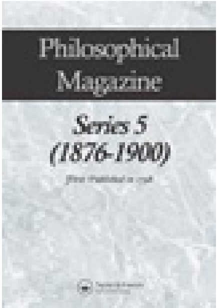

Philosophical Magazine Series 5
Publication details, including instructions for authors and subscription information:
http://www.tandfonline.com/loi/tphm16

XII. On the resultant of a large number of vibrations of the same pitch and of arbitrary phase

Lord Rayleigh F.R.S. [a]
[a] University of Cambridge
Version of record first published: 28 Apr 2009.

To cite this article: Lord Rayleigh F.R.S. (1880): XII. On the resultant of a large number of vibrations of the same pitch and of arbitrary phase , Philosophical Magazine Series 5, 10:60, 73-78

To link to this article: http://dx.doi.org/10.1080/14786448008626893

THE

LONDON, EDINBURGH, AND DUBLIN

PHILOSOPHICAL MAGAZINE

AND

JOURNAL OF SCIENCE.

[FIFTH SERIES.]

AUGUST 1880.

XII. *On the Resultant of a large Number of Vibrations of the same Pitch and of arbitrary Phase.* By LORD RAYLEIGH, *F.R.S., Professor of Experimental Physics in the University of Cambridge**.

VERDET†, in an investigation upon this subject, has arrived at the conclusion that the resultant of n vibrations of unit amplitude and arbitrary phase approaches the definite value $\sqrt{n}$ when n is very great. It can be shown‡, however, that this conclusion is inaccurate, and that the resultant tends to no definite value, however great the number of components may be.

But there is a modified form of the question, which admits of a definite answer, and was perhaps vaguely before Verdet's mind. If we inquire what is the *average* intensity in a great number of cases, or, in the language of the theory of probabilities, what is the *expectation* of intensity in a single case of composition, we shall find that the result is that assigned by Verdet, namely n.

A simple but instructive variation of the problem may be obtained by supposing the possible phases limited to *two opposite* phases, in which case it is convenient to discard the idea of phase altogether, and to regard the amplitudes as at random positive or negative. If all the signs are the same, the resultant intensity is n^2; if, on the other hand, there are as many positive as negative, the result is zero. But although the intensity may range from 0 to n^2, the smaller values are much more *probable* than the greater; and to calculate the ex-

* Communicated by the Author.
† *Leçons d'Optique physique*, t. i. p. 297.
‡ Math. Soc. Proc. May 1871.

pectation of intensity, these different degrees of probability must be taken into account. By well-known rules the expression for the expectation is

$$\frac{1}{2^n}\left\{1\,.\,n^2+n\,.\,(n-2)^2+\frac{n(n-1)}{1\,.\,2}(n-4)^2 + \frac{n(n-1)(n-2)}{1\,.\,2\,.\,3}(n-6)^2+\ldots\right\}.$$

The value of the series, which is to be continued so long as the terms are finite, is simply n, as may be proved by comparison of coefficients of x^2 in the equivalent forms

$$(e^x+e^{-x})^n=2^n(1+\tfrac{1}{2}x^2+\ldots)^n = e^{nx}+ne^{(n-2)x}+\frac{n(n-1)}{1\,.\,2}e^{(n-4)x}+\ldots.$$

The expectation of intensity is therefore n, and this whether n be great or small.

In the more general problem, where the phases are distributed at random over the complete period, the expression for the expectation of intensity is

$$\int_0^{2\pi}\int_0^{2\pi}\int_0^{2\pi}\ldots\frac{d\theta}{2\pi}\frac{d\theta'}{2\pi}\frac{d\theta''}{2\pi}\ldots[(\cos\theta+\cos\theta'+\cos\theta''+\ldots)^2 + (\sin\theta+\sin\theta'+\sin\theta''+\ldots)^2].$$

If we effect the integration with respect to θ, we get

$$\int_0^{2\pi}\int_0^{2\pi}\ldots\frac{d\theta'}{2\pi}\frac{d\theta''}{2\pi}\ldots.[1+(\cos\theta'+\cos\theta''+\ldots)^2 + (\sin\theta'+\sin\theta''+\ldots)^2].$$

Continuing the process by successive integrations with respect to θ', θ'', ..., we see that, as before, the expectation of intensity is n.

So far there is no difficulty; but a complete investigation of this subject involves an estimate of the relative probabilities of resultants lying between assigned limits of magnitude. For example, we ought to be able to say what is the probability that the intensity due to a large number (n) of equal components is less than $\frac{1}{2}n$. It will be convenient to begin by taking the problem under the restriction that the phases are of two opposite kinds only. When this has been dealt with, we shall not find much difficulty in extending our investigation to phases entirely arbitrary.

