CV. *On the Forced Vibrations of Bridges.*

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IT is now generally agreed that imperfect balance of the locomotive driving-wheels is the principal source of impact effect in bridges of long span. The laws governing this effect have not yet been definitely formulated, and much more information is needed on the experimental side †.

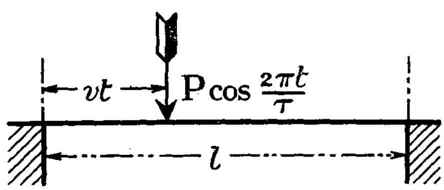
Some idea of the forced vibrations which are thus induced may be obtained by considering the bridge as a beam of constant cross-section with supported ends (fig. 1). The deflexion of the vibrating beam may be represented as follows :—

$$y=\phi_1 \sin\frac{\pi x}{l}+\phi_2 \sin\frac{2\pi x}{l}+\phi_3 \sin\frac{3\pi x}{l}+\ldots \quad . \quad . \quad . \quad (1)$$

where ϕ_1, ϕ_2, ..., etc. are functions of t only. Then if EI denotes the flexural rigidity of the beam, and w its weight per unit length, the expressions for the potential and kinetic energies will be

$$\left.\begin{aligned} V&=\frac{1}{2}EI\int_0^l\left(\frac{\partial^2 y}{\partial x^2}\right)^2 dx=\frac{1}{4}\frac{EI\pi^4}{l^3}\sum_{n=1}^{\infty}[n^4\phi_n^2], \\ T&=\frac{1}{2}\frac{w}{g}\int_0^l \dot{y}^2 dx=\frac{1}{4}\frac{wl}{g}\sum_{n=1}^{\infty}[\dot{\phi}_n^2]. \end{aligned}\right\} \quad . \quad (2)$$

Fig. 1.



We suppose that a single variable force $P\cos 2\pi t/\tau$ moves along the beam with a constant velocity v (fig. 1). The corresponding differential equations may be written in the form

$$\frac{wl}{2g}\ddot{\phi}_n+\frac{EI\pi^4}{2l^3}n^4\phi_n=P\cos\frac{2\pi t}{\tau}\sin\frac{n\pi vt}{l}. \quad . \quad . \quad (3)$$

Then taking $\phi_n=\dot{\phi}_n=0$ at the instant $t=0$, and writing

$$a^2=\frac{gEI}{w},$$

* Communicated by Mr. R. V. Southwell.

† *Cf.* 'Engineering,' vol. cxii. p. 80 (1921).

we obtain

$$\phi_n = \frac{l^2}{n^2\pi^2 a} \cdot \frac{2a}{wl}\int_0^t \mathrm{P}\cos\frac{2\pi t_1}{\tau}\sin\frac{n\pi v t_1}{l}\sin\frac{n^2\pi^2 a(t-t_1)}{l^2}\,dt_1, \quad (4)$$

and the expression (1) may be written as follows :—

$$y = \frac{\mathrm{P}l^3}{\pi^4 \mathrm{EI}}\sum_{n=1}^{\infty}\left[\sin\frac{n\pi x}{l}\left\{\frac{\sin\left(\frac{n\pi v}{l}+\frac{2\pi}{\tau}\right)t}{n^4-(\beta+n\alpha)^2}+\frac{\sin\left(\frac{n\pi v}{l}-\frac{2\pi}{\tau}\right)t}{n^4-(\beta-n\alpha)^2}\right.\right.$$

$$\left.\left.+\frac{\alpha}{n}\left(\frac{\sin\frac{n^2\pi^2 at}{l^2}}{n^2\alpha^2-(n^2-\beta)^2}+\frac{\sin\frac{n^2\pi^2 at}{l^2}}{n^2\alpha^2-(n^2+\beta)^2}\right)\right\}\right], \quad . \quad . \quad . \quad (5)$$

where $\alpha = \frac{vl}{a\pi}$; $\beta = \frac{2l^2}{\pi a \tau}$.

If the period τ of the force is the same as the period $\tau_1\left(=\frac{2l^2}{\pi a}\right)$ of the principal mode of vibration of the beam, resonance will occur, and the amplitude of the forced vibration will increase with t. Under these conditions, we have

$$\beta = 1,$$

and at the instant when the periodic force ceases to act upon the bridge we have

$$t = l/v,$$

so that $\alpha = \frac{1}{2}\frac{\tau_1}{t}$ (in general, a small quantity).

Then the first term (for $n=1$) in the series on the right of (5), which is the most important part of y, may be reduced to the form

$$\frac{2\mathrm{P}l^3}{\pi^4\mathrm{EI}}\sin\frac{\pi x}{l}\cdot\frac{1}{\alpha}\sin\frac{2\pi t}{\tau},$$

and the maximum value of the deflexion is given by the formula

$$f_{\text{max.}} = \frac{2\mathrm{P}l^3}{\alpha\pi^4\mathrm{EI}}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

Since, as we have seen, α is usually a small fraction, we may conclude from (6) that the forced vibration produced by want of balance in locomotives may be of practical importance.

The same method can be applied in other cases, where more complicated expressions are required for the forces which produce the vibration, and also in cases where variable horizontal forces act on the beam.

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