many of them as there are λ -combinations, with repetition, of the eleven matrices of Table I. But we can always pass from any one of the equations (16) to any other by means of the formula (19), which may be employed to express any one of the 4th Theta functions as a product of any other by an exponential factor; although (for brevity) we have supposed that the indices of one of the Theta functions compared in that formula are all equal to zero.

Quaternion Proof of Minding's Theorem. By J. J. WALKER, M.A.

[Read February 13th, 1879.]

At the conclusion of his very able paper on the Conditions of Astatic Equilibrium (Proc. of Lond. Math. Soc., Vol. IX., pp. 116—118), Professor Minchin has given a demonstration of this theorem,* in which recourse has been had to Cartesian coordinates, whereby the unity of treatment of the subject by Quaternions, elsewhere adopted throughout the paper, and to the use of which it owes much of its interest, has been marred. A short note, showing how the pure Quaternion proof may be completed, may, perhaps, be thought worthy of insertion.

Starting with Professor Minchin's equation (32) of the line in which the single resultant and Poinso's axis combine, viz.,

$$\rho = \lambda i + \frac{1}{a} V(Jk - Kj),$$

with (31) $SJk = SKj \dots\dots\dots(31),$

which give $\rho = \lambda i + \frac{1}{a} (Jk - Kj) \dots\dots\dots(32'),$

wherein λ is an arbitrary scalar, the problem is to find the vector of the point in which that line meets the plane of IJ or IK .

Let $i' = UI$, $j' = UJ$, $k' = UK$, while $t_1 = TJ$, $t_2 = TK$.

To determine the λ of ρ in the plane IJ ,

$$Spk' = \lambda Sik' + \frac{t_2}{a} Sj'kk',$$

i.e., $0 = \lambda Sik' - \frac{t_2}{a} Skv',$

* Originally given in a paper which appeared in *Crelle*, Bd. XV. (1835), to which notice appears to have been more recently drawn by Moigno, *Mec. Anal.*, Ch. ix. (1868).

whence
$$\lambda = \frac{t_3}{a} \frac{Sk\check{i}}{Si\check{k}} \dots\dots\dots (34),$$

Assume
$$\rho = x\check{i} + y\check{j},$$

then, taking the scalars of $\rho\check{i}$, $\rho\check{j}$ successively, there result

$$x = -S\rho\check{i}, \quad y = -S\rho\check{j}.$$

Introducing in (32') the value found for λ (34),

$$\rho aSi\check{k}' = it_3Sk\check{i}' + Si\check{k}'(t_3j'k - t_3k'j),$$

whence
$$S\rho\check{i}aSi\check{k}' = Si\check{i}'t_3Sk\check{i}' + Si\check{k}'(t_3Skk' + t_3Sjj'),$$

and
$$S\rho\check{j}aSi\check{k}' = Si\check{j}'t_3Sk\check{i}' + Si\check{k}'(-t_3Sj\check{i}'),$$

supposing j', k', i' to be so designated that $j'k' = i' \dots$

Substituting in
$$\rho = -i'S\rho\check{i} - j'S\rho\check{j},$$

$$\begin{aligned} -\rho aSi\check{k}' &= i' \{ t_3 (Si\check{i}'Sk\check{i}' + Si\check{k}'Skk') + t_3 Si\check{k}'Sjj' \} \\ &\quad + j' \{ t_3 Si\check{j}'Sk\check{i}' - t_3 Si\check{k}'Sj\check{i}' \} \\ &= i' (-t_3 Si\check{j}'Sjk' + t_3 Si\check{k}'Sjj') + j' (\dots) \end{aligned}$$

by (31),
$$= i't_3(-Si\check{j}'Sjk' + Si\check{k}'Sjj') + j't_3 \frac{Si\check{j}'Sjk'Sk\check{i}' - Si\check{k}'Sj\check{i}'Sk\check{j}'}{Skj'}$$

$$= -i't_3Si\check{i}'k - j' \frac{t_3}{Skj'}, \{ (Skj')^2 - (Sjk')^2 \},$$

i.e.,
$$\rho aSi\check{k}' = i't_3Sk\check{i}' + j'\sqrt{t_3^2 - t_2^2} \sqrt{(Skj')^2 - (Sjk')^2},$$

