On the application of the Sylvester-Clifford Graphs to Ordinary Binary Quantics. (Second Part.) By A. B. KEMPE, M.A.,
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This paper is a continuation of that "On the Application of Clifford's Graphs to Ordinary Binary Quantics," printed in the *Proceedings*, Vol. xvii., pp. 107-121, and the divisions and sections are accordingly numbered continuously with those of the preceding part. An alteration has, however, been made in the title. This has been done because, though the subject is regarded from the point of view of Professor Clifford, and therefore, following the precedent set by the late Mr. Spottiswoode in his paper "On Clifford's Graphs" (*Proceedings*, Vol. x., p. 204), his name alone was in the first part attached to the "graphs," it has been thought, on reflection, that by such exclusive association, an impression might be created which would operate unjustly towards the unquestionable originality of the paper by Professor Sylvester "On an application of the new Atomic Theory to the Graphical Representation of the Invariants and Covariants of Binary Quantics" (*American Journal of Mathematics*, Vol. i., p. 64). With regard to the "singularly beautiful series of conceptions" relative to these graphical representations, the late Professor H. J. S. Smith, in his introduction to Clifford's "Mathematical Papers" (Macmillan & Co., 1882), remarks that "it is difficult to say how much belongs to Clifford and how much to Sylvester, the more so because each of them was ready to attribute to the other the largest share." Under these circumstances the proper course would seem to be that, here adopted, of designating the graphs by their joint names.

In the former part it was shown that on multiplying any invariant or covariant of a binary quantic by a proper number of polar elements, it can be expressed as the sum of one or more "pure compound forms," each of which is itself an invariant or covariant of such quantic multiplied by the same polar elements; that to the algebraical expression of a pure compound form there corresponds directly an equivalent "graph"; that these graphs admit of being multiplied and added together, and the syzygetical relations between invariants and covariants of being expressed by purely graphical formulae. In this second part an alternative algebraical method of representing compound forms is given, which, though directly

derivable from the preceding method, and suggested by the graphs, is of a totally distinct character, and is independent of the use of polar elements. It is also shown that seminvariants can be directly represented by compound forms as well as invariants and covariants, a result due to the fact that the *seminvariants* of any quantic $(a_0, a_1, a_2, a_3, \dots a_n \mathfrak{Q} x, y)^n$ are *invariants* of two or more of the series of quantics

$$a_0; (a_0, a_1 \mathfrak{Q} x, y)^1; (a_0, a_1, a_2 \mathfrak{Q} x, y)^2; \dots (a_0, a_1, a_2, a_3, \dots a_n \mathfrak{Q} x, y)^n.$$

The expression of pure compound forms as the sum of products of simpler pure compound forms is also considered, and a theorem given with regard to it. Some other matters are also briefly dealt with which need not be specifically referred to here.

V. Order, Degree, and Weight of Pure Compound Forms.

57. It will be as well to point out that a pure compound form (Secs. 17, 24) which contains

- j simple forms of mark a and valence i ,
- w bonds connecting these forms to each other,
- g simple forms of mark x , and valence 1,
- g bonds connecting these simple forms of mark x to those of mark a ,

represents a covariant of

- order g in the variables,
- degree j in the coefficients,
- weight w in the coefficients,

of the quantic $(a_0, a_1, a_2, \dots a_i \mathfrak{Q} x, y)^i$ of order i in the variables.

58. Since every bond connects two simple forms, and there are j simple forms of valence i , and g simple forms of valence 1, the total number of bonds will be

$$\frac{ij + g}{2}.$$

But, since w bonds connect the simple forms of mark a to each other, and g bonds connect the simple forms of mark a to those of mark x , and there are no bonds connecting the simple forms of mark x to each other, the total number of bonds is also

$$w + g.$$

Thus we have $ij + g = 2w + 2g$, and therefore $g = ij - 2w$.

61. Consider now the two special linear forms

$$\text{---}\bullet = (u)_e = U', \text{ so that } \epsilon_0 = 0 \text{ and } \epsilon_1 = 1;$$

$$\text{---}\circ = (u)_n = U, \text{ so that } \eta_0 = 1 \text{ and } \eta_1 = 0.$$

We have

$$\text{---}\bullet\text{---}\bullet = (u)_n (u)_e = UU'.$$

a pure compound form of which the coefficient (Sec. 17) is 1. Also, since

$$U'(uvw\dots)_a = U'U(vw\dots)_a,$$

we have

$$(u)_e (uvw\dots)_a = (u)_e (u)_n (vw\dots)_a;$$

and therefore

$$\bullet\text{---}\circ = \text{---}\bullet\text{---}\bullet\text{---}\circ = \text{---}\bullet\text{---}\bullet\text{---}\circ$$

Hence, by multiplying any compound form by $(u)_e (u)_n$, we can, without altering the value of its coefficient, increase the valence of any of its component simple forms by 1, and the weight also by 1; and we can repeat the process at will.

62. On the other hand, in cases where we have forms such as that on the left-hand side of the identity of the last section, and we are concerned only with the coefficients of the pure compound forms under consideration, the linear form $\text{---}\bullet$ and its bond may be omitted without affecting such coefficients.

