Note on the Complete Jacobian Elliptic Integrals.

By F. BOWMAN, M.A., M.Sc.Tech.

(Read 13th December 1918. Received 26th December 1918.)

The complete elliptic integrals K and E are functions of their modulus k which satisfy the equations

$$kk'^2 \frac{dK}{dk} = E - k'^2 K$$

$$k \frac{dE}{dk} = E - K$$

respectively.

If we write Π for $\Pi(K, a; k)$ where

$$\Pi(u, a; k) \equiv \int_0^u \frac{k^2 \operatorname{sn} a \operatorname{cn} a \operatorname{dn} a \operatorname{sn}^2 u}{1 - k^2 \operatorname{sn}^2 a \operatorname{sn}^2 u} du$$

is the elliptic integral of the third kind, then Π is a function of a and k , and $\frac{d\Pi}{dk}$ has a meaning only when a and k are connected by a functional relation. In this note the value of the derivate $\frac{d\Pi}{dk}$ is found in certain cases, on the assumption that such a relation does exist. On account of their simplicity, the results appear to be worth recording. They arose in discussing the geodesics on an ellipsoid of revolution.

Write $x = a \sin \theta \cos \phi$

$y = a \sin \theta \sin \phi$

$z = c \cos \theta$

to define a point on the ellipsoid $\frac{x^2 + y^2}{a^2} + \frac{z^2}{c^2} = 1$, and suppose $c > a$ and $c^2 e^2 = c^2 - a^2$ (prolate spheroid).

Consider the geodesic which touches the parallels of latitude $\theta = \pm \alpha$. The first integral of geodesics on a surface of revolution is known to be $r \sin \eta = \text{const}$, where η is the angle between the curve and the meridian. Expressing this fact for the above surface, we obtain for the equation of the geodesic in question

$$d\phi = \frac{c}{a} \frac{\sin \alpha}{\sin \theta} \cdot \frac{\sqrt{1 - e^2 \cos^2 \theta}}{\sqrt{\cos^2 \alpha - \cos^2 \theta}} d\theta$$

and the difference in longitude between a turning point of the geodesic and the point where it crosses the equator is

$$\frac{c}{a} \int_{\alpha}^{\frac{\pi}{2}} \frac{\sin \alpha}{\sin \theta} \frac{\sqrt{1 - e^2 \cos^2 \theta}}{\sqrt{\cos^2 \alpha - \cos^2 \theta}} d\theta = \frac{c}{a} \int_{\alpha}^{\frac{\pi}{2}} f(\theta, \alpha) d\theta = \frac{c}{a} \cdot I, \text{ say.}$$

Now let us find $\frac{dI}{d\alpha}$. We have

$$\frac{\partial f}{\partial \alpha} = \cos \alpha \sin \theta \frac{\sqrt{1 - e^2 \cos^2 \theta}}{(\cos^2 \alpha - \cos^2 \theta)^{3/2}}$$

and notice that it contains an infinity of order $\frac{3}{2}$ at the lower limit of integration. We can write, however, in this case (cf. Hardy: *Quarterly Journal*, Vol. 32, 1901)

$$\frac{dI}{d\alpha} = \lim_{\beta \rightarrow \alpha} \left[\int_{\beta}^{\frac{\pi}{2}} \frac{\partial f}{\partial \alpha} d\theta - f(\beta, \alpha) \right]$$

Now transform to the notation of the Jacobian elliptic functions by writing

$$\cos \theta = \cos \alpha \cdot \text{sn}(u + K) = \cos \alpha \frac{cn u}{dn u}, \text{ where } k = e \cos \alpha;$$

we get

$$f(\theta, \alpha) = \frac{1}{\cos \alpha} \cdot \frac{dn u}{sn u} \cdot \frac{1}{\sqrt{1 + \cot^2 \alpha (1 - e^2) sn^2 u}}$$

$$\int \frac{\partial f}{\partial \alpha} d\theta = \frac{1}{\cos \alpha} \int \frac{du}{sn^2 u} = \frac{1}{\cos \alpha} \left[u - \frac{cn u}{sn u} \frac{dn u}{dn u} - E(u) \right]$$

$$\text{and } I = \frac{1 - e^2 \cos^2 \alpha}{\sin \alpha} \int_0^K \frac{du}{1 + \cot^2 \alpha (1 - e^2) sn^2 u}$$

$$\text{and without difficulty } \frac{dI}{d\alpha} = \frac{1}{\cos \alpha} \cdot (K - E).$$

Substituting for α in terms of k throughout this equation we arrive at the result

$$\frac{d}{dk} \frac{ek'^2}{\sqrt{e^2 - k^2}} \int_0^K \frac{du}{1 + k^2 \frac{1 - e^2}{e^2 - k^2} \operatorname{sn}^2 u} = \frac{e(E - K)}{k \sqrt{e^2 - k^2}} \dots\dots\dots (1)$$

in which we can remove the restriction that e should be less than the unity.

