... $G^2R^2B^2$ will thus be perfectly definite, and will have the same properties as a route composed of black and red lines only; and the theory thus subsists without alteration.

I remark, in conclusion, that the group of 12 considered above is, in fact, the group of 12 positive substitutions upon 4 letters $abcd$; viz., the substitutions are 1, abc , acb , abd , adb , acd , adc , bcd , bdc , $ab.cd$, $ac.bd$, $ad.bc$; the groups each contain unity, three substitutions of the order (or index) 2, and 8 substitutions of the order (or index) 3, and their identity can be easily verified.

On Conjugate Four-piece Linkages.

By A. B. KEMPE, B.A.

[Read May 9, 1878.]

The plane linkage represented in Fig. 1, consisting of eight planes pivoted together [of which triangular portions only are shown in the figure], is in general a framework, *i.e.*, the various links cannot move about among themselves, but are fixed in relative position; and this is still so if any one piece be removed. In certain cases, however, this is not so, and the linkage is said to be "complete." It is the object of this paper to investigate these cases.

Fig. 1.

The investigation is important for this reason. The fundamental idea of linkages is constancy of distance; it is material therefore to investigate the various cases in which points on linkages are at a constant distance apart. Such points can be connected by links, and other links removed, so that we get new linkages having the same relations between certain of their parts as existed in the old ones. This constancy of distance does not prevail while the linkages go through the whole of their configurations,* as in "supplemental linkages," as these linkages may be termed in which points can be joined by new links, the algebraic relations between the

* The *form* of a linkage is the way in which the links are put together. The *dimensions* are the pivot distances. The *configuration* is the relative position of the links in a linkage of given form and dimensions.

parts break up. The factor-relations will be different in the case of the old and the new linkages, one or more relations being common to the two; and if the new link or links be added to the old linkage, and no link taken away, the common relations can alone exist,—we in fact eliminate “dead points.” Now we know that of four-piece linkages the parallelogrammatic linkage alone is supplementable. Proceeding by steps, we come to six-piece linkages. The present paper is a contribution to the consideration of these six-piece linkages and in this way.

An examination of Fig. 1 will show that, if the linkage there shown is complete, twelve different six-piece supplementable linkages can be formed out of it by the removal of two adjacent links; the six-piece supplementable linkages being doubly supplementable by links which are themselves joinable.

The method I adopt for the purpose of investigating this eight-piece linkage is this:—It will be observed that the linkage can be divided in three different ways into two four-piece complete linkages such as that shown in Fig. 2. Let α , β , γ , δ be the pivots which were common to the two linkages when united, and join these by dotted lines, and let the resulting figure be called the “connecting diagram.” The relations between the lines and angles of this connecting diagram must be the same for the two linkages, and if we can find out for what linkages this is the case, we can find the eight-piece linkage. This is what is here done.

Fig. 2.

The investigation is, I venture to think, one of much interest on account of the novelty and symmetry of the results obtained. We shall, in the course of it, encounter both of Mr. Hart's five-bar parallel motions as special cases of more general results.

Before commencing, it will be convenient to define some terms which will be frequently employed. The two linkages which have a common connecting diagram will be termed “conjugate.” It will also be convenient to call the linkage we treat of first in order the “first,” and the next its “conjugate.” The same parts of the conjugate will be affected with the same letters as the first, but they will be dashed. Those points on one connecting diagram which can be pivoted to points on the other connecting diagram are said to “correspond” to them. The figure LMNR, formed by joining the pivots of either linkage, is termed the “quadrilateral.” The eight-piece linkage, formed by connecting the two four-piece linkages, I shall call the “compound” linkage.

In Fig. 3, let

$$LM = a, \quad \widehat{LaM} = \alpha,$$

$$La = a_1,$$

$$Ma = a_2, \quad \text{area of } LaM = A;$$

and let corresponding letters stand for the same parts of the other pieces in which

$$MN = b, \quad NR = c, \quad RL = d.$$

$$\text{Also let both angles at } L = L,$$

$$,, \quad ,, \quad M = M,$$

$$,, \quad ,, \quad N = N,$$

$$,, \quad ,, \quad R = R,$$

this equality of angles at L, M, N , and R being the only restriction upon the otherwise perfect generality of the linkage.