63. Again, by the formula of Sec. 30, we have

$$\begin{array}{c} \circ \\ | \\ \circ \end{array} + \begin{array}{c} \circ \\ | \\ \bullet \end{array} + \begin{array}{c} \circ\text{---}\circ \\ | \quad | \\ \circ\text{---}\bullet \end{array} + \begin{array}{c} \circ\text{---}\circ \\ | \quad | \\ \circ\text{---}\bullet \end{array} = 0;$$

where, as stated in that section, there may be any number of bonds and nuclei besides those represented. It follows from the formula of Sec. 61 that this may be expressed as

$$\begin{array}{c} \circ \\ | \\ \circ \end{array} + \begin{array}{c} \circ \\ | \\ \bullet \end{array} + \begin{array}{c} \circ\text{---}\circ \\ | \quad | \\ \circ\text{---}\bullet \end{array} + \begin{array}{c} \circ\text{---}\circ \\ | \quad | \\ \circ\text{---}\bullet \end{array} = 0;$$

or, removing the common factor $(u)_c(u)_b$, as

$$\begin{array}{c} \textcircled{a} \\ | \\ \textcircled{b} \end{array} + \begin{array}{cc} \textcircled{a} & \textcircled{c} \\ | & | \\ \textcircled{b} & \textcircled{b} \end{array} + \begin{array}{cc} \textcircled{a} & \textcircled{c} \\ & \diagdown \diagup \\ & \textcircled{b} \end{array} = 0,$$

a formula giving syzygetical relations between compound forms in which the corresponding simple forms, though of like marks, may differ in valence in different terms; *e.g.*, in the formula, a is of lower valence in the third term than it is in the first two. Formulæ such as the first and last of this section may be spoken of as *homogeneous* and *non-homogeneous* respectively. Great care must be exercised in the use of non-homogeneous formulæ, for, owing to the fact that there is no such correspondence of the points of egress of the bonds in the terms as exists in the case of the homogeneous formulæ, difficulties arise as to the determination of the signs of the terms (Sec. 29). In cases, however, where we are only concerned to show that certain compound forms may be expressed as the sums of others of certain special descriptions, and the signs are not material, the formulæ may be used freely without special care. The signs, when necessary, can be accurately determined by expressing the formulæ algebraically (see note to Sec. 69).

VII. Algebraical Representation of Forms. Second Method.

64. I proceed to indicate a second algebraical method of representing forms, suggested to me by the graphical method, and directly derivable from the first algebraical method. I propose to call these two algebraical methods the "polar" and "difference" methods respectively. Some reference is made to the latter method by Mr. S. Roberts in his paper "Concerning Semi-invariants," *Proceedings*, Vol. xxi., p. 220, where he states that he had been informed by Mr. Hammond that it had been given by me. I had not, however, published anything on the subject, though I had made verbal communications with respect to it to Mr. Hammond and others. Mr. Roberts states the method in a more restricted form than that in which it has always presented itself to me, and makes no reference to its relation to the graphical mode of representation. I therefore venture to set it forth here, and to indicate some applications of it.

65. The method deals with *pure* compound forms (Sec. 17), and in the first instance with those only which are *primary* (Sec. 15). The nuclei of the corresponding graphs will accordingly be supposed to have each a different mark, and every bond will proceed from one nucleus to another, there being no *free* bonds (Sec. 24).

$$66. \text{ Let } a = a_1 \frac{d}{da_0} + a_2 \frac{d}{da_1} + a_3 \frac{d}{da_2} + \dots \text{ ad inf.,}$$

$$b = b_1 \frac{d}{db_0} + b_2 \frac{d}{db_1} + b_3 \frac{d}{db_2} + \dots \text{ ad inf.,}$$

and similarly in the case of other letters c, d, e , &c. Then we have

$$a_r = a^r a_0, \quad a_r b_s = a^r b^s a_0 b_0.$$

We may therefore write the formula of Sec. 3,

$$(uvw\dots)_a$$

$$= \{ [uvw\dots]^0 + a [uvw\dots]^1 + a^2 [uvw\dots]^2 + \dots + a^n [uvw\dots]^n \} a_0^*$$

$$= \{ U + U'a \} \{ V + V'a \} \{ W + W'a \} \dots a_0;$$

and thus a simple form may be expressed as the product of a number of terms such as $\{ U + U'a \}$ operating upon a_0 .

67. A pure primary compound form, being merely the product of a number of simple forms, may therefore be written

$$\{ U + U'a \} \{ V + V'a \} \dots \{ U + U'f \} \dots \{ V + V'g \} \dots a_0 b_0 c_0 d_0 e_0 f_0 g_0 \dots,$$

where, since the form is pure, to every factor $\{ U + U'a \}$ there corresponds a factor $\{ U + U'f \}$, and one only; and, since it is primary, a and f are different letters. But

$$\{ U + U'a \} \{ U + U'f \} = \{ f - a \} UU'.$$

Hence a pure primary form may be written as

$$\{ a - b \}^\lambda \{ b - c \}^\mu \{ a - c \}^\nu \dots a_0 b_0 c_0 \dots UU'VV'WW' \dots,$$

the exponent λ being the same as the number of the bonds joining the two simple forms whose marks are a and c respectively, and similarly in the case of the exponents of the other factors. This mode of representing a primary compound form is that which I term "the difference method."

68. The expression of a form by the difference method consists then of three sets of factors, viz.,

- (1) The difference, or operating, factors, such as $\{ a - b \}^\lambda$.
- (2) The leading, or operand, factors, such as a_0, b_0 .
- (3) The polar elements, or bond coordinates (Sec. 6), such as U, U' .

* Compare the foot-note to Sec. 10, where the operator a is denoted by D_a . If we there put $A = uvw\dots$, and $\{ a_0 [B]^0 + a_1 [B]^1 + a_2 [B]^2 + \dots \} \equiv (B)_a = (\dots)_a \equiv a_0$ (Sec. 60), we obtain the formula of the text.

69. It will be interesting, for the purpose of comparison, to give the equivalent graphical, polar, difference, and ordinary algebraical representations of a particular form. We have

$$\begin{aligned}
 \begin{array}{c} \textcircled{a} \\ \diagup \quad \diagdown \\ \textcircled{b} \text{---} \textcircled{c} \end{array} &= (uv)_a (uvw)_b (vwx)_c \\
 &= (a-b)(b-c)^2 (a-c) a_0 b_0 c_0 \cdot UU'VV'WW'XX' \\
 &= \{a_2 b_2 c_0 - 2a_2 b_1 c_1 + a_2 b_0 c_2 - a_1 b_3 c_0 + a_1 b_2 c_1 + a_1 b_1 c_2 - a_1 b_0 c_3 \\
 &\quad + a_0 b_3 c_1 - 2a_0 b_2 c_2 + a_0 b_1 c_3\} UU'VV'WW'XX'.
 \end{aligned}$$

It will be observed that in the polar method the factors represent the nuclei, or simple forms, of the graphical representation,* while in the difference method the factors of the essential or distinctive part—the operator—correspond to the bonds.

