The Calculus of Equivalent Statements and Integration Limits.

By HUGH McCOLL, B.A.

[*Read Nov.* 8, 1877.]

The above title seems to be the most suitable for an analytical method which I discovered a few months ago, and to which a short introduction was published in the "Educational Times" for last July, under the name of "Symbolical Language." The chief use of the method, as far as I have yet carried it, is to determine the new limits of integration when we change the order of integration or the variables in a multiple integral, and also to determine the limits of integration in questions relating to probability. This object it will accomplish with perfect certainty, and by a process almost as simple and mechanical as the ordinary operations of elementary algebra. The fundamental principles of the method are as follows:

DEFINITION 1.—Let any symbols, say A, B, C, &c., denote *statements* (or propositions) registered for convenience of reference in a table. Then the equation $A=1$ asserts that the statement A is *true;* the equation $A=0$ asserts that the statement A is *false;* and the equation $A=B$ asserts that A and B are equivalent statements.

DEF. 2.—The symbol $A\times B\times C$ or ABC denotes a *compound* statement, of which the statements A, B, C may be called the *factors.* The equation $ABC=1$ asserts that *all the three statements are true;* the equation $ABC=0$ asserts that all the three statements are *not* true, *i. e.*, that at least *one* of the three is false. Similarly a compound statement of any number of factors may be defined.

DEF. 3.—The symbol $A+B+C$ denotes an *indeterminate* statement, of which the statements A, B, C may be called the *terms.* The equation $A+B+C=0$ asserts that all the three statements are *false;* the equation $A+B+C=1$ asserts that all the three are *not* false, *i. e.*, that at least *one* of the three is true. Similarly an indeterminate statement of any number of terms may be defined.

DEF. 4.—The symbol A' is the *denial* of the statement A. The two statements A and A' are so related that they satisfy the two equations $A+A'=1$ and $AA'=0$; that is to say, one of the two statements (either A or A') must be true and the other false. The same symbol (*i. e.*, a *dash*) will convert any complex statement into its denial. For example, $(AB)'$ is the denial of the compound statement AB.

Note.—The statements A and A' are what logicians call "contradictories"; and the two equations $A+A'=1$ and $AA'=0$ combined express the principle known in logic as the "Law of Excluded Middle."

Def. 5.—When only *one* of the terms of an indeterminate statement $A+B+C+\ldots$ can be true, or when no two terms can be true at the same time, the terms are said to be *mutually inconsistent* or *mutually exclusive*.

Rule 1.—The rule of ordinary algebraical multiplication applies to the multiplication of indeterminate statements, thus:

$$A(B+C) = AB+AC; \quad (A+B)(C+D) = AC+AD+BC+BD;$$

and so on for any number of factors, and whatever be the number of terms in the respective factors.

Note.—It is evident that if the terms of every indeterminate factor be mutually inconsistent, the terms of the product will also be mutually inconsistent.

Rule 2.—Let A be any statement whatever, and let B be any statement which is implied in A (and which must therefore be true when A is true, and false when A is false); or else let B be any statement which is admitted to be true independently of A; then (in either case) we have the equation $A=AB$. As particular cases of this we have $A = AA = AAA =$ &c., as repetition neither strengthens nor weakens the logical value of a statement. Also,

$$A = A(B+B') = A(B+B')(C+C') = \&c.,$$

for $B+B' = 1 = C+C' =$ &c. (See Def. 4.)

Rule 3.
$$\begin{aligned}(AB)' &= AB'+A'B+A'B' \\ &= AB'+A'(B+B') = AB'+A' \\ &= A'B+B'(A+A') = A'B+B',\end{aligned}$$

for $A+A'=1$ and $B+B'=1$. Similarly we may obtain various equivalents (with mutually inconsistent terms) for $(ABC)'$, $(ABCD)'$, &c.

Rule 4. $(A+B)' = A'B'$; $(A+B+C)' = A'B'C'$;

and so on.

Rule 5.
$$\begin{aligned}A+B &= \{(A+B)'\}' = (A'B')' \\ &= AB'+A'B+AB \\ &= AB'+(A'+A)B = AB'+B \\ &= A'B+A(B'+B) = A'B+A.\end{aligned}$$

Similarly we get equivalents (with mutually inconsistent terms) for $A+B+C$, $A+B+C+D$, &c.