Let

$$\delta\alpha = \lambda,$$

$$\alpha\beta = \mu, \quad MR = V,$$

$$\beta\gamma = \nu,$$

$$\gamma\delta = \rho, \quad LN = W.$$

We can readily find the following,

$$\alpha + \gamma = \beta + \delta \dots\dots\dots(1),$$

$$\frac{a^2}{A} + \frac{c^2}{O} = \frac{b^2}{B} + \frac{d^2}{D} \dots\dots\dots(2),$$

$$\frac{a^2 c^2}{AO \sin \alpha \sin \gamma} = \frac{b^2 d^2}{BD \sin \beta \sin \delta} = P \dots\dots\dots(3),$$

$$\begin{aligned} & \left[\frac{a^2}{A} \cot \alpha + \frac{c^2}{O} \cot \gamma \right] - \left[\frac{b^2}{B} \cot \beta + \frac{d^2}{D} \cot \delta \right] \\ &= -\frac{P}{2} \sin (\beta - \alpha) \sin (\gamma - \delta) \dots\dots\dots(4), \end{aligned}$$

$$\frac{a_2}{b_1} = \frac{bA}{aB} \quad \text{and three other similar equations} \dots\dots\dots(5),$$

$$a_1 a_2 = \frac{2A}{\sin \alpha} \quad ,, \quad ,, \quad \dots\dots\dots(6),$$

$$\cos \alpha = \frac{a_1^2 + a_2^2 - a^2}{2a_1 a_2} \quad ,, \quad ,, \quad \dots\dots\dots(7),$$

$$\sin^2 L = \frac{\frac{D}{d^2} - \frac{A}{a^2}}{\cot \delta - \cot \alpha} \quad ,, \quad ,, \quad \dots\dots\dots(8),$$

Fig. 3.

and, from (5), $a_1 b_1 c_1 d_1 = a_2 b_2 c_2 d_2 \dots\dots\dots(9).$

We have also

$$\frac{\lambda^2 \sin \alpha \sin \delta}{2 \sin (\alpha - \delta)} \left[\frac{a^2}{A} - \frac{d^2}{D} \right] = V^2 - \left[\frac{a^2}{A} - \frac{d^2}{D} \right] (A - D) \dots\dots(10),$$

and similar equations giving μ , ν , and ρ . Also the angle between μ and

$$\rho \text{ is } \widehat{MNL} - \widehat{NLR} - \widehat{M\beta\alpha} + \widehat{R\delta\gamma} + (M - R)$$

$$= \widehat{MNL} - \widehat{NLR} - \widehat{M\beta\alpha} + \widehat{R\delta\gamma} + (\gamma - \beta),$$

and we have a similar expression for the angle between λ and $\nu\dots(11).$

CASE I.

Let $\alpha = \beta = \gamma = \delta$ and therefore the four triangles be similar. The connecting diagram becomes a parallelogram, and if the conjugate linkage be a duplicate of the first, we have a correspondence between

$$\begin{array}{ll} \alpha & \text{and } \gamma', \\ \beta & \text{,, } \delta', \\ \gamma & \text{,, } \alpha', \\ \delta & \text{,, } \beta', \end{array}$$

and we thus get a complete eight-piece linkage, which is shown in Fig. 4.

Fig. 4.

The other pairs of conjugate linkages into which the compound linkage can be divided are related to each other in precisely the same way as the first pair, their connecting diagrams being parallelograms.

CASE II.

Let $A + C = B + D$ (12),

so that $\left. \begin{aligned} \lambda^3 \sin \alpha \sin \delta &= \nu^3 \sin \beta \sin \gamma \\ \mu^3 \sin \alpha \sin \beta &= \rho^3 \sin \delta \sin \gamma \end{aligned} \right\}$ (13).