70. It is obvious that, though the polar elements are necessary factors in expressing the equivalence of the polar representation and the differential one, if we confine ourselves to the difference method we need not trouble about the polar elements, and may regard our graphs as representing the operator and operand terms only; e.g., we may regard the graph of the last section as equal to $(a-b)(b-c)^2 (a-c) a_0 b_0 c_0$. Occasionally it will be convenient, instead of omitting the polar elements, to use the symbol Π to denote the factor which is the product of the polar elements, however many of such polar elements there may be. Thus, if C_0 be the coefficient of any pure compound form C , we shall have $C = C_0 \Pi$.

71. In some cases it will be unnecessary to express either the polar elements or the operands $a_0, b_0, c_0, \dots$, and we may regard the graphs as representing merely the product of a number of differences of operators. Here the operators may be regarded just as if they were ordinary quantities, and the discussion of pure primary forms reduces itself to the discussion of expressions which are the product of differences of quantities—say, are difference terms—and the relations of such expressions to each other. It may be here pointed out that the fundamental homogeneous syzygetical identity of Sec. 30

* It is obvious that there is nothing to determine whether the bond joining a nucleus of mark a to one of mark b corresponds to a factor $a-b$ or a factor $b-a$. It has been suggested by Mr. Hammond that the correspondence may be made determinate by the addition of an arrowhead to each bond, with the convention that where the barb points from a to b , the bond shall be understood to correspond to the factor $a-b$, and *vice versa*. The adoption of this plan would remove the difficulty as to signs, referred to in Secs. 29 and 63.

will, in the difference method, be expressed by

$$(a-b)(c-d) + (a-c)(d-b) + (a-d)(b-c) \equiv 0,$$

while the non-homogeneous one of Sec. 63 will be expressed by

$$(a-b) + (b-c) + (c-a) \equiv 0.$$

72. Observe that, since

$$(\epsilon - a) a_0 \epsilon_0 = (\epsilon_1 - a \epsilon_0) a_0 = \epsilon_1 a_0 = \epsilon \cdot a_0 \epsilon_0 \text{ (Sec. 61),}$$

we have

$$\epsilon - a \equiv \epsilon.$$

Also, since

$$(a - \eta) a_0 \eta_0 = (a \eta_0 - \eta_1) a_0 = a \cdot a_0 \eta_0,$$

we have

$$a - \eta \equiv a.$$

73. The difference method is not directly applicable to the case of a degraded primary form (Sec. 16); but it can be made applicable by taking the corresponding primary form from which the degraded form is derivable (*i.e.*, the primary form obtained by replacing like marks on the degraded form by unlike), and appending an indication that the unlike marks are, after the operator has operated upon the operand $a_0 b_0 c_0 \dots$, to be replaced by like marks. Thus the form $(uv)_a (uvw)_b (vwx)_b$ may be represented in the difference method by

$$(a-b)(b-c)^2(a-c) a_0 b_0 c_0 \Pi, \quad (b=c);$$

the indication $b=c$ being construed to mean that after the operator $(a-b)(b-c)^2(a-c)$ has operated upon $a_0 b_0 c_0$, we are to put

$$b_0 = c_0, \quad b_1 = c_1, \quad b_2 = c_2, \quad b_3 = c_3.$$

If this be done, we get

$$2 \{ a_2 b_2 b_0 - a_2 b_1^2 + a_1 b_1 b_2 - a_1 b_3 b_0 + a_0 b_3 b_1 - a_0 b_2^2 \} \Pi.$$

74. For some purposes, *e.g.*, where we are considering the syzygetical relations between compound forms, it is sufficient to regard the direction $b=c$ as meaning that in the operator T to which it applies, or in any one or more of the terms in any expression which is equal to T , we may interchange c and b without affecting the value of T .

75. In accordance with the notation of Sec. 66, we have

$$x = x_1 \frac{d}{dx_0} + x_2 \frac{d}{dx_1} + \dots,$$

$$y = y_1 \frac{d}{dy_0} + y_2 \frac{d}{dy_1} + \dots$$

Here the x, y , are not, of course, the x, y of the quantic which in the

ordinary notation would be written

$$(a_0, a_1, a_2, \dots a_n \mathfrak{X} x, y)^n,$$

these latter being respectively the x_1 and $-x_0$ of the notation of this paper (Sec. 33). In future the notation proper to this paper will be adhered to, and the variables of the binary quantic will be represented accordingly by x_1 and $-x_0$. Also, since

$$(a_0, a_1, \dots a_n \mathfrak{X} x_1, -x_0)^n = (x_1 - ax_0)^n a_0,$$

the binary quantic will in general be expressed in the latter, and shorter, form.

76. In speaking of invariants it is not necessary always to refer to them as being invariants of a particular quantic or quantics, but, in accordance with the fact that they are fully represented by pure compound forms consisting of simple forms of marks a, b, c , &c., it will suffice to speak of them simply as invariants of marks a, b, c , &c., or as pure compound forms of those marks. The invariants of $(x_1 - ax_0)^n a_0$ may accordingly be spoken of as invariants of mark a ; the quantic itself and its covariants being spoken of as invariants of marks a and x . (Sec. 39.)

