $\frac{x+iy}{2}$ and Y for $\frac{x-iy}{2}$ in (1), and taking the coefficient of Y^2 , we get the tangents at one of the circular points, and they are found to be

$$x+iy=0, \quad x-\gamma+iy=0, \quad x-\gamma \frac{\sin B}{\sin C} \cos A + i \left(y - \gamma \frac{\sin A \sin B}{\sin C} \right).$$

Hence the chord OO' subtends the angle C at the third focus, and we see that the triangle of foci is similar to the traversing triangle.

We also see, at once, how it is impossible to describe by these mechanical means curves having coincident tangents at the circular points, e. g., the Limaçon of Pascal; for, in that case, two or more of the foci would coincide. The foci do indeed coincide when the directrix-circles are concentric, and the locus degenerates.

Since the absolute term of (2) is $-\frac{\sin^2 A \gamma^2 (b^2 - r^2)^2}{\sin^2 C}$, if $b^2 = r^2$, the focus at the origin is also a node; if also $a^2 = s^2$, the point O' is a node; and in this case the third focus is also a node. If $a = b = c$, we have a remarkably symmetrical curve described in three ways by the same linkage with different pairs of centres.

5. We are thus led to remark that, since the singular foci are similarly related to the locus, we might have taken as fixed centres the focus O , and the third focus, say O'' , and by means of links of suitable lengths we should obtain the same locus. And in like manner we might have taken as centres O' and O'' . Moreover, the triangle of foci may be said to be not only similar to the traversing triangle, but similarly disposed with regard to the centres about which the traversing triangle rotates. For the angle A of the triangle of foci is at O the centre of the circle on which the angle A of the traversing triangle moves, and the angle B of the triangle of foci is at O' .

We conclude, then, that the locus in question can be described in three different ways, by similar traversing triangles moving with an angle on each of two circles about two singular foci as centres, and the angle moving on a circle is equal to the angle subtended at its centre by the two remaining singular foci.

6. I have already had occasion to notice a particular case of the foregoing threefold generation, namely, when the locus breaks up into a circle and the pedal of a conic.

Thus, let F , F' , and E be the foci and centre of the conic of which part of the locus is the pedal. Then, if Q , R , S are the singular foci of the complete locus, the last being the centre of the extraneous circle, I have shewn that, O being the node of the pedal,

$$\angle ROS = \angle QSO,$$

$$\angle QOS = \angle RSO.$$

Also, if Q , S are taken as centres of the directrix circles, the radius of the circle about Q as centre is $\frac{OF'}{2}$, and the radius of the circle about

S as centre is $\frac{OF'}{2e}$, where e is the eccentricity of the conic. Similarly,

if we take R and S as centres, the radius of the circle about R as centre is $\frac{OF}{2}$, and that of the circle about S as centre is $\frac{OF}{2e}$. If, again, we

take circles about Q and R as centres for directrices, their radii are equal to semi-major axis of the conic. The traversing triangle is similar, in each case, to the triangle QRS , and similarly disposed in the sense previously mentioned. In the second mode of generation, the node O is determined as at a distance $OD = FE$ from the extremity D of a diameter of the circle about R as centre, where it intersects OS , which is parallel to QR and FF' . This is the determination of the node given in effect in my paper on the Pedals of Conics (*Proc*,

No. 26). M. Mannheim has found it in a different way (*Proc.*, No. 77, p. 35), where the figure gives a hyperbola as primitive, while the above implies an ellipse.

7. It remains to determine the relations existing among the radii of the different directrix circles in the general case. Fortunately we can avoid the tedious process of deriving these from the equation by making use of Dr. Sylvester's discovery of the "Plagiograph," or Skew Pantograph, which is, in fact, the key to the theory of three-bar motion.

In Fig. 3, OA , $O'B$ are the rotating links, AB is the traversing link, and ABC is the traversing triangle. Then, completing the parallelogram $OA'BA$, and taking AP in $AB = AC$, draw OP meeting $A'B$ in Q . Make $\angle BA'C' = \angle BAC$, and $A'C' = A'Q$; then C' describes a curve similar to that described by C , but twisted through an angle A .