—in virtue of $t_3Skj' = t_3Sjk'$ (31)—which is the equation to a central

conic, having $\frac{t_3}{a}i'$ and $\frac{\sqrt{t_3^2 - t_2^2}}{a}j'$ as axes, since

$$(Si\check{k}')^2 = (Sk\check{i}')^2 + (Skj')^2 - (Sjk')^2.$$

In fact
$$(Si\check{k}')^2 + (Sjk')^2 = \frac{1}{2} - \frac{1}{2}Skk'kk' = (Vkk')^2 = (S\check{i}k')^2 + (S\check{j}k')^2.$$

I may observe, also, that the quaternion equation of Poincot's axis has been very simply worked out by pure quaternion process from its definition, without the assumption of any property deduced by other methods, by Prof. Tait, *Quaternions*, 2nd ed., Ch. xi.

[The Secretaries have received the following further Note on the same subject from Prof. Tait, which is printed here by order of the Council.]

Quaternion Proof of Minding's Theorem. By Prof. P. G. TAIT.

1. Since the set of forces β_1 at α_1 , &c., is to be turned about, let them be treated in connection with a rectangular system i', j', k' , which is to turn with them. We thus avoid the use of $q(\)q^{-1}$, a weapon

obviously of immense power, the full art of managing which is unfortunately still unknown.

Then, taking the portions parallel to i' alone, they give

$$-\Sigma i' S i' \beta = -i' S i' \beta, \text{ say,}$$

and a couple $-\Sigma V i' a S i' \beta$, in the same plane.

They are therefore equivalent to a single force at the point

$$\frac{\Sigma a S \beta i'}{S i' \beta} = \frac{\phi i'}{S i' \beta}, \text{ suppose, [centre of parallel forces].}$$

The three such points obviously lie in the plane $S \beta \phi^{-1} \rho = 1$, which passes through the point $\phi \beta^{-1}$. This, which we take as origin, is called the centre of the plane of centres, and will be the centre of the k' parallel forces if $\beta = b k'$. In that case the other forces vanish, and their points of application go off to an infinite distance.

2. Hence (changing the a 's in ϕ to suit the new origin) we may state the system as equivalent to the force $b k'$ at the origin, and the couple $-V i' \phi i' - V j' \phi j'$. Since, by our choice of origin, we have made $\phi k' = 0$, $\phi i'$, $\phi j'$ are fixed vectors in the plane of centres, and we may take them as perpendicular to one another; for the condition is

$$S i' \phi' \phi j' = 0,$$

and $\phi' \phi$ is essentially a self-conjugate function. They are the central lines of the plane of centres. Let them be called $e_1 i$ and $e_2 j$, and let $\varpi \sigma = -e_1 i S i \sigma - e_2 j S j \sigma$. Thus we have now the force $b k'$ at the origin, and the couple

$$V \varpi i i' + V \varpi j j'.$$

The condition that these shall be equivalent to a single force is, of course,

$$S k' (\varpi i i' + \varpi j j') = 0.$$

This enables us to express i' and j' in terms of k' , and thus to eliminate them. For we have at once

$$t i' = V k' \varpi j + k' V k' \varpi i,$$

$$t j' = -V k' \varpi i + k' V k' \varpi j,$$

where t is the common tensor, a function of k' . One simple form is

$$t = S i' \varpi i + S j' \varpi j.$$

Now the equation of the line in which the single force acts is

$$b V k' \rho = V (\varpi i i' + \varpi j j'),$$

or, with the above values of i' , j' ,

$$-b t \rho = (\varpi^2 + u) k' - e_1 e_2 k,$$

where u is wholly indeterminate.