VIII. Seminvariants.

77. The coefficient of a pure primary compound form of which the factor simple forms have respectively the marks $a, b, c, \dots$ and are respectively of valence $a', b', c', \dots$ is an invariant of the set of quantics

$$\begin{aligned} (x_1 - ax_0)^{a'} a_0, \\ (x_1 - bx_0)^{b'} b_0, \\ (x_1 - cx_0)^{c'} c_0, \\ \dots \quad \dots \quad \dots \\ \dots \quad \dots \quad \dots \end{aligned}$$

linear in the coefficients of each quantic (Sec. 35). If the marks $a, b, c, \dots$ are all replaced by the same mark p , then the coefficient of the pure primary compound form becomes an invariant I of the system of quantics

$$\begin{aligned} (x_1 - px_0)^{a'} p_0, \\ (x_1 - px_0)^{b'} p_0, \\ (x_1 - px_0)^{c'} p_0, \\ \dots \quad \dots \quad \dots \\ \dots \quad \dots \quad \dots \end{aligned}$$

where, of course, if any of the indices $a', b', c', \dots$ are the same, the quantics will be so also.

78. But this invariant I , besides being an invariant of the system of quantics, also bears a definite relation to a single quantic, viz., the quantic

$$(x_1 - px_0)^n p_0,$$

where n is the greatest of the indices $a', b', c', \dots$. I is, in fact, a *seminvariant* of $(x_1 - px_0)^n p_0$.

The proof is as follows. I may be expressed in the difference method as

$$I = (a-b)^\lambda (a-c)^\nu (b-c)^\rho \dots, \quad a_0 b_0 c_0 \dots, \quad (a = b = c = \dots = p),$$

and therefore
$$\left(\frac{d}{da} + \frac{d}{db} + \frac{d}{dc} + \dots \right) I = 0.$$

But, since
$$\frac{d}{da} a_r = \frac{d}{da} a^r a_0 = r a^{r-1} a_0 = r a_{r-1},$$

we may write

$$\frac{d}{da} = a_0 \frac{d}{da_1} + 2a_1 \frac{d}{da_2} + 3a_2 \frac{d}{da_3} + \dots \text{ad inf.},$$

and we have corresponding expressions for $\frac{d}{db}$, $\frac{d}{dc}$, &c. Hence, if we make

$$a = b = c = \dots = p,$$

and therefore
$$a_0 = b_0 = c_0 = \dots = p_0,$$

and
$$a_1 = b_1 = c_1 = \dots = p_1,$$

and
$$a_2 = b_2 = c_2 = \dots = p_2,$$

$$\&c., \quad \&c.,$$

so that
$$I = \phi(p_0, p_1, p_2 \dots p_n),$$

we have
$$\left(p_0 \frac{d}{dp_1} + 2p_1 \frac{d}{dp_2} + 3p_2 \frac{d}{dp_3} \dots \right) I = 0,$$

i.e., I is a *seminvariant* of $(x_1 - px_0)^n p_0$.

It will be borne in mind that, since I is a *seminvariant* of $(x_1 - px_0)^n p_0$, it is also a *seminvariant* of $(x_1 - px_0)^{n+k} p_0$, where k is any positive integer; but, since I contains p_n , it will not be a *seminvariant* of any quantic $(x_1 - px_0)^{n-k} p_0$.

79. It can also be shown that every seminvariant of $(x_1 - px_0)^n p_0$ can be exhibited as the sum of one or more expressions, such as

$$(a-b)^1 (a-c)^2 (b-c)^3 \dots a_0 b_0 c_0 \dots \quad (a = b = c = \dots = p),$$

and can therefore be represented as the coefficient of the sum of one or more pure compound forms. The proof is as follows.

Let S be any seminvariant of $(x_1 - px_0)^n p_0$ of degree m . Then

$$\left(\sum_{r=0}^{\infty} r p_{r-1} \frac{d}{dp_r} \right) S = 0,$$

and therefore

$$\left(\sum a_r \frac{d}{dp_r} \right) \left(\sum b_r \frac{d}{dp_r} \right) \left(\sum c_r \frac{d}{dp_r} \right) \dots \text{to } m \text{ factors} \left(\sum r p_{r-1} \frac{d}{dp_r} \right) S = 0,$$

where the summation is in each case from $r = 0$ to $r = \infty$. Now

$$\left(\sum q_r \frac{d}{dp_r} \right) \left(\sum r p_{r-1} \frac{d}{dp_r} \right) S = \left(\sum r q_{r-1} \frac{d}{dq_r} + \sum r p_{r-1} \frac{d}{dp_r} \right) \left(\sum q_r \frac{d}{dp_r} \right) S.$$

Hence we have (*)

$$\begin{aligned} & \left(\sum a_r \frac{d}{dp_r} \right) \left(\sum b_r \frac{d}{dp_r} \right) \left(\sum c_r \frac{d}{dp_r} \right) \dots \text{to } m \text{ factors} \left(\sum r p_{r-1} \frac{d}{dp_r} \right) S \\ &= \left(\sum r a_{r-1} \frac{d}{da_r} + \sum r b_{r-1} \frac{d}{db_r} + \sum r c_{r-1} \frac{d}{dc_r} + \dots \text{to } m \text{ terms} \right) \\ & \quad \times \left(\sum a_r \frac{d}{dp_r} \right) \left(\sum b_r \frac{d}{dp_r} \right) \left(\sum c_r \frac{d}{dp_r} \right) \dots \text{to } m \text{ factors } S. \end{aligned}$$

But $\left(\sum a_r \frac{d}{dp_r} \right) \left(\sum b_r \frac{d}{dp_r} \right) \left(\sum c_r \frac{d}{dp_r} \right) \dots \text{to } m \text{ factors } S = I,$

a function of the m sets of quantities

$$\begin{array}{ccccccc} a_0, & a_1, & a_2, & a_3, & \dots, \\ b_0, & b_1, & b_2, & b_3, & \dots, \\ c_0, & c_1, & c_2, & c_3, & \dots, \\ & \&c., & & \&c., \end{array}$$

linear as to the quantities of each set. Hence, since $a_r = a^r a_0$, we may write

$$I = f(a, b, c, \dots) a_0 b_0 c_0 \dots,$$

and, since $\sum r a_{r-1} \frac{d}{da_r} = \frac{d}{da}$, $\sum r b_{r-1} \frac{d}{db_r} = \frac{d}{db}$, &c., &c.,

we have, by (*),

$$\left(\frac{d}{da} + \frac{d}{db} + \frac{d}{dc} \dots \right) f(a, b, c, \dots) = 0,$$

of which the solution is

$f(a, b, c, \dots)$ = the sum of terms such as $(a-b)^{\lambda} (a-c)^{\mu} (b-c)^{\nu} \dots$,

i.e. (Sec. 70), III = the sum of one or more pure primary compound forms.

