Expansions of Trigonometrical Functions
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Source: *The Mathematical Gazette*, Vol. 3, No. 52 (Jul., 1905), pp. 202-206
Published by: Mathematical Association
Stable URL: http://www.jstor.org/stable/3604037
Accessed: 11-02-2016 18:03 UTC

I.—EXPANSIONS OF TRIGONOMETRICAL FUNCTIONS.

In the following articles the method of Averages, used in the Oct. 1903 issue of the *Gazette* to effect the expansion of the simple Algebraical functions, is employed for the Trigonometrical Expansions and for Taylor's Theorem, and the results are applied to the question of convergence. The method was adopted as the simplest I could devise for even the Binomial, subject to the condition that no infinite series should be used unless the limits of error after a finite number of terms were explicitly stated. I would suggest that this condition might well be laid down in a school course, as the uncertainty that comes over our Algebra as soon as we leave Geometrical Progressions 'to infinity' delays a clear grasp of Mathematics more than a fuller consideration of the simpler infinite series would. The method is, of course, essentially integral calculus, but I have found—and I hope my subjects have not been specially favourable—that the proofs can be readily followed by students to whom the ideas and notation of the Differential and Integral will not be familiar for a considerable time, and in whom perhaps they will never inspire a sense of absolute security.

1. *Expansion of the sine.*

$$\sin x - \sin y = 2\sin\frac{x-y}{2}\cos\frac{x+y}{2},$$

which

$$= 2\tan\frac{x-y}{2}\cos\frac{x-y}{2}\cos\frac{x+y}{2}.$$

Now, if the angles are in circular measure,

$$2\sin\frac{x-y}{2} < (x-y), \quad \text{and } 2\tan\frac{x-y}{2} > (x-y).$$

Moreover, if x and y are less than $\frac{\pi}{2}$,

$$\cos\frac{x+y}{2} < \cos y, \quad \text{and } \cos\frac{x-y}{2}\cos\frac{x+y}{2} = \tfrac{1}{2}(\cos x + \cos y) > \cos x;$$

$$\therefore \sin x - \sin y \overset{*}{<} (x-y)\left|\begin{matrix}\cos y \\ \cos x\end{matrix}\right|.$$

Now consider the interval between 0 and a broken up into N equal intervals, the intermediate angles being

$$\frac{a}{N}, \frac{2a}{N}, \ldots \frac{(N-2)a}{N}, \frac{(N-1)a}{N},$$

which we may write shortly

$$a_1, a_2, \ldots a_{N-2}, a_{N-1}.$$

* "Lies between."

Then, $\sin a \equiv (\sin a - \sin a_{N-1}) + (\sin a_{N-1} - \sin a_{N-2}) + \ldots + (\sin a_1 - \sin 0)$,

that is, $$\sin a = \sum_{i=1}^{i=N} (\sin a_i - \sin a_{i-1}).$$

Therefore, provided $a < \frac{\pi}{2}$ (so as to ensure that $\cos a_{i-1}$ is always greater than $\cos a_i$),

$$\sin a < \sum_1^N \frac{a}{N} \left| \begin{matrix} \cos a_{i-1} \\ \cos a_i \end{matrix} \right|,$$

that is, $$< \frac{a}{N} \left| \begin{matrix} \cos 0 + \cos a_1 + \cos a_2 + \ldots + \cos a_{N-2} + \cos a_{N-1} \\ \cos a_1 + \cos a_2 + \ldots \qquad + \cos a_{N-1} + \cos a \end{matrix} \right|. \quad (1)$$

Now the difference between the upper and lower limits is

$$\frac{a}{N}(\cos 0 - \cos a) \text{ or } \frac{a}{N} \cdot 2 \sin^2 \frac{a}{2}, \text{ which } < \frac{a^3}{2N}.$$

But N is quite independent of all the other magnitudes concerned, and therefore we may consider it indefinitely increased, and may neglect this small error, writing for either of the limits in (1) $a \cdot \underset{\theta=0}{\overset{\theta=1}{\text{Av}}} (\cos \theta a)$, θ representing the fractions $\frac{1}{N}, \frac{2}{N}, \ldots$ and 'Av' the average of the values when N is *indefinitely* increased.

Thus $$\sin a = a \cdot \underset{0}{\overset{1}{\text{Av}}} \cos \theta a. \quad \text{(s)}$$

An exactly similar piece of work leads to

$$\cos a - 1 = -a \cdot \underset{0}{\overset{1}{\text{Av}}} \sin \theta a. \quad \text{(c)}$$

Hence, substituting for $\cos \theta a$ in (s) by means of (c),

$$\sin a = a \underset{\theta=0}{\overset{\theta=1}{\text{Av}}} \left[1 - \theta a \underset{\theta'=0}{\overset{\theta'=1}{\text{Av}}} \sin \theta\theta' a \right] = a - a^2 \cdot \underset{0}{\overset{1}{\text{Av}}}\, \theta \cdot \underset{0}{\overset{1}{\text{Av}}} \sin \theta\theta' a.$$