The foregoing principles constitute the elementary *basis* of the method. We now come to the more important part of the subject, namely, the application of the method to multiple integrals and probability.

DEF. 6.—The symbol p prefixed to any algebraical (or arithmetical) expression converts the expression into a statement, namely, that the expression is *real and positive;* the symbol p' in like manner asserts that the expression affected by it is *real and negative.* For example, if we know that x and y are both real, we have the equations:

$$p(xy) = px\,py + p'x\,p'y,$$
$$p'(xy) = p'x\,py + px\,p'y.$$

Again, suppose we know that x, y, a are all three real and positive, it is easy to see the identities,

$$p(x^2+y^2-a^2) = p\{y-\sqrt{(a^2-x^2)}\}\,p'(x-a) + p(x-a),$$
$$p'(x^2+y^2-a^2) = p'\{y-\sqrt{(a^2-x^2)}\}\,p'(x-a).$$

Note.—We might also use a symbol q to denote the statement that the expression affected by it was imaginary, but I do not think that the need for the symbol would often arise.

DEF. 7.—The symbols px, $p'x$, py, $p'y$, &c. occur so frequently that it is convenient to replace them respectively by x_0, $x_{0'}$, y_0, $y_{0'}$, &c. Another reason for the employment of these last symbols will appear later.

DEF. 8.—The symbols x_1, x_2, x_3, &c. denote the 1st, 2nd, 3rd, &c. limits of x registered in any convenient order in a table of reference. The limits of the other variables of the expression (or expressions) under consideration are denoted similarly.

Note.—Among the limits of the variables thus registered, the limit *zero* is not included (see Defs. 7 and 9); but we may denote it either by the usual symbol 0, or (for the sake of uniformity in the notation) by any of the symbols x_0, y_0, z_0, &c.

DEF. 9.—The symbols x_1, x_2, x_3, &c. also denote *statements*, namely, the statements that the *limits* x_1, x_2, x_3, &c. are *inferior* limits of x. Similarly y_1, y_2, &c., z_1, z_2, &c. are to be interpreted.

Note.—The symbol x_m has thus two meanings: it denotes the m^{th} *limit* of x, and it also denotes the *statement* that this limit is an *inferior limit;* in other words, x_m is an abbreviation for $p(x-x_m)$. The context will always prevent any confusion of ideas resulting from this double signification of the same symbol.

DEF. 10.—The symbols $x_{1'}$, $x_{2'}$, $x_{3'}$, &c. denote the *statements* that the limits x_1, x_2, x_3, &c. are *superior* limits of x. Similarly $y_{1'}$, $y_{2'}$, &c., $z_{1'}$, $z_{2'}$, &c. are to be interpreted.

Note.—The symbol $x_{m'}$ is thus an abbreviation for the statement $p'(x-x_m)$.

Def. 11.—The symbol $x_{m'n'rs}$ is an abbreviation for the symbol $x_{m'}\,x_{n'}\,x_r\,x_s$, and denotes the compound statement

$$p'(x-x_m)\,p'(x-x_n)\,p(x-x_r)\,p(x-x_s).$$

Similarly a compound statement of any number of factors, and having reference to the limits of x or of any other variable, may be abbreviated.

Rule 6.—The compound statement

$$x_{m'n'r's'\dots} = x_{m'}\,\alpha + x_{n'}\,\beta + x_{r'}\,\gamma + x_{s'}\,\delta + \dots,$$

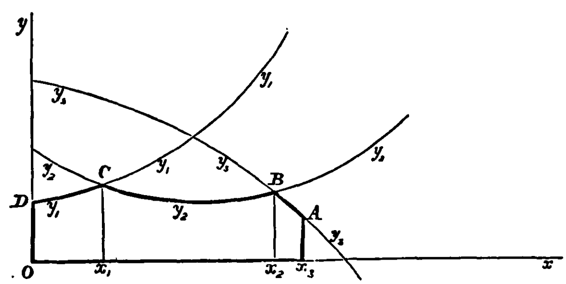
in which $x_{m'}$, $x_{n'}$, &c. are abbreviations for the statements that the m^{th}, n^{th}, &c. limits of x are all *superior* limits; while α, β, γ, &c. respectively denote the statements that amongst these x_m is the nearest superior limit of x, that x_n is the nearest superior limit, that x_r is the nearest superior limit, and so on. In other words, α is an abbreviation for the compound statement

$$p'(x_m-x_n)\,p'(x_m-x_r)\,p'(x_m-x_s)\,\dots.$$

The value of β is obtained from this expression by simply interchanging m and n; the value of γ is obtained from the expression for β by interchanging n and r; and so on.