Now, the triangle $A'BC'$ is similar to the triangle ABC , but differently placed with regard to the traversing link $A'B$, viz., $\angle BA'C' = \angle A$, $\angle A'CB = \angle B$, $\angle A'BC' = \angle C$. In order to make the transformed figure equal as well as similar to the primitive, we must alter its magnitude in the ratio $\frac{AC}{AB}$. Hence we see at once that, if OA_1 is the

radius of the other directrix circle with centre O , we must have $OA_1 = AC$. That is to say, the radius of a directrix circle about O as centre is equal to the adjacent side of the traversing triangle, when the conjugate circle about the same centre is employed.

In order better to realise the result, suppose that the three singular foci form a triangle OO_1O_2 , and that ABC , $A_1B_1C_1$, $A_2B_2C_2$ are the traversing triangles corresponding to the centres (O_1, O_2) , (O, O_2) , (O, O_1) respectively.

Then we have, using the usual notation,

$$\begin{array}{lll} O_1A = c_2, & O_2B_1 = a, & OC_1 = a_2, \\ O_1A_2 = b, & O_2B = c_1, & OC_2 = b_1. \end{array}$$

Hence $OC_1 \cdot O_1A_2 \cdot O_2B = OC_2 \cdot O_1A \cdot O_2B_1$;

and, in terms of a, b, c , $O_1A = r$, and $O_2B = s$, the sides of the three triangles are (a, b, c) , $(\frac{a}{c}s, \frac{b}{c}s, s)$, $(\frac{a}{c}r, \frac{b}{c}r, r)$.

8. We are now in a position to draw for any point of the locus the corresponding positions of the traversing triangles, and determine a remarkable linkage, of which I think it is worth while to give a figure, omitting some interesting connections, which can be easily supplied.

I observed in a paper "On the Motion of a Plane under certain conditions,"* that, from the form of the analytical equations for obtaining the locus of a point rigidly connected with a limited line moving with its extremities on two given curve directrices, the curve so generated was also generated by the intersections of the two curves supposed to move always parallel to themselves, and translated to the extremities of two links, inclined at a constant angle, and rotating about two points.

That is to say, in the case we are dealing with, O_1, O_2 being the fixed centres, O_1O_2FH is a parallel ruler; D is a point rigidly connected with O_1H ; DC, CF are two links moving about D, C , and F ; then C describes our locus. The thing is evident immediately we complete the parallelograms O_1DCA, O_2FCB , and join AB ; for the triangle ABC is equal to the constant triangle O_1DH . Dr. Sylvester's Skew Pantograph can now be applied; that is to say, making $\angle EDC = \angle BAC, \angle DCE = \angle ABC$, we get another constant triangle DCE , whose vertex E moves on a circle about O as centre, O_1O_2O being the triangle of foci.

Similarly we determine the triangle CFG , moving with G on a circle about O as centre.

If the figure were completed, we should have a system of nine triangular links, all similar and equal in threes, a pair rotating about each of the foci O, O_1, O_2 , and the other three joining at the describing point and connected with the rotating links in the manner above indicated.

Many of the relations of this linkage are obvious on inspection, and I need not dwell on them. It will be seen that if we take two of the foci and the two links rotating about each of them, the two links which do not belong to the three-bar linkage remain inclined at a constant angle, and their intersection, when prolonged, lies on the circle of nodes and foci. The intersection of the others gives the instantaneous centre. Thus for the system we get three such centres, which lie on and determine the normal. The locus of instantaneous centres for each three-bar movement is a curve of the eighth order, having quadruple points at the two corresponding singular foci, passing twice through the circular points, and having the deficiency unity. The corresponding loci on the moving planes are of the same kind. When the describing point is on the traversing link, the foregoing constructions are, of course, much simplified. The rotating links are parallel in pairs, and the line joining two extremities of parallel links passes through the remaining focus, as follows from similar triangles.

