we have

$$\int_0^\infty \int_0^{2\pi} \exp \{ -r^n (a^n \cos^n \theta + b^n \sin^n \theta) \} r dr d\theta = \frac{1}{ab} \Gamma^3 \left(1 + \frac{1}{n} \right);$$

and therefore

$$\int_0^{2\pi} \frac{d\theta}{\sqrt[n]{(a^n \cos^n \theta + b^n \sin^n \theta)}} = \frac{2}{ab} \frac{\Gamma^3 \left(1 + \frac{1}{n} \right)}{\Gamma \left(1 + \frac{2}{n} \right)}$$

$$= \frac{1}{nab} \frac{\Gamma^3 \left(\frac{1}{n} \right)}{\Gamma \left(\frac{2}{n} \right)}.$$

On the Tangents drawn from a Point to a Nodal Cubic.

By R. A. ROBERTS, B.A.

[Read Jan. 13th, 1881.]

The equation of a nodal cubic, referred to the triangle formed by the inflexional tangents, may be written $x^{\frac{1}{3}} + y^{\frac{1}{3}} + z^{\frac{1}{3}} = 0$, or in tangential coordinates $\lambda^{-\frac{1}{3}} + \mu^{-\frac{1}{3}} + \nu^{-\frac{1}{3}} = 0$. If we combine with this last equation the equation of a point $x\lambda + y\mu + z\nu = 0$, we get a biquadratic which determines the tangents drawn from (x, y, z) to the cubic.

Let us consider the cubics $f = au^3 + bv^3 + cw^3$, $f' = a'u^3 + b'v^3 + c'w^3$, where $u + v + w = 0$. Then the discriminant of $f + kf'$ is seen to be $(a + ka')^{-\frac{1}{3}} + (b + kb')^{-\frac{1}{3}} + (c + kc')^{-\frac{1}{3}} = 0$. Let us suppose that this equation in k coincides with the biquadratic found above; we must have, then, $\lambda = a + ka'$, $\mu = b + kb'$, $\nu = c + kc'$; and, since $x\lambda + y\mu + z\nu = 0$, identically,

$$\frac{x}{(bc')} = \frac{y}{(ca')} = \frac{z}{(ab')} \dots\dots\dots (1).$$

Now, the invariants, S and T , of the equation in k are expressed in terms of the combinants, P and Q , of the cubics f and f' , thus (Salmon's *Higher Algebra*, Art. 204),

$$S = 3P(P^3 - 24Q),$$

$$T = -(P^6 - 36P^3Q + 216Q^2).$$

But

$$P = (bc') + (ca') + (ab'),$$

$$Q = (ab')(bc')(ca').$$

Hence, from (1),

$$P = x + y + z,$$

$$Q = xyz;$$

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therefore $S = 3(x+y+z) \{(x+y+z)^2 - 24xyz\},$

$$T = - \{(x+y+z)^3 - 36xyz(x+y+z) + 216x^2y^2z^2\},$$

and
$$\frac{S^3}{T^2} = \frac{27\lambda(\lambda-24)^3}{(\lambda^3-36\lambda+216)^2} \dots\dots\dots (2),$$

where
$$(x+y+z)^3 - \lambda xyz = 0.$$

Let us calculate the invariants of the cubic $(x+y+z)^3 - \lambda xyz = 0$. We find $S' = 3\lambda^3(\lambda-24)$, $T' = -\lambda^4(\lambda^3-36\lambda+216)$. Hence we infer that the absolute invariant $\frac{S^3}{T^2}$ of the tangents drawn from any point to the nodal cubic $(x+y+z)^3 - 27xyz = 0$, is equal to the absolute invariant $\frac{S^3}{T^2}$ of the cubic of the system $(x+y+z)^3 - \lambda xyz = 0$, which passes through the point.

In particular, the tangents form a harmonic pencil from any point of either of the harmonic cubics $(x+y+z)^3 - 6(3 \pm \sqrt{3})xyz = 0$, and S vanishes for any point of the line of inflexions, and the cubic $(x+y+z)^3 - 24xyz = 0$.

2. The cubic, being referred to the triangle formed by the nodal tangents and the line of inflexions, can be written $x^3 + y^3 + 6xyz = 0$. Eliminating z between the equations of the curve and the polar conic of (x', y', z') , $U \equiv x'(x^2 + 2yz) + y'(y^2 + 2zx) + 2x'y = 0$, we obtain

$$y'x^4 - 2x'x^3y - 6z'x^2y^2 - 2y'xy^3 + x'y^4 = 0 \dots\dots\dots (3).$$

Multiplying this equation (3) by $x'x^3 + y'y^3$, it becomes

$$x'y'(x^6 + y^6 - 4x^3y^3) + (x'^3 - 6y'z')x^2y^4 + (y'^3 - 6z'x')x^4y^2 - 2x'^2x^5y - 2y'^2xy^5 = 0.$$

But, from the equation of the curve,

$$x^6 + y^6 - 4x^3y^3 = 6x^2y^3(6z^3 - xy), \quad x^5y = -x^2y^2(y^3 + 6zx), \\ xy^5 = -x^3y^2(x^3 + 6yz);$$

hence, substituting and dividing by $3x^2y^3$, we have

$$(y^3 - 2z'x')x^3 + (x^3 - 2y'z')y^3 + 12x'y'z^2 + 4y'^2yz + 4x'^2zx - 2x'y'xy = 0,$$

which represents a conic passing through the points of contact of tangents from (x', y', z') to the curve. If we call this conic V , $V + \lambda U = 0$ represents any conic passing through the points of contact of the tangents.