Rule 7.—This is obtained from the preceding Rule by simply copying all the words in it (except *superior*, for which we must write *inferior*), and *omitting all the accents*, both on the numbers m, n, r, s, ... and on the symbol p.

Note.—Rule 6 may be illustrated by a plane figure as follows:—



Let $y=\phi(x)$, $y=\psi(x)$, $y=\chi(x)$ be the three equations for the curves marked respectively $y_1y_1y_1$, $y_2y_2y_2$, $y_3y_3y_3$. Then all the points contained within the thick boundary DOx_3ABCD will be expressed by the statement $y_{1'.2'.3'.0}\,x_{3'.0}$;

and this statement is evidently equivalent to the statement

$$y_{1'.0}\,x_{1'.0}+y_{2'.0}\,x_{2'.1}+y_{3'.0}\,x_{3'.2},$$

x and y being the Cartesian coordinates of any point whatever within the boundary.

A similar geometrical illustration may be given of Rule 7.

RULE 8.—The statement $x_{m'n} = x_{m'n}\,a$, in which $a = p\,(x_m - x_n)$.

Note.—This is a particular case of Rule 2.

Rules 6 and 7 may also be brought under Rule 2 as follows:—Attaching the same meaning to a, β, γ, &c. as in Rule 6, it is evident that $a+\beta+\gamma+\delta+\ldots = 1$. Hence, by Rule 2, we get

$$x_{m'n'r's'\ldots} = x_{m'n'r's'\ldots}\,(a+\beta+\gamma+\delta+\ldots)\,;$$

and since $x_{m'}\,a = x_{m'n'r's'\ldots}\,a$, and so on for the β, γ, δ, &c. terms, we get Rule 6 by the suppression of implied factors. We may similarly show Rule 7 to be a particular case of Rule 2.

The last three Rules, 6, 7, 8 (combined with Rule 1), constitute the pivot on which the whole process turns, whether in its application to the transformation of multiple integrals or to probability. By repeated application of these three Rules to the several variables in succession, any compound statement of the form

$$x_{1'.2'.0'\ldots 3.4.6\ldots}\,y_{5'.7'.2'\ldots 0.3.4\ldots}\,z_{3'.4'.0'\ldots 1.2.6\ldots}\,\cdots,$$

with any number of variables, and any number of factors for each variable, will finally be reduced either to a single *elementary** term of the form $x_{m'n}\,y_{r's}\,z_{t'u}\,\ldots$, in which x_m and x_n are the *nearest* limits (superior and inferior) of x, and so on for the variables y, z, &c.; or (as will generally happen in complicated cases) it will be reduced to an indeterminate statement consisting of several such terms. The work may generally be much abbreviated by dropping *zero terms* (*i. e.*, terms with inconsistent factors) as we go along, when mere inspection of the table of limits (without having recourse to Rule 8) will suffice to detect them. But if we overlook these zero terms, they will eventually disappear of themselves in the subsequent evolutions of the process. We may also shorten the process by cancelling factors which mere inspection of the table (instead of having recourse to Rule 6 or 7) will show to be implied in their co-factors *of the same variable.*

To give a practical illustration of the method, we will take a problem of some complexity. Suppose we are required to *reverse the order of integration* in the multiple integral

$$\int_{-a}^{2a} du \int_{-u}^{2u} dx \int_{-x}^{2x} dy \int_{-2x}^{\frac{y^2}{2x}} dz\ \phi\,(u, x, y, z).$$

* A term of this form may be called *elementary* when the application of Rule 8 will introduce no fresh factor except unity, or such factors as Rules 6 or 7 would afterwards reject as unnecessary.