If now we put $a = b = c = \dots = p$,

and therefore $a_0 = b_0 = c_0 = \dots = p_0$,

and $a_1 = b_1 = c_1 = \dots = p_1$,

and $a_2 = b_2 = c_2 = \dots = p_2$,

&c., ... &c.,

I becomes $|n S$, and each pure primary compound form becomes a pure degraded compound form, each factor simple form of which has the same mark p . Thus S will be represented by the sum of a number of seminvariants of $(x_1 - px_0)^n p_0$, each of which is a pure degraded compound form.

80. A seminvariant of the quantic $(x_1 - ax_0)^n a_0$ may be spoken of as a seminvariant "of mark a ."

81. The coefficient of a pure compound form of which the factor simple forms are some of like and some of unlike marks, some of those of like marks being of different valence, will be a seminvariant of two or more quantics, and may be spoken of as a seminvariant of marks a, b, c , &c., where a, b, c , &c., are the marks of the component simple forms. It may be shown, by a proof similar to that in Sec. 79, that every seminvariant of two or more quantics can be represented by the coefficient of the sum of one or more pure compound forms containing simple forms of various marks and various valences.

82. The recognition of the fact that the seminvariants of a quantic $(x_1 - ax_0)^n a_0$ are invariants of two or more quantics of the series

$$\begin{aligned} & a_0, \\ & (x_1 - ax_0) a_0, \\ & (x_1 - ax_0)^2 a_0, \\ & (x_1 - ax_0)^3 a_0, \\ & \dots \quad \dots \quad \dots \\ & (x_1 - ax_0)^n a_0, \end{aligned}$$

or, as it may otherwise be expressed, that seminvariants of mark a differ from invariants of mark a , in this only, viz., that the simple forms which compose the pure compound forms representing the latter are all of the same valence, while those which compose the pure compound forms representing the former may differ from each other in valence, throws important light upon the structure of seminvariants and invariants. Thus we see that it is an essential feature of the seminvariant which is the source of the cubic-covariant J of $(x_1 - ax_0)^3 a_0$ that the compound form $\textcircled{a} \text{---} \textcircled{a} \text{=} \textcircled{a}$, which represents it, consists of three simple forms of valence 1, 2, and 3 respectively, so that J is an invariant of the three quantics

$$\begin{aligned} (x_1 - ax_0) a_0 &\equiv a_0 x_1 - a_1 x_0, \\ (x_1 - ax_0)^2 a_0 &\equiv a_0 x_1^2 - 2a_1 x_1 x_0 + a_2 x_0^2, \\ (x_1 - ax_0)^3 a_0 &\equiv a_0 x^3 - 3a_1 x_1 x_0 + 3a_2 x_1 x_0^2 - a_3 x_0^3; \end{aligned}$$

while the source of the Hessian H , being represented by $\textcircled{a} \text{=} \textcircled{a}$, is seen to consist essentially of two simple forms of the same valence 2, and is therefore an invariant of the single quantic

$$(x_1 - ax_0)^2 a_0.$$

IX. Breaking up of Pure Compound Forms.

83. A pure compound form consists either of two or more pure compound forms unconnected by bonds, or of one continuous portion. By successive applications of the formulæ of Secs. 13 and 30, a continuous pure compound form may, without altering the valence of the factor simple forms, be expressed in various ways as the sum of other

pure compound forms, some or all of which may consist of two or more detached pure compound forms. If a continuous pure compound form is expressed as the sum of a number of pure compound forms, each of which thus consists of two or more detached portions, the continuous pure compound form may conveniently be said to be "broken up" into pure compound forms of lower degree and weight.

84. The coefficient of a pure compound form which consists of two or more distinct portions, will consist of two or more distinct coefficients multiplied together, and the expression of a pure compound form, as the sum of terms each of which consists of two or more detached pure compound forms, amounts to the expression of its coefficient as the sum of terms each of which is the product of two or more distinct coefficients, *i.e.*, amounts to the expression of an invariant as the sum of terms each of which is the product of two or more invariants.

85. From Gordan's theorem that the number of invariants of a quantic of given order, or a system of quantics of given orders, is finite, it follows immediately that a pure compound form containing simple forms of given valences and marks will always break up, if the number of such simple forms (*i.e.*, the degree of the pure compound form) is sufficiently large.

86. If we use the non-homogeneous formula of Sec. 63 (see also Sec. 71), we can increase the number of ways in which a pure compound form can be broken up; but, as the valence of individual factor simple forms will be altered in the process, the factor pure compound forms into which the original pure compound form breaks up, may, when algebraically expressed, contain letters which were not comprised in the algebraical expression of the original pure compound form; *viz.*, if the original pure compound form contained a simple form of mark a and valence n , its algebraical expression would contain $a_0, a_1, a_2, \dots a_n$, but not a_{n+1}, a_{n+2} , &c. If, however, in the process of breaking up, the valence of the simple form a becomes increased to $n+k$, we shall have $a_{n+1}, a_{n+2}, \dots a_{n+k}$ in one or other of the pure compound forms into which the original form breaks up.