By Bernoulli's theorem* we find that the probability that

* Todhunter's 'History of the Theory of Probability,' § 993.

of n vibrations, which are at random positive or negative, the number of positive vibrations lies between

$$\tfrac{1}{2}n-\tau\sqrt{(\tfrac{1}{2}n)} \text{ and } \tfrac{1}{2}n+\tau\sqrt{(\tfrac{1}{2}n)}$$

is, when n is great,

$$\frac{2}{\sqrt{\pi}}\int_0^\tau e^{-t^2}\,dt,$$

where $\tau=r\sqrt{(2n)}$, and r must not surpass $\sqrt{n}$ in order of magnitude. In the extreme cases the amplitude is $\pm 2\tau\sqrt{(\tfrac{1}{2}n)}$, and the intensity is $2\tau^2 n$. Thus, if we put $\tau=\tfrac{1}{2}$, we see that the chance of intensity less than $\tfrac{1}{2}n$ is

$$\frac{2}{\sqrt{\pi}}\int_0^{\frac{1}{2}} e^{-t^2}\,dt=\cdot 5205\,;$$

so that however great n may be, there is always more than an even chance that the intensity will be less than $\tfrac{1}{2}n$. This, of course, is inconsistent with any such tendency to close upon the value n as Verdet supposes.

From the tables of the definite integral, given in De Morgan's 'Differential Calculus,' p. 657, we may find the probabilities of intensities less than any assigned values. The probility of intensity less than $\tfrac{1}{8}n$ is $\cdot 2764$.

Again, the chance that in a series n the number of positive vibrations lies between

$$\tfrac{1}{2}n+\tau\sqrt{(\tfrac{1}{2}n)} \text{ and } \tfrac{1}{2}n+(\tau+\delta\tau)\sqrt{(\tfrac{1}{2}n)}$$

is

$$\frac{1}{\sqrt{\pi}}e^{-\tau^2}\,\delta\tau,$$

which expresses accordingly the chance of a positive amplitude lying between

$$2\tau\sqrt{(\tfrac{1}{2}n)} \text{ and } 2(\tau+\delta\tau)\sqrt{(\tfrac{1}{2}n)}.$$

Let these limits be called x and $x+\delta x$, so that $\tau=x/\sqrt{(2n)}$; then the chance of amplitude between x and $x+\delta x$ is

$$\frac{1}{\sqrt{(2\pi n)}}e^{-\frac{x^2}{2n}}\,\delta x.$$

The expectation of intensity is expressed by

$$\frac{1}{\sqrt{(2\pi n)}}\int_{-\infty}^{+\infty} e^{-\frac{x^2}{2n}}x^2\,dx=n,$$

as before.

It will be convenient in what follows to consider the vibrations to be represented by lines (of unit length) drawn from a fixed point O, the intersection of rectangular axes Ox and Oy.

G 2

If n of these lines be taken at random in the directions $\pm x$, the probability of resultants also along $\pm x$, and of various magnitudes, is given by preceding expressions. We will now suppose that $\frac{1}{2}n$ are distributed at random along $\pm x$, and $\frac{1}{2}n$ along $\pm y$, and inquire into the probabilities of the various resultants. The probability that the end of the representative line, or, as we may consider it, the representative *point*, lies in the rectangle $dx\,dy$ is evidently

$$\frac{1}{\pi n} e^{-\frac{x^2+y^2}{n}} dx\,dy.$$

Substituting polar coordinates r, θ and integrating with respect to θ, we see that the probability of the representative point of the resultant lying between the circles r and $r+dr$ is

$$\frac{2}{n} e^{-\frac{r^2}{n}} r\,dr.$$

This is therefore the probability of a resultant vibration with amplitude between the values r and $r+dr$. In this case there are n components distributed in four rectangular directions; and we have supposed that $\frac{1}{2}n$ exactly are distributed along $\pm x$, and $\frac{1}{2}n$ along $\pm y$. It is important to remove this restriction, and to show that the result is the same when the distribution is perfectly arbitrary in respect to all four directions.