For greater facility of reference throughout the process, the annexed table of limits may be conveniently made out on a card or moveable slip of paper. The values severally entered in the table during the course of the operations are found to arise spontaneously in reducing the various factors to convenient symbols. Each limit is registered in the table as soon as it is ascertained, so that the table *grows* as the process proceeds. Sometimes a limit which has already been registered as x_m may again inadvertently be registered as x_n; when this happens the oversight will be detected later by the appearance of an anomalous statement, such as

$$p(x_m - x_n) = p(0).$$

Table of Limits.

$u_1 = -a$	$x_1 = -y$	$y_1 = \frac{1}{2}z$	$z_1 = -8a$
$u_2 = 2a$	$x_2 = \frac{1}{2}y$	$y_2 = -z$	$z_2 = 2a$
$u_3 = -x$	$x_3 = -\frac{1}{2}z$	$y_3 = \sqrt{8az}$	$z_3 = 8a$
$u_4 = \frac{1}{2}x$	$x_4 = \frac{y^2}{2z}$	$y_4 = -\sqrt{8az}$	
	$x_5 = a$	$y = -2z$	
	$x_6 = -2a$	$y_6 = -4a$	
	$x_7 = 4a$	$y_7 = z$	
		$y_8 = 8a$	

From the integral we get a compound statement of 8 factors (2 for each variable); so that, if we denote the compound statement by A, and the 8 factors by A_1, A_2, A_3, &c., we have

$$\begin{aligned}
A_1 &= p(u+a) = u_1, \\
A_2 &= p'(u-2a) = u_{2'}, \\
A_3 &= p(x+u) = u_3, \\
A_4 &= p'(x-2u) = p(u-\tfrac{1}{2}x) = u_4, \\
A_5 &= p(y+x) = x_1, \\
A_6 &= p'(y-2x) = p(x-\tfrac{1}{2}y) = x_2, \\
A_7 &= p(z+2x) = p(x+\tfrac{1}{2}z) = x_3, \\
A_8 &= p'\left(z-\frac{y^2}{2x}\right) = p'\left\{\frac{(x-x_4)z}{x}\right\}.
\end{aligned}$$

But evidently $p'\left(\dfrac{x-x_4}{x}\right) = x_{4'.0} + x_{0'.4}$; $pz = z_0$;

$$p\left(\frac{x-x_4}{x}\right) = x_{4.0} + x_{4'.0'}; \quad p'z = z_{0'};$$

and therefore

$$\begin{aligned}
p'\left\{\frac{x-x_4}{x}z\right\} &= (x_{4'.0}+x_{0'.4})z_0 + (x_{4.0}+x_{4'.0'})z_{0'} \\
&= x_{4'.0}z_0 + x_0 z_{0'} + x_{4'} z_{0'},
\end{aligned}$$

for by inspection of the table of limits we see that $x_{0'.4}z_0 = 0$, $x_{4.0}z_{0'} = x_0 z_{0'}$, $x_{4'.0'}z_{0'} = x_{4'}z_{0'}$. Hence we have

$$A = u_{2'.1.3.4}\, x_{1.3.8}\,(x_{4'.0}z_0 + x_0 z_{0'} + x_{4'} z_{0'}).$$

Now the simplest and generally the most advantageous order in the application of Rules 6, 7, 8 would be to apply these rules first to the u-factors; then (after multiplying) to the x-factors; then to the y-factors which would arise; and lastly to the z-factors. But a little saving of labour will be effected in this instance by slightly departing from this order, which is never an absolutely necessary order of application. We will first apply Rule 7 to the compound statements $u_{1.3.4}$ and $x_{1.2.3}$, and then Rule 8 to the u-factors, thus:

$$u_{1.3.4} = u_1 \alpha + u_3 \beta + u_4 \gamma,$$

in which
$$\begin{aligned} \alpha &= p(u_1 - u_3)\ p(u_1 - u_4) \\ &= p(-a+x)\ p(-a-\tfrac{1}{2}x) = x_{5.6'} = 0, \\ \beta &= p(u_3 - u_1)\ p(u_3 - u_4) \ = x_{5'.0'} = x_{0'}, \\ \gamma &= p(u_4 - u_1)\ p(u_4 - u_3) \ = x_{6.0} = x_0. \end{aligned}$$