Then, if a second linkage be formed in which

$$\left. \begin{aligned} a' &= ka, & A' &= A, & \alpha' &= \pi - \gamma \\ b' &= kb, & B' &= B, & \beta' &= \pi - \delta \\ c' &= kc, & C' &= C, & \gamma' &= \pi - \alpha \\ d' &= kd, & D' &= D, & \delta' &= \pi - \beta \end{aligned} \right\} \text{ (14),}$$

which, it will be seen by equations (1) to (8), can always be done if, (4),

$$\left[\frac{a^3}{A} \cot \gamma + \frac{c^3}{C} \cot \alpha \right] - \left[\frac{b^3}{B} \cot \delta + \frac{d^3}{D} \cot \beta \right] \\ = -\frac{P}{2} k^3 \sin (\beta - \alpha) \sin (\gamma - \beta),$$

$$\text{i. e., } k^3 - 1 = \frac{\frac{A}{a^3} + \frac{C}{c^3}}{\sin \beta \sin \delta} \sin (\beta + \delta) = \frac{\frac{B}{b^3} + \frac{D}{d^3}}{\sin \alpha \sin \gamma} \sin (\alpha + \gamma) \text{ (15):}$$

Then we have, by (13),

$$\left. \begin{aligned} \lambda' &= \nu \\ \mu' &= \rho \\ \nu' &= \lambda \\ \rho' &= \mu \end{aligned} \right\} \text{ (16);}$$

and, by (11), (1), and (5), the angles between λ' and ν' , μ' and ρ' are the same as those between ν and λ , ρ and μ .

Thus the connecting diagram of the linkage $[abcd]$ is the same as that of $[a'b'c'd']$, there being a correspondence between

$$\begin{aligned} \alpha &\text{ and } \gamma', \\ \beta &\text{ and } \delta', \\ \gamma &\text{ and } \alpha', \\ \delta &\text{ and } \beta'. \end{aligned}$$

The quadrilaterals of the two linkages are similar, but are in the compound linkage laid face to face.

The other pairs of conjugate linkages into which the compound linkage is divisible have the same relations to each other as the linkages $[abcd]$ and $[a'b'c'd']$. It is worthy of notice that the quadrilaterals of all the conjugate linkages have only six variable angles.

We now proceed to some special cases of this Case II.

CASE II. (a).

This is the case in which the linkage of Case II. becomes a bar linkage.

In this case,

$$\left. \begin{aligned} a &= a_1 + a_2 \\ b &= b_1 + b_2 \\ c &= c_1 + c_2 \\ d &= d_1 + d_2 \end{aligned} \right\} \dots\dots\dots (17),$$

and we have similar expressions for a' , &c.

The areas A, B, C, D vanish, and $\alpha, \beta, \gamma, \delta$ each equal π .

The equation (12), being put into the form

$$\frac{ABO}{BOD} - \frac{BC}{OD} + \frac{O}{D} - 1 = 0,$$

becomes
$$\frac{aa_2b_2c_2}{db_1c_1d_1} - \frac{bb_2c_2}{dc_1d_1} + \frac{cc_2}{dd_1} - 1 = 0 \dots\dots\dots (18),$$

and we have a similar equation in a' , &c.

We have also, from (5),

$$\frac{a_1}{b_1} = \frac{a'_1}{b'_1}, \text{ and similar equations } \dots\dots\dots (19);$$

and from (6), recollecting that $\alpha' = \pi - \gamma$, &c.,

$$a_1a_2c_1c_2 = a'_1a'_2c'_1c'_2, \quad b_1b_2d_1d_2 = b'_1b'_2d'_1d'_2 \dots\dots\dots (20).$$

By the aid of (5), (6), and (7), we can put (15) into the form

$$k^2 = \frac{abcd - a_1b_1cd - b_1c_1ad - c_1d_1ba - d_1a_1bc}{abcd},$$

which can be further simplified into

$$k^2 = - \frac{(a_1c_1 - a_2c_2)(b_1d_1 - b_2d_2)}{abcd} \dots\dots\dots (21).$$

CASE II. (b).

Let $k = 0$.

Then the sides of $[a'b'c'd']$ vanish, and therefore the areas A, B, C, D vanish, and the linkage becomes a bar one.

Hence, by (21), and (9),
$$\left. \begin{aligned} a_1c_1 &= a_2c_2 \\ b_1d_1 &= b_2d_2 \end{aligned} \right\}.$$