But we can write

$$\begin{aligned} I &= \frac{ek'^2}{\sqrt{e^2 - k^2}} \int_0^K \left[du - \frac{k^2 \frac{1 - e^2}{e^2 - k^2} \operatorname{sn}^2 u}{1 + k^2 \frac{1 - e^2}{e^2 - k^2} \operatorname{sn}^2 u} du \right] \\ &= \frac{ek'^2}{\sqrt{e^2 - k^2}} \cdot K + i \sqrt{1 - e^2} \int_0^K \frac{k^2 \frac{ik'^2 e \sqrt{1 - e^2}}{(e^2 - k^2)^{3/2}}}{1 + k^2 \frac{1 - e^2}{e^2 - k^2} \operatorname{sn}^2 u} du \\ &= \frac{ek'^2}{\sqrt{e^2 - k^2}} \cdot K + i \sqrt{1 - e^2} \Pi(K, a; k) \end{aligned}$$

$$\text{where } \operatorname{sn}^2 a = -\frac{1 - e^2}{e^2 - k^2}.$$

Taking I in this form and differentiating again with respect to k and equating the result to the right side of (1), we finally obtain

$$\frac{d\Pi}{dk} = \operatorname{sn} a \operatorname{cn} a \operatorname{dn} a \cdot \frac{k}{k'^2} \cdot K. \dots\dots\dots (2)$$

$$\text{This is the value of } \frac{d\Pi}{dk} \text{ under the condition } \operatorname{sn}^2 a = -\frac{1 - e^2}{e^2 - k^2},$$

which is easily seen to be equivalent to

$$\frac{\operatorname{cn}^2 a}{\operatorname{dn}^2 a} = \operatorname{sn}^2(a + K) = \frac{1}{e^2} = \text{const.}$$

$$\text{or } \operatorname{sn}(a + K, k) = \text{const.}$$

Now if we carry out the same work for the case of the oblate spheroid, using the transformation given by Forsyth (*Differential*

Geometry, p. 139), we arrive at the result

$$\frac{d}{dk} \frac{e}{\sqrt{e^2 - k^2}} \int_0^K \frac{dn^2 u}{1 + k^2 \cdot \frac{1 - e^2}{e^2 - k^2} sn^2 u} du = \frac{e}{\sqrt{e^2 - k^2}} \frac{E - k'^2 \cdot K}{kk'^2} \dots\dots\dots(3)$$

corresponding to (1) above.

After dividing by e , subtract (1) from (3) and get

$$\frac{d}{dk} \frac{1}{\sqrt{e^2 - k^2}} \int_0^K \frac{dn^2 u - k'^2}{1 + k^2 \frac{1 - e^2}{e^2 - k^2} sn^2 u} \cdot du = \frac{1}{\sqrt{e^2 - k^2}} \cdot \frac{k}{k'^2} \cdot E$$

which easily reduces to

$$\frac{d}{dk} \int_0^K \frac{\sqrt{e^2 - k^2} \cdot \frac{k^2}{e^2} \frac{cn^2 u}{dn^2 u}}{1 - \frac{k^2}{e^2} \frac{cn^2 u}{dn^2 u}} du = \frac{1}{\sqrt{e^2 - k^2}} \cdot \frac{k}{k'^2} E$$

or, since $sn(u + K) = \frac{cn u}{dn u}$

and $sn(2K - u) = sn u$

$$\frac{d}{dk} \int_0^K \frac{k^2 \frac{\sqrt{e^2 - k^2}}{e} \cdot \frac{1}{e} \cdot sn^2 u}{1 - k^2 \cdot \frac{1}{e^2} \cdot sn^2 u} du = \frac{1}{\sqrt{e^2 - k^2}} \cdot \frac{k}{k'^2} \cdot E. \dots\dots\dots(4)$$

Put $sn a = \frac{1}{e}$, and after multiplying by $cn a$ we get

$$\frac{d\Pi}{dk} = \frac{sn a \, cn a}{dn a} \cdot \frac{k}{k'^2} \cdot E. \dots\dots\dots(5)$$

This then is the value of $\frac{d\Pi}{dk}$ under the condition $sn a = \text{const.}$

We may notice in passing that the values of $\frac{dK}{dk}$ and $\frac{dE}{dk}$ may be derived as special cases of the equations (1), (3) or (4). For example, putting $e = 1$ in (1) we get $kk'^2 \frac{dK}{dk} = E - k'^2 K$, and $e = \infty$ gives $k \frac{dE}{dk} = E - K$.

The results (2) and (5) could have been obtained in a more straightforward manner as follows, though one could hardly have predicted that they would turn out to be as simple as they are.