Hence, the locus of points, from which the tangents have their points of contact on a conic passing through two fixed points, is a cubic $U_1V_2 - V_1U_2 = 0$. By taking for the fixed points the circular points at infinity, we have the locus of the points whence the tangents have their points of contact on a circle.

3. Putting $\lambda = -2z'$ in the equation of the conic $V + \lambda U = 0$, we obtain the equation of the conic of the system which passes through (x', y', z') ,

$$(y'^2 - 4x'z')x^2 + (x'^2 - 4y'z')y^2 + 12x'y'z'^2 + 4(y'^2 - z'x')yz \\ + 4(x'^2 - y'z')zx - 2(x'y' + 2z'^2)xy = 0.$$

The tangent to this conic at (x', y', z') is

$$x'(x^2 + 2y'z')x + x'(y^2 + 2z'x')y - (x'^2 + y'^2 + 4x'y'z')z = 0,$$

which coincides with the tangent to the cubic

$$x^3(x^3 + y^3 + 6xyz) - (x^3 + y^3 + 6x'y'z')z^3 = 0,$$

as it ought (see Salmon's *Higher Plane Curves*, Art. 169).

4. The discriminant of $V + \lambda U$ is found to be, after dividing by the Hessian, $\lambda^3 - 6z'\lambda^2 + 4(x'^2 + y'^2 + 6x'y'z') = 0$. By means of this result we can find the locus of the intersection of tangents at the extremities of a chord which passes through a fixed point. Forming the equation of the chords of intersection of U and V , and expressing that this equation is satisfied by the coordinates of the fixed point, we obtain the equation of the locus,

$$W^3 + 6zPW^2 - 4(x^3 + y^3 + 6xyz)P^3 = 0,$$

where

$$W = (y'^2 + 4x'z')x^2 + (x'^2 + 4y'z')y^2 - 2y'^2yz - 2x'^2zx + 2(6z'^2 - x'y')xy,$$

$$P = (x'^2 + 2y'z')x + (y'^2 + 2z'x')y + 2x'y'z',$$

(x, y, z) being now the coordinates of a point on the locus, and (x', y', z') of the fixed point.

When the fixed point is on the curve, W becomes divisible by P , and therefore also the locus, which becomes, in this case,

$$4x^3y^3(x^3 + y^3) + (x^3x + y^3y)^3 - 6x'y'(x^2x - y^2y)^2z = 0.$$

Thursday, March 10th, 1881.

S. ROBERTS, Esq., F.R.S., President, in the Chair.

Prof. Cayley read a paper "On the Flexure and Equilibrium of a Skew Surface."

Mr. Tucker communicated portions of papers, viz., "An Application of Elliptic Functions to the Nodal Cubic," Mr. R. A. Roberts; and "Note on Prof. Peirce's Probability Notation of 1867," Mr. Hugh McColl.

Mr. Glaisher, Vice-President, having taken the Chair, the President communicated a Theorem which was the Analogue in Space of the Theorem relating to three Circles intersecting in a Point.

The following presents were made to the Library :—

“Educational Times,” March, 1881.

“An Introduction to the Ancient and Modern Geometry of Conics,” by C. Taylor, M.A. : from the Author.

“American Journal of Mathematics,” Vol. iii., No. 3.

“Various Papers and Notes that have appeared in the ‘Quarterly Journal of Mathematics,’ and the ‘Messenger of Mathematics,’ during the year 1880, by J. W. L. Glaisher, F.R.S.”

“On the Method of Least Squares,” from the “Monthly Notices of the Royal Astronomical Society,” Vol. xl., No. 9, and Vol. xvi., No. 1.

“Note on a Method of obtaining the q -formula for the Sine-amplitude in Elliptic Functions.” (Proceedings of London Mathematical Society, Vol. xi., No. 158.)

“On the Deduction of Trigonometrical from Elliptic Function Formulæ,” and two other Notes. (Report of British Association, 1880.)

Papers extracted from the “Proceedings of the Cambridge Philosophical Society,” Vol. iii., Pt. viii.

“Report of Committee of British Association on Mathematical Tables.” (Report of British Association, 1880.)

The above from the Author, J. W. L. Glaisher, F.R.S.

“Bulletins de l’Académie Royale . . . de Belgique,” 49^{me} année, 2^e série, tomes xlvi., xlvii., xlviii., xlix., l., 1880.

“Annuaire de l’Académie Royale . . . de Belgique,” 1879, 1880, 1881; Bruxelles.

A Note on Prof. C. S. Peirce’s Probability Notation of 1867.

By HUGH MCCOLL, B.A.

[The Secretaries were directed by the Council to state that they had received a note from Mr. McColl which showed them that the apparent coincidence of notation, in some few particulars, between himself and Prof. Peirce, was entirely accidental, and that Mr. McColl was not at the time acquainted with Prof. Peirce’s paper. In fact, the revised fourth paper (“On Probability Notation,” Proceedings of London Mathematical Society, Vol. xi., No. 163) was communicated to the Society about nine months before the Author read Prof. Peirce’s paper.]