Let
$$a_1 = ea, \quad b_1 = e'b.$$

Then (18) becomes

$$\frac{a^2}{e} + \frac{c^2}{1-e} = \frac{b^2}{1-e} + \frac{d^2}{e} \dots\dots\dots (22),$$

and (19) becomes

$$\frac{(1-e)}{e'} \cdot \frac{a}{b} = -\frac{a'_1}{b'_1},$$

$$\frac{(1-e)}{(1-e)} \cdot \frac{b}{c} = -\frac{b'_1}{c'_1},$$

$$\frac{e}{(1-e)} \cdot \frac{c}{d} = -\frac{c'_1}{d'_1},$$

$$\frac{e'}{e} \cdot \frac{d}{a} = -\frac{d'_1}{a'_1};$$

whence

$$\frac{a_1'^2}{c_1'^2} = \frac{a^2}{c^2} \left[\frac{1-e'}{e'} \right]^2,$$

$$\frac{b_1'^2}{d_1'^2} = \frac{b^2}{d^2} \left[\frac{e}{1-e} \right]^2,$$

and (20) becomes

$$a_1'^2 c_1'^2 = a^2 c^2 e^2 (1-e)^2,$$

$$b_1'^2 d_1'^2 = b^2 d^2 e^2 (1-e)^2.$$

Thus,

$$a'^4 = a^4 \frac{e^2}{e'^2} (1-e)^2 (1-e')^2,$$

and we can get corresponding values for b' , c' , d' .

The form of the compound linkage is shown in Fig. 5. Fixing LR , and removing the links $P\beta$ and $P\delta$, we have a circle-describing apparatus. If we transpose the bars αP and γP , we get a second such apparatus, the two forms corresponding to the two values of e' given by (22). It is to be noticed that this removal of the links $P\beta$ and $P\delta$, which are jointed to opposite sides of the quadrilateral $LMNR$, cannot be effected in the general form of the compound linkage given in Fig. 1.

One special case of this form is of particular interest.

CASE II. (b).

Let $e' = \infty$.

Then, by (22),

$$e = \frac{a^2}{d^2 - b^2},$$

and

$$a_1'^4 = a^4 e^2 (1-e^2);$$

i. e.,

$$a'_1 = \frac{adb}{d^2 - b^2}; \quad c'_1 = \frac{cbd}{d^2 - b^2};$$

and d_1' and b_1' are infinite.

The apparatus becomes Mr. Hart's second parallel motion, see Vol. VIII., p. 288.

Fig. 5.

The bars NR and $L'R$ become superfluous, and may be removed, and Mr. Hart's linkage remains. The connecting diagram is in this case a triangle.

CASE II. (d).

Let $k = 1$.

Here, by (15), we have either

$$\sin(\alpha + \gamma) = 0,$$

or

$$\frac{B}{b^3} + \frac{D}{d^3} = 0.$$

I. Let $\sin(\alpha + \gamma) = 0$;

$$\therefore \alpha = n\pi - \gamma;$$

$\therefore$ by (1), $\beta = n\pi - \delta.$

Hence, by (13), $\lambda^3 = \nu^3,$
 $\mu^3 = \rho^3.$

The connecting diagram must therefore be a contra-parallellogram. It cannot be a parallelogram, because that would involve all the triangles on a, b, c, d being similar, as in Case I., which would be inconsistent with $\sin(\alpha + \gamma) = 0.$

A bar form of one conjugate is shown in Fig. 8; the other is a duplicate.

Fig. 8.

There is a special form of the bar case which is of interest.

Let $a\beta$ be always parallel to $\gamma\delta$. For this we must have (Fig. 3)

$$M = R, \text{ i. e., } M + N = N + R;$$

$$\therefore \beta = \gamma, \text{ and } \therefore \alpha = \delta.$$

But $\sin^3 L = \frac{\frac{D}{d^3} - \frac{A}{a^3}}{\cot \delta - \cot \alpha} = \infty, \text{ unless } \frac{D}{d^3} = \frac{A}{a^3};$

so $\sin^3 N = \frac{\frac{B}{b^3} - \frac{C}{c^3}}{\cot \beta - \cot \gamma} = \infty, \text{ unless } \frac{B}{b^3} = \frac{C}{c^3}.$

We must therefore, by (12), either have

$$a^2 + c^2 = b^2 + d^2,$$

or

$$A = 0, \quad B = 0, \quad C = 0, \quad D = 0.$$

The former case involves the linkage, being one with perpendicular diagonals, and the triangles on the sides of the quadrilateral being all similar, and therefore the connecting diagram a parallelogram, which is, as before pointed out, inconsistent with

$$\sin(a + \gamma) = 0.$$

In the latter case the linkage becomes a bar one. It is shown in Fig. 9.