Using the same process on $\sin \theta\theta' a$, and introducing subscripts to θ, we get

$$\sin a = a - a^3 \cdot \underset{0}{\overset{1}{\text{Av}}}\, \theta_1^2 \cdot \underset{0}{\overset{1}{\text{Av}}}\, \theta_2$$
$$+ a^4 \cdot \underset{0}{\overset{1}{\text{Av}}}\, \theta_1^3 \cdot \underset{0}{\overset{1}{\text{Av}}}\, \theta_2^2 \cdot \underset{0}{\overset{1}{\text{Av}}}\, \theta_3 \cdot \underset{0}{\overset{1}{\text{Av}}} (\sin \theta_1\theta_2\theta_3\theta_4 a).$$

Now, by a simple process,* we find that $\underset{0}{\overset{1}{\text{Av}}}\, \theta^n = \frac{1}{n+1}$;

$$\therefore \sin a = a - \frac{a^3}{\underline{|3}} + a^4 \cdot (\text{the same expression as before}).$$

* *Gazette*, July, 1903, § v.

By repetition of the process we obtain, if n is odd,

$$\sin a = a - \frac{a^3}{\underline{|3}} + \frac{a^5}{\underline{|5}} - \dots \pm \frac{a^n}{\underline{|n}} \mp \text{ an error which is}$$

$$a^{n+1} . \text{Av}\, \theta_1{}^n . \text{Av}\, \theta_2{}^{n-1} \dots \text{Av}\, \theta_n . \text{Av}\, \sin(\theta_1\theta_2 \dots \theta_n\theta_{n+1}a).$$

The value of this error can be obtained by the consideration that, as every θ is a fraction, the sine that occurs is less than $\sin a$.

$\therefore$ the error lies between $\dfrac{a^{n+1}}{\underline{|n+1}} \sin a$ and 0.

By the use of equation (S) above, it is equally easy to show that the error lies between $\dfrac{a^{n+2}}{\underline{|n+2}} \left| \begin{matrix} 1 \\ \cos a \end{matrix} \right|$, n still being odd.

In either case it diminishes indefinitely as n increases.

2. *The expansion of the cosine* follows from equation (C), or can be established *à priori* by similar work.

$$\cos a = 1 - \frac{a^2}{\underline{|2}} + \frac{a^4}{\underline{|4}} - \dots \pm \frac{a^n}{\underline{|n}} \; (n \text{ even}) \mp \text{ an error}$$

which lies between $\dfrac{a^{n+1}}{\underline{|n+1}} \left| \begin{matrix} \sin a \\ 0 \end{matrix} \right|$, or between $\dfrac{a^{n+2}}{\underline{|n+2}} \left| \begin{matrix} 1 \\ \cos a \end{matrix} \right|$.

[NOTE. For the sake of simplicity these expansions are here proved only for acute angles: it is shown in § II., that a slight modification of the proof makes it general.]

3. *Expansion of the inverse tangent.*

We have $\tan x - \tan y = \dfrac{\sin(x-y)}{\cos x . \cos y} = \tan(x-y) . \dfrac{\cos(x-y)}{\cos x \cos y}$.

Hence, $\tan x - \tan y < (x-y) \left| \begin{matrix} \sec^2 x \\ \sec^2 y \end{matrix} \right|$.

[We could get from this, as before,

$$\tan a = a . \underset{0}{\overset{1}{\text{Av}}} (\sec^2 \theta a) = a + a^2 \underset{0}{\overset{1}{\text{Av}}} (\tan^2 \theta a);$$

but the square under the average introduces difficulties.]

Since we may write the above in the form

$$(x-y) < (\tan x - \tan y) \left| \begin{matrix} \cos^2 y \\ \cos^2 x \end{matrix} \right|,$$

we have $\tan^{-1} a - \tan^{-1} b < (a-b) \left| \begin{matrix} (1+b^2)^{-1} \\ (1+a^2)^{-1} \end{matrix} \right|$

leading, by the usual process, to $\tan^{-1} x = x . \underset{0}{\overset{1}{\text{Av}}} (1+\theta^2 x^2)^{-1}$.

Now $\dfrac{1 \mp \theta^{2n}x^{2n}}{1+\theta^2x^2} = 1 - \theta^2x^2 + \ldots \pm \theta^{2n-2}x^{2n-2}$, the sign depending on n even or odd.

$\therefore \tan^{-1}x = x - \dfrac{x^3}{3} + \dfrac{x^5}{5} \ldots \mp \dfrac{x^{2n-1}}{2n-1} \pm$ an error $x^{2n+1} . \text{Av} \dfrac{\theta^{2n}}{1+\theta^2x^2}$.

This error is less than*

$$x^{2n+1} . \text{Av}\, \theta^{2n} . \text{Av}\, (1+\theta^2x^2)^{-1},$$

which

$$= \frac{x^{2n+1}}{2n+1} \cdot \frac{\tan^{-1}x}{x},$$

and is greater than

$$\frac{x^{2n+1}}{2n+1} \cdot \frac{1}{1+x^2},$$

and therefore diminishes indefinitely if x is less than 1.