* *Proc.*, Vol. III., p. 287.

9. The equation (1) has been written in a form which shows at once that, if $b^2 - r^2$, $a^2 - s^2$ remain constant, and the foci are given, the curves form a faisceau of sextics with the parameter k . The fundamental points are the nodes and triple circular points, multiple in all the curves, and the points of contact of the tangents at the triple points.

For $\frac{1}{k} = 0$, the locus is a circle twice and the line at infinity twice.

The traversing triangle is infinite, but the describing angle is C , and in effect the rotating links OA , $O'B$ prolonged constitute two sides of the triangle, and C describes the circle of nodes and foci.

Again, if $k = 0$, we have two imaginary cubics, each having a node at a circular point, and passing through the remaining one, the nodes being different in the two curves, the tangents at the nodes and the other circular points constitute the six circular asymptotes of the sextic, and give the singular foci.

It can now be shown that the family of sextics in question is defined by the conditions—(1) the circular points triple; (2) three other nodes and the real singular foci lie on a circle; (3) the triple points and nodes, and the six points of contact of tangents at the triple points, form a complete intersection of sextics.*

For, as one fundamental curve of such a faisceau we may take a doubled circle and the line at infinity twice, since this combination fulfils the analytical conditions. And again, if we write X for $\frac{x+iy}{2}$,

Y for $\frac{x-iy}{2}$; and Z for unity, so that

$$tY(X+aZ) - sX(Y+aZ) = 0 \dots\dots\dots(a)$$

$$\text{is a circle, and } XY(X+aZ) + bX^2Z + cXZ^2 + dZ^3 = 0 \dots\dots\dots(b)$$

is a cubic having a node at XZ , with tangents X and $X+aZ$, and a simple point at YZ . We may determine b, c, d so that the cubic passes through three arbitrary points on the circle (a). Suppose this to be done. We may next multiply (b) by $\frac{a}{a-b}$ or $1 + \frac{b}{a-b}$, and write it in the form

$$(X+aZ) \left(XY + \frac{d}{a-b} Z^2 \right) + \frac{bX}{a-b} \left\{ XY + a(X+Y)Z + \frac{ac-d}{b} Z^2 \right\} = 0 \dots\dots\dots(b'),$$

* In a paper containing further developments, which will shortly appear, Professor Cayley has stated more directly and simply the connection between the nodes and foci thus: (1) with regard to the centre and a diameter of the circle, the sum of the angular distances of the nodes is equal to the corresponding sum for the foci; (2) otherwise, the triangles of nodes and of foci circumscribe a parabola.

and then the cubic

$$(Y + aZ) \left(XY + \frac{d}{a-b} Z^2 \right) + \frac{tb}{s(a-b)} \left\{ XY + a(X+Y)Z + \frac{ac-d}{b} Z^2 \right\} = 0 \dots\dots(c)$$

will pass through the same three finite points on the circle (*a*), and has a node at *YZ* and a simple point at *XZ*, the tangents at the node being *Y* = 0, *Y* + *aZ* = 0, which intersect the corresponding tangents to the other cubic upon the given circle.

The cubics are now determined, but it will be observed that the tangents to (*b'*) at *YZ*, and to (*c'*) at *XZ*, namely,

$$Y + \frac{b}{a-b} (Y + aZ) = 0,$$

$$X + \frac{tb}{s(a-b)} (X + aZ) = 0,$$

also intersect on the circle.

Hence, to determine the faisceau, we can always get the form for three-bar motion.