Fig. 9.

In it we have

$$a_1 = (1-s) a,$$

$$b_1 = (1-s) b,$$

$$c_1 = (1+s) c,$$

$$d_1 = (1+s) d,$$

where

$$s = \frac{[a^2 - b^2] + [c^2 - d^2]}{[a^2 - b^2] - [c^2 - d^2]}.$$

The four points $\alpha, \beta, \gamma, \delta$ lie at the angles of a rectangle, whatever be the configuration of the linkage. It is noticeable that, if we merely desire to make the linkage without wishing for any special values of a, b, c, d , we can draw any rectangle $\alpha\beta\gamma\delta$, and any diameter LN parallel to $\alpha\beta$, and then complete the figure. The resulting linkage will be such that $\alpha\beta\gamma\delta$ is always a rectangle. The lengths a, b, c, d are, of course, entirely at our disposition if we desire it, the points $\alpha, \beta, \gamma, \delta$ being determined by the value of s as given. It is somewhat remarkable that it should be possible, in a linkage of which the pivot-distances are arbitrary, to obtain four points making a figure with constant angles.

We may from this bar-linkage obtain, by a very simple geometrical construction, a piece-linkage, which falls under this case, and in which the connecting diagram has this property of angular constancy.

In Fig. 10 the figure $LMNR$ is the same as in Fig. 9.

Fig. 10.

Bisect La in P , $N\beta$ in Q , $N\gamma$ in S , $L\delta$ in T (see Fig. 9).

$$\begin{aligned} \text{Make} \quad Pa_1 &= Pa, \quad \text{and} \quad \widehat{aPa_1} = \epsilon; \\ Q\beta_1 &= Q\beta, \quad \text{and} \quad \widehat{\beta Q\beta_1} = \epsilon; \\ S\gamma_1 &= S\gamma, \quad \text{and} \quad \widehat{\gamma S\gamma_1} = \epsilon; \\ T\delta_1 &= T\delta, \quad \text{and} \quad \widehat{\delta T\delta_1} = \epsilon. \end{aligned}$$

Then $a_1\gamma_1$ is parallel to $\beta_1\delta_1$, and $a_1\beta_1 = \gamma_1\delta_1$, and the angles of $a_1, \beta_1, \delta_1, \gamma_1$ are constant, the figure being symmetrical.

The other correspondences which are suggested by the symmetry of the connecting diagrams in Figs. 9 and 10, fall under Case III.

$$\text{II. Let} \quad \frac{B}{b^3} + \frac{D}{d^3} = 0,$$

$$\therefore \frac{A}{a^3} + \frac{C}{c^3} = 0.$$

$$\begin{aligned} \text{Thus} \quad A &= \frac{\sigma a^3}{a^3 - c^3}, \\ B &= \frac{\sigma b^3}{b^3 - d^3}, \\ C &= \frac{\sigma c^3}{c^3 - a^3}, \\ D &= \frac{\sigma d^3}{d^3 - b^3}. \end{aligned}$$

In this case, the quadrilaterals of the two conjugates being the same, their connecting diagrams may be exhibited on one of them. This

was of course also the case in the example just treated of, but there the two connecting diagrams coincided; here it is not so.

Fig. 11.

In Fig. 11, a bar case, the two connecting diagrams are shown. This case does not present any points of special interest.

CASE III.

Let (12) be satisfied, and let

$$M = R,$$

$$\therefore M + N = N + R,$$

$$\therefore \beta = \gamma,$$

$$\text{and} \quad \therefore \alpha = \delta.$$

Then $\mu = \rho$, by (13),

and the triangles B and C are similar, as also the triangles A and D .

Also the angle between μ and ρ is

$$N - L = \overline{N + M} - \overline{M + L} = \alpha - \beta.$$

Thus, if we turn all the triangles of Fig. 3 over to opposite sides of a, b, c , and d , the figure $\alpha\beta\gamma\delta$ remains the same in dimensions, and its form is merely changed in the manner shown in Figs. 12 and 13.

Fig. 12.

Fig. 13.

If therefore we construct the two linkages which give these two connecting diagrams, and dash the letters in one, we can make a correspondence between

α and δ' ,

δ and α' ,

β and γ' ,

γ and β' ,

by turning them face to face. The same thing may be done with the triangles in Case I.