4. *Expansion of the inverse sine and cosine.*

Since

$$\sin x - \sin y < (x-y) \left| \begin{matrix} \cos y \\ \cos x \end{matrix} \right|,$$

$$\sin^{-1}a - \sin^{-1}b < (a-b) \left| \begin{matrix} (1-a^2)^{-\frac{1}{2}} \\ (1-b^2)^{-\frac{1}{2}} \end{matrix} \right|.$$

Hence, as before, $\sin^{-1}x = x . \underset{0}{\overset{1}{\text{Av}}} (1-\theta^2x^2)^{-\frac{1}{2}}$.

We here need the error after the n term of a Binomial Expansion.

In the case of $(1+u)^p$ (see §§ VI, VIII, *Gazette*, Oct. 1903) it lies between 0 and

$$(p-1)_n . u^n \{(1+u)^n - 1\},$$

$$(p-1)_n \text{ denoting } \frac{(p-1)(p-2)\ldots(p-n)}{\lfloor n},$$

therefore the error in the case of $(1-\theta^2x^2)^{-\frac{1}{2}}$,

$$< (-\tfrac{3}{2})_n (-\theta^2x^2)^n \left| \begin{matrix} (1-\theta^2x^2)^{-\frac{1}{2}} - 1 \\ 0 \end{matrix} \right|;$$

$\therefore \sin^{-1}x = \sum\limits_0^n \dfrac{1 . 3 \ldots \overline{2n-1}}{2 . 4 \ldots 2n} \cdot \dfrac{x^{2n+1}}{2n+1} +$ an error which lies between

$$\frac{3 . 5 \ldots \overline{2n+1}}{2 . 4 \ldots 2n} \cdot x^{2n+1} . \text{Av}\, \theta^{2n} \{(1-\theta^2x^2)^{-\frac{1}{2}} - 1\} \text{ and } 0.$$

But $(1-\theta^2x^2)^{-\frac{1}{2}} < (1-x^2)^{-\frac{1}{2}}$;

$$\therefore \text{ the error } < \frac{1 . 3 . 5 \ldots \overline{2n-1}}{2 . 4 . 6 \ldots 2n} x^{2n+1} \{(1-x^2)^{-\frac{1}{2}} - 1\},$$

which diminishes indefinitely as n increases, if $x < 1$.

* The following inequality can be easily established: $\text{Av} . ab . \gtreqless \text{Av}\, a . \text{Av}\, b$ according as the a's and b's increase together or contrariwise.

5. *The hyperbolic functions.*

Exactly similar work serves for these: for

$$\sinh x - \sinh y = 2\sinh\frac{x-y}{2}\,.\cosh\frac{x+y}{2}$$

$$= 2\tanh\frac{x-y}{2}\,.\cosh\frac{x-y}{2}\cosh\frac{x+y}{2},$$

and therefore $$< (x-y)\left|\begin{matrix}\cosh x\\ \cosh y\end{matrix}\right|.$$

$$\cosh x - \cosh y = 2\sinh\frac{x-y}{2}\sinh\frac{x+y}{2} < (x-y)\left|\begin{matrix}\sinh x\\ \sinh y\end{matrix}\right|,$$

$$\tanh x - \tanh y = \frac{\sinh(x-y)}{\cosh x\cosh y} < (x-y)\left|\begin{matrix}\operatorname{sech}^2 y\\ \operatorname{sech}^2 x\end{matrix}\right|.$$

The method suggests the following answer to the difficult question 'what does "expansion" mean?'

The idea of continuity leads naturally to the expression of the increase of a function as a definite integral: expansion in a power series is the simplest form of integration by parts.

II.—EXPANSION OF FUNCTIONS IN GENERAL.

To prove that if f represents a function satisfying certain conditions for all values of the variable from a to $a+x$, then $f(a+x)$ can be expanded in powers of x.

Assume that, as in § I., $fu - fv < (u-v)\left|\begin{matrix}f'u\\ f'v\end{matrix}\right|$ where f' is another function which we call the *derived* function of f.

Now in the special cases investigated above $f'u$ was always greater or always less than $f'v$; in other words, f' was throughout an increasing or throughout a diminishing function. But this condition may not be fulfilled (*e.g.* $\cos x$ is diminishing from 0 to π, increasing from π to 2π, diminishing from 2π to 3π, etc.). In this general article slight modifications are introduced to provide for such cases.

If $x_1, x_2 \ldots$ are *any* intermediate values of the variable between x_0 and x_N, we obtain as before $fx_N - fx_0 < \Sigma(x_i - x_{i-1})\left|\begin{matrix}f'x_i\\ f'x_{i-1}\end{matrix}\right|$; but the upper limit sometimes has $f'x_i$ and sometimes $f'x_{i-1}$.

Now choose the simplest series of values for x_i. Let x_0 be a, and let x_i be $a+\frac{ix}{N}$, so that the intermediate values divide the whole range a to $a+x$ into N equal parts, and $x_i - x_{i-1} = \frac{x}{N}$.