The expression of $\Pi(u, a; k)$ in terms of the Jacobian Θ function is $\Pi(u, a; k) = \frac{1}{2} \log \frac{\Theta(u - a)}{\Theta(u + a)} + u \cdot Z(a)$ and on writing $u = K$ it becomes

$$\Pi(K, a; k) = K \cdot Z(a) = K \cdot E(a) - a \cdot E. \dots\dots\dots(6)$$

And since
$$E(a) = \int_0^a dn^2 u \, du,$$

and therefore
$$\frac{dE(a)}{dk} = dn^2 a \cdot \frac{da}{dk} + \int_0^a \frac{\partial}{\partial k} (dn^2 u) \, du,$$

we obtain on differentiating

$$\frac{d\Pi}{dk} = E(a) \cdot \frac{dK}{dk} - a \frac{dE}{dk} + (K dn^2 a - E) \frac{da}{dk} + K \int_0^a \frac{\partial}{\partial k} (dn^2 u) \, du. \quad (7)$$

Now, from the equation
$$u = \int_0^{sn u} \frac{dt}{\sqrt{(1-t^2)(1-k^2 t^2)}}$$

it is easy to prove that, while u remains constant

$$\frac{\partial sn u}{\partial k} = - \frac{cn u \, dn u}{k k'^2} \left[E(u) - k'^2 \cdot u - k^2 \frac{sn u \cdot cn u}{dn u} \right] \dots\dots(8)$$

and hence that

$$\frac{\partial dn^2 u}{\partial k} = \frac{2k}{k'^2} \left[sn u \, cn u \, dn u \{ E(u) - k'^2 \cdot u \} + sn^2 u \, dn^2 u \right], \dots(9)$$

and after a little reduction we obtain

$$\int_0^a \frac{\partial}{\partial k} (dn^2 u) du = \frac{k}{k'^2} \left[sn a \, cn a \, dn a - cn^2 a \cdot E(a) - k'^2 \cdot a \cdot sn^2 a \right] \quad (10)$$

Substituting in (7)

$$\frac{d\Pi}{dk} = (K dn^2 a - E) \left[\frac{da}{dk} - \frac{1}{k k'^2} \{ E(a) - k'^2 \cdot a \} \right] + sn a \, cn a \, dn a \cdot \frac{k}{k'^2} \cdot K \dots\dots\dots(11)$$

This equation gives the value of $\frac{d\Pi}{dk}$ when a and k are connected by any relation of the form $f(a, k) = \text{const.}$, provided we introduce the appropriate value of $\frac{da}{dk}$.

For example, if we make the assumptions

$$\operatorname{sn} a = \text{const.}, \quad \operatorname{sn}(a + K) = \text{const.},$$

and evaluate the right side of (11) in each case, we arrive at the results (5) and (2) respectively.

Again, since we are led to define

$$\Pi(u, a; k) = \int_0^u \frac{k^2 \operatorname{sn} a \operatorname{cn} a \operatorname{dn} a \operatorname{sn}^2 u}{1 - k^2 \operatorname{sn}^2 a \operatorname{sn}^2 u} du$$

from a consideration of the integral $\int \frac{\alpha + \beta \operatorname{sn}^2 u}{1 + \gamma \operatorname{sn}^2 u} du$, it is natural to discuss the value of $\frac{d\Pi}{dk}$ when $\gamma [= -k^2 \operatorname{sn}^2 a]$ remains constant; that is, when $\operatorname{dn} a$ is constant.

Making use of (9) to differentiate $\operatorname{dn}^2 a = \text{const.}$, we get

$$kk'^2 \frac{da}{dk} = E(a) - k'^2 \cdot a - \frac{\operatorname{sn} a \operatorname{dn} a}{\operatorname{cn} a}$$

and substituting in (11), we have

$$\left(\frac{d\Pi}{dk} \right)_{\operatorname{dn} a = \text{const.}} = \frac{\operatorname{sn} a \operatorname{dn} a}{\operatorname{cn} a} \cdot \frac{dK}{dk} \dots \dots \dots (12)$$

Finally, when $\operatorname{dn}(a + K, k) = \text{const.}$, we obtain

$$kk'^2 \frac{da}{dk} = E(a) - k'^2 \cdot a + \frac{\operatorname{cn} a \operatorname{dn} a}{\operatorname{sn} a},$$

and in this case

$$\left(\frac{d\Pi}{dk} \right)_{\operatorname{dn}(a+K) = \text{const.}} = - \frac{\operatorname{cn} a \operatorname{dn} a}{\operatorname{sn} a} \frac{1}{k'^2} \cdot \frac{dE}{dk} \dots \dots \dots (13)$$

Collecting the results (11), (5), (2), (12), and (13), we may summarise the foregoing work in the following table:—

Relation between a and k .	Value of $\frac{d\Pi}{dk}$.
$a = \text{const.}$	$\frac{(E - dn^2 a \cdot K)}{kk'^2} (E(a) - k'^2 \cdot a)$ $+ sn a \, cn a \, dn a \, \frac{k}{k'^2} \cdot K$
$sn(a \cdot k) = \text{const.}$	$\frac{sn a \, cn a}{dn a} \cdot \frac{k}{k'^2} \cdot E$
$sn(a + K, k) = \text{const.}$	$sn a \, cn a \, dn a \cdot \frac{k}{k'^2} \cdot K$
$dn(a \cdot k) = \text{const.}$	$\frac{sn a \, dn a}{cn a} \cdot \frac{dK}{dk}$
$dn(a + K, k) = \text{const.}$	$- \frac{cn a \, dn a}{sn a} \cdot \frac{1}{k'^2} \cdot \frac{dE}{dk}$