The resulting compound linkage may be dissected into two quadru-

planes, and we may obtain the compound linkage by connecting any two quadruplanes, in which the area and angles of the connecting diagrams are the same. The quadruplane is so well known that I shall

Fig. 14.

merely give a figure of it, Fig. 14, and point out that

$$LR = MN,$$

$$ML = RN,$$

the triangles are all similar,

$$\alpha\delta \cdot \delta\gamma = D - A = [-B] - [-C],$$

and the angles of the connecting diagram are all constant.

In the particular case in which the sides of one quadruplane become infinite, we get a double Sylvester-Kempe parallel motion, of which the bar case was discovered by Mr. Hart and is known by his name.

In the bar case the connecting diagram in Fig. 12 takes the symmetrical form shown in Fig. 15.

Fig. 15.

Figure 9 is a special case of this.

The special case in which β and γ move up and coincide with N , the connecting diagram becoming an isosceles triangle, is shown in Fig. 16, which exhibits the compound linkage resulting from joining the two

Fig. 16.

conjugates. Any two of the bars meeting at N may be removed.

CASE IV.

The special case requires notice, in which

$$a^2 + c^2 = b^2 + d^2,$$

i.e., in which the diagonals of the quadrilateral of the four-piece linkage are at right angles to each other.

In this case, if we make the four triangles of Fig. 3 similar, they satisfy the equation

$$A + C = B + D \dots \dots \dots (13),$$

the connecting diagram being a parallelogram. The *sides* of the triangles of the conjugate linkage of Case III. are the same as those of the first, the vertical angles being, as required by (14), the supplements of those of the first. Consequently, if we take the conjugates of Case I. and Case II., and connect them together,—which we can do, as the connecting diagram is the same in both cases,—we get, by combining them, a compound linkage which will dissect into conjugate “Kites.”

The bar form of this compound linkage is shown in Fig. 17.

Fig. 17.

It is noticeable that, if the vertical angles of the similar triangles on the four sides of the quadrilateral in Case IV. are right angles, the connecting diagram becomes a straight line.

CASE V.

In the cases hitherto discussed, the angles at a pivot have all been equal or supplements. In the present case there is no such restriction. In Fig. 18 any quadrilateral figure $a\beta\gamma\delta$ is taken as the connecting diagram, and similar triangles $aM\beta$, $\beta N\gamma$, $\gamma R\delta$, $\delta L a$, are described on the sides of it.

The vertices L, M, N, R then form a parallelogram. If then the parallelogrammatic linkage be constructed as in the figure, the connecting diagram will always remain

Fig. 18.

similar to itself. Hence, if we form another linkage upon the same diagram in the same manner, the two will be conjugate. The other quadrilaterals of the compound linkage are all similar, and may be of any dimensions.

CASE VI.

This case does not belong to the same class as those hitherto treated of, as the compound linkage can only be dissected into two pairs of conjugate linkages. As, however, it is directly suggested by Case V., I have thought it better to include it here.

It is clear, from Fig. 18, that not only does the figure $a\beta\gamma\delta$ remain similar to itself, but, if we join $a\gamma$ and $\beta\delta$, the figure $a\gamma\beta\delta$ does so also. Hence, treating this figure as we did $a\beta\gamma\delta$, we get a conjugate linkage, which, if connected to the first, gives us an eight-piece compound linkage. The correspondence is not, as in former cases, between adjacent points and adjacent points, but is between

a and δ' ,
 β and β' ,
 γ and γ' ,
 δ and a' .

The resulting linkage is only divisible into four-piece linkages in one other way.

The compound linkage is shown in Fig. 19.

Fig. 19.

I have not as yet found any other linkages of this description.

June 13th, 1878.

Prof. H. J. S. SMITH, F.R.S., Vice-President, in the Chair.

Mr. J. D. H. Dickson, M.A., Fellow and Tutor of St. Peter's College, Cambridge, was proposed for election, and Mr. T. R. Terry was admitted into the Society.

Dr. Hirst, F.R.S., gave an account of a paper by M. Halphen, on "the Characteristics of Systems of Conics."

Mr. J. J. Walker read a paper on "a method in the Analysis of Plane Curves."