But Minding's theorem refers to the intersection of this line with the plane

$$Si\rho = 0,$$

or

$$Sj\rho = 0.$$

Take the first, and we get

$$(e_1^2 + u) SiK = 0,$$

which determines u , and then at once

$$-bt(\varpi^2 - e_1^2)^{-1}\rho = k' + \frac{e_2}{e_1}k.$$

Now we see easily that

$$i^2 = S \left(k' + \frac{e_2}{e_1}k \right) [(\varpi^2 - e_1^2)k' - e_1e_3k];$$

so that, finally,
$$Sp(\varpi^2 - e_1^2)^{-1}\rho = \frac{1}{b^2},$$

which is one of the focal conics of the system

$$Sp(\varpi^2 - h)^{-1}\rho = \frac{1}{b^2}.$$

Minding's theorem might have been anticipated from the general equation above; for it gives at once the transformation

$$b^2Sp(\varpi^2 + u)^{-1}\rho = 1 - \frac{1}{i^2u}(e_1^2 + u)(e_2^2 + u),$$

which is obviously much simplified when $u = -e_1^2$ or $= -e_2^2$. This is a good example of the suggestiveness of quaternions.

March 13th, 1879.

C. W. MERRIFIELD, Esq., F.R.S., President, in the Chair.

Mr. J. D. H. Dickson, M.A., was admitted into the Society. Mr. R. Hargreaves and Prof. W. E. Story were elected Members. Mr. Donald McAlister, B.A., B.Sc., Fellow of St. John's College, Cambridge, was proposed for election.

Prof. Cayley made a few remarks appreciative of the late Prof. Clifford's work as a mathematician, instancing more particularly his papers on the Riemann's Surface, the Contact Problems of Mr. Spottiswoode, and the Classification of Loci.

The Chairman, the Rev. A. Freeman, and Dr. Hirst, also added a few words on the loss the Society and the mathematical world generally

had sustained,* and expressed a hope that steps would be taken to secure the publication, if necessary, of any MSS. Prof. Clifford might have left.

Dr. Hirst made a statement to the Society of the present position of the movement for presenting a De Morgan memorial Medal to the Society; it appeared that the bust and die for the Medal had been executed by Mr. Woolner, and that, after all claims for these had been met, there would still be a small sum required to make up the total requisite for investment in the Funds to bring in the necessary interest for the above purpose. It was agreed that a subscription list be opened for any Members who might wish to contribute to the Fund, and that Mr. Tucker be authorized to receive such subscriptions as might be paid in in this way, and to hand them over to Mr. A. Wills, Q.C., the Hon. Sec. to the Fund. Copies of the obverse and reverse of the Medal were exhibited.

The following communications were made:—

“On Differential Equations, Total and Partial; and on a new Soluble Class of the First, and an Exceptional Case of the Second;” by Sir James Cockle, F.R.S.

“Discussion of two Double Series arising from the number of Terms in Determinants of certain Forms;” by Mr. J. D. H. Dickson.

“Two Geometrical Notes relating to Surfaces of the Second Order;” by Prof. H. J. S. Smith, F.R.S.

The following presents were received:—

“Educational Times,” March, 1879.

“Atti della R. Accademia dei Lincei . . . Transunti,” Vol. iii., Fasc. 3°, Febb. 1879; Roma 1879.

“Bulletin de la Société Mathématique de France,” Tome vii., No. 2; Paris, 1879.

“The Physical Society of London, Proceedings,” Vol. ii., Pt. v., July to December, 1878; London, 1879.

“Proceedings of Royal Society,” Vol. xxviii., No. 193.

“Crelle's Journal,” Bd. 86, viertes Heft; Bd. 87, erstes Heft und zweites Heft.

“Monatsbericht,” December, 1878; Berlin, 1879.

“New Solar Elements;” by Rev. G. T. Carruthers; Nagpur, 1879: from the Author.

“Proceedings of Royal Society of Edinburgh,” Session 1877–78, Vol. ix., No. 100, Nov. 1877 to July 1878.

* Prof. Clifford died at Madeira, March 3rd, 1879.