10. In conclusion, I refer shortly to the inverse curves.

The inverse of the general locus with respect to an arbitrary point is also a sextic curve, having the circular points at infinity for triple points and three other nodes. The singular foci, however, do not correspond as inverses to those of the primitive, and the inverse curve is not in general a case of three-bar motion. If the describing point is on the traversing link, the circle of nodes and foci degenerates to a line, or rather breaks up into a finite line, and the line at infinity and the locus is symmetrical with regard to the finite line. The inverse with respect to a point on the axis is also symmetrical with regard to it, and the nodes and singular foci lie on that line. If, then, we take the inverse with respect to (1) any point on the axis, or (2) a node, it is the same family as the primitive, and included analytically as a case of three-bar motion.

I shall verify (1) by identifying the form of the inverse and that of the primitive.

(i.) Multiply equation (1) by k^2 , and write a, b for $k \sin A, k \sin B$ respectively; then, making $C = 0$, we get for the equation of the locus, when the describing point is on the link,

$$y^2 \{ a(x^2 + y^2 + b^2 - r^2) - b[(x - \gamma)^2 + y^2 + a^2 - s^2] \}^2 + \{ a(x^2 + y^2 + b^2 - r^2) - bx[(x - \gamma)^2 + y^2 + a^2 - s^2] \}^2 - 4a^2b^2\gamma^2y^2 = 0,$$

which, if ρ^2 is written for $x^2 + y^2$, is of the form

$$y^2(\rho^2 + 2Px + Q)^2 + [x(\rho^2 + 2Px + Q) + R\rho^2 + S]^2 + Ty^2 = 0 \dots\dots(3),$$

where *P, Q, R, S, T* are independent constants.

For a new origin on the axis, the general form with six constants is

$$(Ax\rho^3 + B\rho^3 + Cx + D)^2 + y^2 (A^2\rho^4 + E^2\rho^3 + F^2x^2 + G^2) = 0.$$

The inverse with regard to the new origin, the modulus being unity, is

$$(Ax + B\rho^3 + Cx\rho^3 + D\rho^4)^2 + y^2 (A^2 + E^2\rho^3 + F^2x\rho^3 + G^2\rho^4) = 0;$$

or, dividing out ρ^2 , we get

$$(A + Bx + C\rho^3 + Dx\rho^3)^2 + y^2 \left\{ D^2\rho^4 + (G^2 - C^2 + 2BD)\rho^3 + F^2x \right. \\ \left. + E^2 + B^2 - 2AC \right\} = 0,$$

the same form as that of the primitive.

(ii.) If we take as origin a node of the symmetrical locus, the inverse is a quartic curve, symmetrical with regard to the same axis. It is binodal and circular, having its nodes on the axis.

I have elsewhere drawn attention to the interesting class of quartics described by a point rigidly connected with a limited line moving with its extremities on a circle and straight line. In the general case the locus is binodal and circular, having its singular focus collinear with the nodes, and having two parallel double tangents, whose points of contact lie on a conic through the nodes.

If the describing point is on the traversing link, we have a family of quartics, symmetrical, circular, and having two nodes on the axis.

The analytical equation includes all such quartics.

For this equation is easily found in the form

$$[a(x^2 + y^2 + b^2 - r^2) - 2b(x - k)x]^2 - 4b^2y^2[a^2 - (x - k)^2] = 0 \dots (4),$$

the centre of the circle directrix being the origin, the equation of the rule line being $x - k = 0$, and a, b being the distances of the describing point from the circle and line respectively. Now, if the singular focus is the origin, a symmetrical circular quartic with two nodes on the axis is represented by

$$y^2[y^2 + (1 + P^2)x^2 + 2PQx + R] + (Px^2 + Qx + S)^2 = 0,$$

$$\text{or } [y^2 + Px^2 + Qx + S]^2 + (1 - P)^2x^2 + 2(P - 1)Qxy + (R - 2S)y^2 = 0,$$

$$\text{or } \left[x^2 + y^2 + S - (1 - P)x \left(x - \frac{Q}{1 - P} \right) \right]^2 \\ - (1 - P)^2y^2 \left[\frac{2S - R - Q}{(1 - P)^2} - \left(x - \frac{Q}{1 - P} \right)^2 \right] = 0,$$

which is of the form (4).