Mr. Tucker communicated papers by Mr. McColl, on "the Calculus of Equivalent Statements, II.,"; by Mr. C. J. Monro, on "the Flexure of Spaces"; by Mr. S. Roberts, F.R.S., on "the Decomposition of certain Numbers into sums of two Square Integers by continued Fractions"; by Mr. R. Rawson, on "a new method of finding Differential Resolvents of Algebraical Equations."

Questions were asked by Prof. Cayley, F.R.S. (Has a solution been given of the statement that in colouring a map of a country, divided into counties, only four distinct colours are required, so that no two adjacent counties should be painted in the same colour?); by Mr. Merrifield, F.R.S., on "the uniform Distribution of Points in Space"; by Mr. Tucker, in connection with the announcement made in "Nature" (Vol. xvii., p. 104), of a second exception to Fermat's statement that all numbers of the form $2^k + 1$ are primes.*

The following presents were made to the Society, some at the Meeting and some in the vacation:—

"Educational Times," June, July, 1878.

"Ueber Gleichungen siebenten Grades," zweite Mittheilung von F. Klein, in München.

"Proceedings of the Royal Society," Vol. xxvii., No. 187.

"Monatsbericht," März, April, Mai, 1878.

"Bulletin de la Société Mathématique de France," Tome vi., No. 4, 1878.

"Bulletin des Sciences Mathématiques," 2^{me} Série, Tome ii., Janvier 1878.

"Physical Society of London, Proceedings," Vol. ii., Pt. iv., Jan. to June, 1878.

"Crelle's Journal," 85 Band, 2^{te} Heft; Berlin, 1878.

Rayleigh's (Lord) "Theory of Sound," Vol. ii.; from the Author.

"Transactions of the Connecticut Academy of Arts and Sciences," Vol. iii., Pt. 2; New Haven, 1878.

"Atti della R. Accad. dei Lincei," anno cclxxv., 1877-8, Serie terza; "Transunti," Vol. ii., Fasc. 6^o, Maggio 1878.

* A third case of exception, discovered by the same Russian clergyman, has subsequently been given in "Nature" (No. 456, July 25th).

"American Journal of Mathematics, Pure and Applied" (published under the auspices of the Johns Hopkins University), Vol. i., Nos. 1, 2; Baltimore, 1878.

Sur la Théorie des Caractéristiques pour les Coniques.

Par M. HALPHEN.

[Read June 13th, 1878.]

Dans deux communications adressées à l'Académie des Sciences de Paris,* j'ai indiqué des modifications qu'il est nécessaire de faire subir à la théorie connue des caractéristiques pour les coniques. Je me propose de donner ici un aperçu des résultats que j'ai obtenus dans l'étude de cette théorie, et que j'ai développés dans un mémoire qui n'a pu être publié jusqu'à présent.

I. *Position de la question.*

1. On donne, je le rappelle, le nom de *système* de coniques dans un plan à l'ensemble des coniques de ce plan qui satisfont à quatre conditions communes. Le problème principal auquel se rapporte la théorie des caractéristiques est de trouver le nombre des coniques, appartenant à un système donné, qui satisfont, d'autre part, à une condition également donnée. En conséquence, cette théorie doit enseigner quels sont les éléments numériques, ou *caractéristiques*, qu'il faut connaître pour le système et pour la condition, et comment on doit combiner ces éléments numériques afin d'obtenir le nombre cherché. Pour qu'il y ait lieu à une théorie sur ce sujet, il faut évidemment supposer entre ces deux données, le système et la condition, une certaine *indépendance*; et on n'a jamais manqué de le faire. Mais cette indépendance doit être définie avec précision, et c'est par là que j'entrerai en matière.

2. Disons tout d'abord que la question est portée sur le terrain de la géométrie projective; par suite, toute condition imposée à une conique devra être mise sous forme projective. Pour la mettre sous une telle forme, on peut convenir d'employer un procédé unique, comme je vais l'expliquer.

Soient, dans un plan, quatre points fixes t, u, v, w et un autre point x , et désignons par $t, t', \dots X, X'$ les coordonnées de ces points par rapport à deux axes. Pour abrégé, employons la notation

$$(tuv) = \begin{vmatrix} t & t' & 1 \\ u & u' & 1 \\ v & v' & 1 \end{vmatrix},$$

* 4 Septembre et 13 Novembre 1876. V. Comptes Rendus, T. lxxxiii., pp. 537 et 586.