87. For example, to take a very simple case, the continuous form

$$\begin{aligned}
 F\Pi &= 2(a_1b_1c_0 - a_1b_0c_1 - a_0b_2c_0 + a_0b_1c_1)\Pi \\
 &= 2\{(a-b)(b-c)a_0b_0c_0\}\Pi \\
 &= \{(a-c)^2a_0c_0\}\{b_0\}\Pi \\
 &\quad - \{(a-b)^2a_0b_0\}\{c_0\}\Pi \\
 &\quad - \{(b-c)^2b_0c_0\}\{a_0\}\Pi \\
 &= \{a_2c_0 - 2a_1c_1 + a_0c_2\}\{b_0\}\Pi \\
 &\quad + \{2a_1b_1 - a_2b_0 - a_0b_2\}\{c_0\}\Pi \\
 &\quad + \{2b_1c_1 - b_2c_0 - b_0c_2\}\{a_0\}\Pi;
 \end{aligned}$$

and thus $F\Pi$ breaks up, being expressed as the sum of three terms, each of which is the product of two wholly detached pure forms; but these forms contain the letters a_2 and c_2 which do not appear in the coefficient F .

88. It is not proposed to further discuss here the general question of the breaking up of pure compound forms. There is, however, an interesting and important theorem with regard to the breaking up of the product of any pure compound form and a special form of the same weight, which will be considered.

89. Let $a, b, c, \dots$ and $P, Q, R, \dots$ be the marks of the simple forms composing the two pure compound forms ϕ and Φ respectively. Then it is the immediate consequence of the identity

$$(a-b)(P-Q) = (a-P)(b-Q) - (a-Q)(b-P)$$

that, if ϕ and Φ are of the same weight w , so that the number of bonds in each is w , the product $\phi\Phi$ can be expressed as the sum of 2^w terms, each term being a pure compound form in which no bonds connect the simple forms $a, b, c, \dots$ to each other, or the simple forms $P, Q, R, \dots$ to each other, but each bond connects one or other of the simple forms $a, b, c, \dots$ to one or other of the simple forms $P, Q, R, \dots$.

90. Hence, if Φ be such that the simple forms $P, Q, R, \dots$ composing it are all of valence 1, the product $\phi\Phi$ will be broken up into what may be called "star forms," each of which will consist of one of the simple forms $a, b, c, \dots$ connected by radiating bonds to uni-valent terminal simple forms of the set $P, Q, R, \dots$, the number of

such terminal simple forms being of course the same as the valence of the central simple form; *e.g.*, one of such star forms would be

this being represented in the difference method by

$$(a-P)(a-Q)(a-R)(a-S)(a-T) a_0 P_0 Q_0 R_0 S_0 T_0 \cdot \Pi.$$

In order that the simple forms $P, Q, R, \dots$ may be all of valence 1, Φ must consist of w pairs of such simple forms connected by single bonds, *i.e.*, of w pairs, such as

$$\begin{aligned} \textcircled{P} \text{---} \textcircled{Q} &= (u)_P (u)_Q = (Q-P) P_0 Q_0 U U' \\ &= (P_0 Q_1 - P_1 Q_0) U U', \end{aligned}$$

which may conveniently be termed "single bond forms." We have here the important result that the form resulting from the multiplication of any pure compound form ϕ , of weight w , by w single bond forms, can be broken up into star forms each containing one simple form only of those composing ϕ .

91. If the w single bond forms are all exactly alike, the two simple forms in each having the marks P and Q respectively, a star form σ containing the simple form a of ϕ may be expressed by

$$\begin{aligned} &(a-A)(a-B) \dots (a-P)(a-L)(a-M) \dots (a-Q) a_0 A_0 B_0 \dots \\ &\dots P_0 L_0 M_0 \dots Q_0 \Pi \cdot \begin{cases} A = B = \dots = P \\ L = M = \dots = Q \end{cases} \\ &= (aA_0 - A_1)(aB_0 - B_1) \dots \\ &(aP_0 - P_1)(aL_0 - L_1)(aM_0 - M_1) \dots (aQ_0 - Q_1) a_0 \Pi \\ &= (aP_0 - P_1)^i (aQ_0 - Q_1)^{i-\lambda} a_0 \Pi = (-1)^{i-\lambda} (aP_0 - P_1)^i (Q_1 - aQ_0)^{i-\lambda} a_0 \Pi \\ &= (-1)^{i-\lambda} \sigma_0 \Pi, \end{aligned}$$

where i is the valence of the simple form a . We shall have such star forms for all values of λ from 0 to i .

92. If $P_0Q_1 - P_1Q_0 = \omega$, and ϕ_0 is the coefficient of ϕ , we have

$$\phi\Phi = \phi_0\omega^w\Pi,$$

and the process we are considering shows how the invariant $\phi_0\omega^w$ may be expressed as the sum of products of "star invariants," such as

$$(aP_0 - P_1)^\lambda (Q_1 - aQ_0)^{i-\lambda} a_0 = \sigma_0.$$

I proceed to consider some special cases. In all these I shall take $Q = \epsilon$ (Sec. 61), so that $Q_0 = 0$ and $Q_1 = 1$, and therefore $\omega = P_0$, and

$$\begin{aligned}\sigma_0 &= (aP_0 - P_1)^\lambda a_0 \\ &= (a_0, a_1, a_2, \dots, a_\lambda \chi - P_1, P_0)^\lambda.\end{aligned}$$

93. First, then, let $P_0 = 1$, so that

$$\omega = 1, \quad \phi_0\omega^w = \phi_0, \quad \text{and} \quad \sigma_0 = (a - P_1)^\lambda a_0.$$

Here the process under consideration shows how any invariant ϕ_0 may be expressed as a rational integral function of star invariants such as $(a - P_1)^\lambda a_0$, *i.e.*, invariants which are the coefficients of star forms consisting each of one or other of the simple forms contained in ϕ , with bonds radiating therefrom to univalent simple forms, each of which is one or other of the simple forms

$$(u)_\epsilon = U' \quad \text{or} \quad (u)_P = U + P_1 U';$$

or, more simply, since by Sec. 62 the simple forms such as $(u)_\epsilon$ may be omitted without affecting the value of the coefficient, we see that ϕ_0 may be expressed as a rational integral function of the coefficients of star forms which consist each of one or other of the simple forms of ϕ with bonds radiating to λ univalent forms of mark P , where $\lambda = 0, 1, 2, 3 \dots i$, the valence of the simple form as a member of ϕ being i .