A line directrix is, of course, a limiting case of a circle directrix. Accordingly, there are two ways of describing the same quartic, the third failing by the traversing triangle becoming infinite.

The foregoing result is interesting as shewing that a parallel motion may be established by adding an inverter and two other links to a symmetrical three-bar linkage. The construction, however, is clearly not always practicable.

Since writing the foregoing I have observed that a class of curves of

three-bar motion, not capable of being so described in a finite manner, are yet given by the following construction:—

If a constant angle moves with a fixed point of one side on a circle, while the other side touches another circle, the locus of the apex is a sextic curve with three finite nodes. The circular points are triple, and at each two tangents are coincident. The two singular foci lie on the circle of nodes. Thus several modes of mechanical description, seemingly far removed from each other, are closely associated, since, analytically, the curves described are included in the equation of three-bar motion curves. I cannot do more here than allude to this result. (See *Proc.*, Vol. II., § 10.)

Values of Certain Infinite Products, with an application to the Summation of the Geometrical Series of the n^{th} Order as a Definite Integral. By J. W. L. GLAISHER, M.A., F.R.S.

[Read November 11th, 1875.]

Starting with the identity

$$1 + x^n = 1 - \omega x \cdot 1 - \omega^3 x \dots 1 - \omega^{2n-1} x,$$

in which $\omega = \cos \frac{\pi}{n} + i \sin \frac{\pi}{n}$ (or any other prime $2n^{\text{th}}$ root of unity),

and writing for x successively $x^3, (\frac{1}{2}x)^3, (\frac{1}{3}x)^3$, &c., we have

$$1 + x^{2n} = 1 - (\zeta x)^2 \cdot 1 - (\zeta^3 x)^2 \dots 1 - (\zeta^{2n-1} x)^2,$$

$$1 + \left(\frac{x}{2}\right)^{2n} = 1 - \left(\frac{\zeta x}{2}\right)^2 \cdot 1 - \left(\frac{\zeta^3 x}{2}\right)^2 \dots 1 - \left(\frac{\zeta^{2n-1} x}{2}\right)^2,$$

...

and

$$x^n = (-1)^n \cdot \zeta x \cdot \zeta^3 x \dots \zeta^{2n-1} x,$$

$$\zeta \text{ being } = \sqrt{\omega} = \cos \frac{\pi}{2n} + i \sin \frac{\pi}{2n};$$

whence

$$\begin{aligned} x^n \cdot 1 + x^n \cdot 1 + \frac{x^{2n}}{2^{2n}} \cdot 1 + \frac{x^{2n}}{3^{2n}} \dots &= (-1)^n \frac{1}{\pi^n} \sin \pi \zeta x \sin \pi \zeta^3 x \dots \sin \pi \zeta^{2n-1} x \\ &= (-1)^n \frac{1}{\pi^n} \sin \pi (\beta_1 + i\alpha_1) x \sin \pi (\beta_3 + i\alpha_3) x \dots \sin \pi (\beta_{2n-1} + i\alpha_{2n-1}) x; \end{aligned}$$

whence, pairing the first and last factors, the second and the last but one, &c., and observing that

$$\sin(a + bi) \sin(-a + bi) = -\frac{1}{2} (\cosh 2b - \cos 2a)$$

and that, if n be uneven, the middle factor $\sin \pi \zeta^n x = i \sinh \pi x$, we obtain—

$$\begin{aligned} \text{I. } x^n \cdot 1 + x^n \cdot 1 + \frac{x^{2n}}{2^{2n}} \cdot 1 + \frac{x^{2n}}{3^{2n}} \dots &= \frac{1}{2^{1n} \pi^n} (\cosh 2\pi\alpha_1 x - \cos 2\pi\beta_1 x) \\ &\times (\cosh 2\pi\alpha_3 x - \cos 2\pi\beta_3 x) \dots (\cosh 2\pi\alpha_{n-1} x - \cos 2\pi\beta_{n-1} x); \end{aligned}$$

c 2