In order to see this, let us suppose that $\frac{1}{2}n+m$ are distributed along $\pm x$ and $\frac{1}{2}n-m$ along $\pm y$, and imagine how far the result is influenced by the value of m. The chance of the representative point of the resultant lying in the rectangle $dx\,dy$ is now expressed by

$$\frac{1}{\pi\sqrt{(n^2-4m^2)}} e^{-\frac{x^2}{n+2m}-\frac{y^2}{n-2m}} dx\,dy$$

$$=\frac{1}{\pi\sqrt{(n^2-4m^2)}} e^{-\frac{n(x^2+y^2)+2m(y^2-x^2)}{n^2-4m^2}} dx\,dy$$

$$=\frac{1}{\pi\sqrt{(n^2-4m^2)}} e^{-\frac{nr^2}{n^2-4m^2}} e^{-\frac{2mr^2}{n^2-4m^2}\cos 2\theta} r\,dr\,d\theta.$$

Also

$$\int_0^{2\pi} e^{-\frac{2mr^2\cos 2\theta}{n^2-4m^2}} d\theta = 2\pi\left\{1+\frac{m^2r^4}{(n^2-4m^2)^2}+\dots\right\},$$

as we find on expanding the exponential and integrating. Thus the chance of the representative point lying between the circles r and $r+dr$ is

$$\frac{2r\,dr}{\sqrt{(n^2-4m^2)}} e^{-\frac{nr^2}{n^2-4m^2}}\left\{1+\frac{m^2r^4}{(n^2-4m^2)^2}+\dots\right\}.$$

Now, if the distribution be entirely at random, all the values of m of which there is a finite probability are of order not higher than $\sqrt{n}$, n being treated as infinite. But if m be of this order, the above expression is the same as if m were zero, and thus it makes no difference whether the numbers of components along $\pm x$ and along $\pm y$ are limited to be equal or not. The previous result, viz.

$$\frac{2}{n} e^{-\frac{r^2}{n}} r\,dr,$$

is accordingly applicable to a thoroughly arbitrary distribution among the four rectangular directions.

The next point to notice is that the result is symmetrical, and independent of the direction of the axes, so long as they are rectangular, from which we may conclude that it has a still higher generality. If a total of n components, to be distributed along one set of rectangular axes, be divided into any number of groups, it makes no difference whether we first obtain the probabilities of various resultants of the groups separately and afterwards of the final resultant, or whether we regard the whole n as one group. But the resultant of each group is the same, notwithstanding a change in the system of rectangular axes; so that the probabilities of various resultants are unaltered, whether we suppose the whole number of components restricted to one set of rectangular axes or divided in any manner between any number of sets of axes. This last state of things, however, is equivalent to no restriction at all; and we thus arrive at the important conclusion that, if n unit vibrations of equal pitch and of arbitrary phases be compounded, the probability of a resultant intermediate in amplitude between r and $r+dr$ is

$$\frac{2}{n} e^{-\frac{r^2}{n}} r\,dr,$$

a similar result applying, of course, in the case of any other vector quantities.

The probability of a resultant of amplitude less than r is

$$\int_0^r \frac{2}{n} e^{-\frac{r^2}{n}} r\,dr = 1 - e^{-\frac{r^2}{n}};$$

or, which is the same thing, the probability of a resultant greater than r is

$$e^{-\frac{r^2}{n}}.$$

The following table gives the probabilities of intensities less than the fractions of n named in the first column. For example, the probability of intensity less than n is ·6321.

·05	·0488	·80	·5506
·10	·0952	1·00	·6321
·20	·1813	1·50	·7768
·40	·3296	2·00	·8647
·60	·4512	3·00	·9502

It will be seen that, however great n may be, there is a reasonable chance of considerable relative fluctuations of intensity in consecutive trials.

The *average* intensity, expressed by

$$\int_0^\infty \frac{2}{n} e^{-\frac{r^2}{n}} . r^2 . r\,dr,$$

is, as we have seen already, equal to n.

If the amplitude of each component be α, instead of unity, as we have hitherto supposed for brevity, the probability of a resultant amplitude between r and $r+dr$ is

$$\frac{2}{n\alpha^2} e^{-\frac{r^2}{n\alpha^2}} r\,dr.$$

The result is therefore in all respects the same as if, for example, the amplitude of the components had been $\frac{1}{2}\alpha$ and their number equal to $4n$. From this we see that the law is not altered, even if the components have different amplitudes, provided always that the whole number of each kind is very great; so that if there be n components of amplitude α, n' of amplitude β, and so on, the probability of a resultant between r and $r+dr$ is

$$\frac{2}{n\alpha^2+n'\beta^2+\ldots} e^{-\frac{r^2}{n\alpha^2+n'\beta^2+\ldots}} r\,dr.$$

The conclusion that the resultant of a large number of independent sounds is practically, and to a considerable extent, uncertain may appear paradoxical; but its truth, I imagine, cannot be disputed. Perhaps even the appearance of paradox will be removed if we remember that with two sounds of equal intensity the degree of uncertainty is far greater, as is evidenced in the familiar experiment with tuning-forks in approximate unison. That the beats should not be altogether obliterated by a multiplication of sources can hardly be thought surprising.

June 1880.