94. If we make $P_1 = 0$, since $P_0 = 1$, we have $P = \eta$ (Sec. 61), and the single bond form $(u)_P(u)_Q$ becomes the form

$$(u)_\eta(u)_\epsilon = UU' = \text{---} \bigcirc \text{---} \bullet \text{---}.$$

Also we have

$$\sigma_0 = a^\lambda a_0 = a_\lambda,$$

and thus the coefficients of the star forms become simply

$$a_0, a_1, a_2, \dots; b_0, b_1, b_2, \dots; c_0, c_1, c_2, \dots; \&c., \&c.,$$

viz., we have, omitting the univalent forms $(u)_e$,

$$\begin{aligned}
 (a) &= a_0, \\
 \text{---} (a) &= a_1 \Pi, \\
 \text{---} (a) \text{---} &= a_2 \Pi, \\
 \begin{array}{c} \text{---} (a) \text{---} \\ | \\ \text{---} \end{array} &= a_3 \Pi, \\
 &\&c., \quad \&c.;
 \end{aligned}$$

and the process gives us the invariant ϕ_0 in its usual algebraical form, viz., as a function of

$$a_0, a_1, a_2, \dots; b_0, b_1, b_2, \dots; c_0, c_1, c_2, \dots; \&c., \&c.$$

95. Observe here that, the star forms of the last section being all pure compound forms, their coefficients are invariants; e.g., a_k is an invariant, not however of a single quantic, but of the two quantics

$$\begin{aligned}
 (x_1 - ax_0)^k a_0, \\
 (x_1 - \eta x_0) \eta_0 = x_1,
 \end{aligned}$$

since

$$\eta_0 = 1 \quad \text{and} \quad \eta_1 = 0.$$

96. Next let all the simple forms of ϕ be of mark a , so that ϕ_0 is an invariant or seminvariant of mark a . Also let $P = a$, and therefore $P_0 = a_0$, and $P_1 = a_1$. Here $\omega = a_0$, and

$$\begin{aligned}
 \sigma_0 &= (a_0 a - a_1)^k a_0 \\
 &= (a_0, a_1, a_2, \dots, a_k \chi - a_1, a_0)^k,
 \end{aligned}$$

and the process shows how the invariant or seminvariant ϕ_0 of mark a and weight w , when multiplied by a_0^w , may be expressed as a rational integral function of the quantities

$$\begin{aligned}
 a_0 &= a_0, \\
 a_1 &= (a_0 a - a_1) a_0 = 0, \\
 a_2 &= (a_0 a - a_1)^2 a_0 = a_0^2 a_2 - a_0 a_1^2, \\
 a_3 &= (a_0 a - a_1)^3 a_0 = a_0^3 a_3 - 3a_0^2 a_2 a_1 + 2a_0 a_1^3, \\
 &\&c., \quad \&c.
 \end{aligned}$$

Here the star form $a_\kappa \Pi$ consists [if we omit the forms of mark e (Sec. 62)] of $\kappa+1$ simple forms of mark a , viz., one of valence κ and κ of valence 1, *e.g.*,

Hence a_κ is an invariant of the two quantics

$$(x_1 - ax_0)^\kappa a_0 \quad \text{and} \quad (x_1 - ax_0) a_0 = (a_0 x_1 - a_1 x_0),$$

and is a seminvariant of the single quantic $(x_1 - ax_0)^\kappa a_0$. The seminvariants $a_0, a_2, a_3, \dots$ &c. are those given by Professor Cayley in his paper "On Seminvariant Tables," *Amer. Jour. of Math.*, Vol. VII., p. 59.

97. Since $S_0 \Pi$, where S_0 is any seminvariant of mark a and weight w , can be expressed as the sum of a number of pure compound forms such as ϕ_0 (Sec. 79), $S_0 a^w$ can be expressed as a rational integral function of $a_0, a_2, a_3 \dots$.

98. Consider now the series of seminvariants $A_1 (= 0), A_2, A_3 \dots$ &c., where

$$A_{p+1} \Pi = \textcircled{a} \overset{p}{\text{---}} \textcircled{a} \text{---} \textcircled{a} = (a-b)^p (a-c) a_0 b_0 c_0 \Pi. \quad (a=b=c.)$$

The compound form here, though of weight $p+1$, breaks up into star forms on multiplication by p of the single-bond forms $(u)_e (u)_a$, and we have

$$A_{p+1} a_0^p = a_{p+1} + f(a_p, a_{p-1} \dots a_0),$$

where f is a rational integral function. Hence

$$a_{p+1} = A_{p+1} a_0^p - f(a_p, a_{p-1} \dots a_0),$$

from whence it immediately follows that

$$a_{p+1} = F(A_{p+1}, A_p, A_{p-1} \dots A_2, a_0),$$

where F is also a rational integral function. Thus $S_0 a_0^w$ may be expressed as a rational integral function of the series of semin-

variants $A_2, A_3, A_4 \dots$ and a_0 . Observe, however, that the degree and weight of A_3 are the same, viz., 3, and that for all values of κ greater than 3, the weight of A_κ , which is κ , exceeds the degree, which is 3. Also

$$\begin{aligned}
 2A_3 &= 2(a-b)(a-c) a_0 b_0 c_0 & (a=b=c.) \\
 &= (a-b)^2 a_0 b_0 c_0 + (a-c)^2 a_0 b_0 c_0 - (b-c)^2 a_0 b_0 c_0 & ,, \\
 &= (a-b)^2 a_0 b_0 c_0 & ,, \\
 &= a_0 \cdot (a-b)^2 a_0 b_0 & (a=b.) \\
 &= a_0 \cdot A'_3,
 \end{aligned}$$

where A'_3 is a seminvariant of equal degree and weight, viz.,

$$A'_3 \Pi = \textcircled{a} \text{---} \textcircled{a}.$$

Hence $S_0 a_0^w$ may be expressed as a rational integral function of $a_0, A'_2, A_3, A_4, \dots$, i.e., of a_0 and a number of seminvariants of mark a , such that the degree of no one of them exceeds the weight. Now, if the degree of S_0 is j , the degree of $S_0 a_0^w$ will be $w+j$, and its weight will be w ; hence every term of the expression of $S_0 a_0^w$ as a rational integral function of $a_0, A'_2, A_3, A_4, \dots$, &c., must contain a factor a_0^j . Dividing this out, we see that $S_0 a_0^{w-j}$ can be expressed as a rational integral function of $a_0, A'_2, A_3, A_4, \dots$.

99. If κ be even, $= 2\kappa'$, we have

$$\begin{aligned}
 A_\kappa &= A_{2\kappa'} = (a-b)^{2\kappa'-1} (a-c) a_0 b_0 c_0 & (a=b=c.) \\
 &= (a-b)^{2\kappa'-1} (\overline{a-b} + \overline{b-c}) a_0 b_0 c_0 & ,, \\
 &= (a-b)^{2\kappa'} a_0 b_0 c_0 + (a-b)^{2\kappa'-1} (b-c) a_0 b_0 c_0 & ,, \\
 &= (a-b)^{2\kappa'} a_0 b_0 c_0 - (a-b)^{2\kappa'-1} (a-c) a_0 b_0 c_0, & ,,
 \end{aligned}$$

by interchanging b and a in the second term (Sec. 74),

$$= a_0 \cdot (a-b)^{2\kappa'} a_0 b_0 - A_\kappa. \quad (a=b.)$$

Hence

$$\begin{aligned}
 2A_\kappa &= a_0 \cdot (a-b)^{2\kappa'} a_0 b_0 & (a=b.) \\
 &= a_0 A'_\kappa \text{ say,}
 \end{aligned}$$

where

$$A'_\kappa = \textcircled{a} \text{---}^\kappa \textcircled{a}.$$

We can therefore express $S_0 a_0^{w-j}$ as a rational integral function of the seminvariants $a_0, A_2, A_3, A'_4, A_5, \dots$.*

X. On the Relation of the Graphical Representations of Covariants and their Sources.

100. Any compound form representing a covariant C of order g in the variables, and of degree j and weight w in the coefficients (i.e., of total degree $g+j$ and weight $g+w$), will consist of g univalent simple forms of mark x , and j other simple forms representing the coefficients. If we multiply $C\Pi$ by g single-link forms, such as

$$\textcircled{x} \text{---} \textcircled{x} = (u)_\epsilon (u)_\eta,$$

the coefficients of which are 1, then, since

$$(a-x)(\epsilon-\eta) = (a-\epsilon)(x-\eta) - (a-\eta)(x-\epsilon),$$

we may express the product, the coefficient of which will also be C , as the sum of 2^g terms, each of which will consist of (1) a compound form $C_r\Pi$ precisely similar to $C\Pi$, but having univalent forms of marks ϵ and η (r of the former and $g-r$ of the latter), in lieu of those of mark x , and (2) of g single-bond forms, some (r) such as

$$\textcircled{x} \text{---} \textcircled{x} = U(x_0U + x_1U') = x_1U'U,$$

and the rest ($g-r$) such as

$$\bullet \text{---} \textcircled{x} = U'(x_0U + x_1U') = x_0U'U;$$

and thus we have a number of terms such as $C_r x_1^r x_0^{g-r} \Pi$. Here, since there are $\frac{|g|}{|r| |g-r|}$ different ways in which we can substitute r univalent forms of mark ϵ , and $g-r$ of mark η for the g univalent forms of mark x , there will be $\frac{|g|}{|r| |g-r|}$ terms such as $C_r x_1^r x_0^{g-r} \Pi$ for any value of r , which terms may all differ in value; and there will be terms for all values of r from 0 to g ; in all, as before stated, 2^g terms. The sum of all the C_r 's for any given value of r will be the coefficient of $x_1^r x_0^{g-r}$ in the expansion of the covariant C . In certain cases all the C_r 's for any given value of r will be equal; e.g., if $C\Pi$ contains merely

* See Sylvester "On Clebsch's Theory, &c.," *Amer. Jour. of Math.*, Vol. I., p. 118, and Hammond "On the Solution of the Differential Equation of Sources," *ibid.*, Vol. V., p. 218.

one simple form of mark a besides those of mark x , so that O is the ordinary quantic $(x_1 - ax_0)^g a_0$, all the terms C_r will be equal, each being $= a_r$, in which case the coefficient of $x_1^g x_0^{g-r}$ is, apart from sign,

$\frac{g!}{r! (g-r)!} a_r$. In every case the coefficient of x_1^g , i.e., the leading term or source of the covariant C , will be represented by a single term $C_0 \Pi$, all the univalent forms substituted for those of mark x being of mark e . Since we may omit all simple forms of mark e and their bonds (Sec. 62), without affecting the value of the coefficient of the compound form containing them, we see that the graphical representation of the source C_0 of any covariant C is obtained by omitting from the graphical representation of the covariant all the nuclei of mark x , and the bonds connecting them to the other nuclei. Thus the source of the cubic-covariant which is the coefficient of

is the coefficient of

On the Vibrations of an Elastic Circular Ring. (Abstract.)

By Mr. A. E. H. LOVE. Communicated January 12th, 1893.

The ring is supposed to be of small circular section of radius c , and the elastic central-line a circle of radius a . There are four ways of displacing the ring. A point on the central-line may move along the radius of the circle which is its primitive form, or perpendicularly to the plane of this circle, or along the tangent to this circle; and the circular sections may be displaced by rotation about the central-line. The modes of vibration fall into four classes, of which two are physically important.