Expanding the compound statement $x_{1.2.3}$ in the same way into an indeterminate statement, and substituting, we get

$$A = u_{2'}(u_3 x_{0'} + u_4 x_0)(x_1 y_{0'.1'} + x_2 y_{0.2} + x_3 y_{2'.1})(x_{4'.0} z_0 + x_0 z_{0'} + x_{4'} z_{0'}).$$

Multiplying the three indeterminate factors, omitting the zero terms in the result, and cancelling those factors in each term which mere inspection of the table will show to be implied in their co-factors of the same variable, we get (see Appendix, Note α)

$$A = u_{2'.4}(x_{4'.1} y_{0'} + x_{4'.2} y_0) z_0 + u_{2'.4}(x_1 y_{1'} + x_2 y_2 + x_3 y_{2'.1}) z_{0'}.$$

Applying now Rule 8 to the compound factor $u_{2'.4}$, we get

$$u_{2'.4} = u_{2'.4} x_{7'},$$

so that a fresh factor $x_{7'}$ is introduced among the factors in x (see Appendix, Note β).

We have now done with the limits of the variable u, so we apply our rules (when mere inspection of the table is not sufficient) to the x-statements, and we get (after cancelling implied factors in y, see Appendix, Note β)

$$\begin{aligned} A = \ & u_{2'.4}(x_{4'.1} y_{5'.4} + x_{7'.1} y_{4'.6} + x_{4'.2} y_{3'.7} + x_{7'.2} y_{8'.3}) z_0 \\ & + u_{2'.4}(x_{7'.1} y_{1'.6} + x_{7'.2} y_{8'.2} + x_{7'.3} y_{2'.1} z_1) z_{0'}. \end{aligned}$$

Having now done with the limits of u and x, we apply Rule 8 (Rules 6 and 7, not being required as implied factors in y, have already been cancelled) to the y statements, when we shall get finally

$$\begin{aligned} A = \ & z_{2'.0}(y_{5'.4} x_{4.1} + y_{4'.6} x_{7.1}) u_{2'.4} \\ & + z_{3'.0}(y_{3'.7} x_{4'.2} + y_{8'.3} x_{7'.2}) u_{2'.4} \\ & + z_{0'.1}(y_{1'.6} x_{7'.1} + y_{8'.2} x_{7'.2} + y_{2'.1} x_{7'.3}) u_{2'.4}, \end{aligned}$$

altogether 7 elementary terms. The first is $z_{2'.0}y_{5'.4}x_{4'.1}u_{2'.4}$, the corresponding term of the transformed integral being

$$\int_0^{z_2} dz \int_{y_4}^{y_5} dy \int_{x_1}^{x_4} dx \int_{u_4}^{u_2} du\, \phi(u, x, y, z);$$

and so on for the remaining 6 terms and the corresponding 6 terms of the transformed integral.

The mode of applying the process to find the limits of integration when we change the *variables* in a multiple integral is so obvious from the mode of applying it in finding the limits when we change the *order of integration*, that it is unnecessary to illustrate it by a separate example. We shall therefore end this article by applying the method to an easy question in probability.

In the quadratic equation $x\theta^2 - y\theta + z = 0$, if the coefficients x, y, z be each taken at random between a and 0, what is the chance that the roots of the equation will be real, all values of x, y, z between the given limits being equally probable?

Let A denote the statement whose truth is taken for granted, namely, the statement $x_{1'.0}y_{1'.0}z_{1'.0}$ (see the table); and let Q denote the statement which may be true or false, namely, $p\,(y^2 - 4xz)$. Then the required chance is

$$\frac{AQ \iiint dx\,dy\,dz}{A \iiint dx\,dy\,dz},$$

the statements A and AQ fixing the limits of integration for the denominator and numerator respectively.

The denominator of the above fraction is evidently

$$\int_0^{x_1} dx \int_0^{y_1} dy \int_0^{z_1} dz = a^3,$$

TABLE OF LIMITS.

$z_1 = a$	$y_1 = a$	$x_1 = a$
$z_2 = \dfrac{y^2}{4x}$	$y_2 = \sqrt{4ax}$	$x_2 = \dfrac{a}{4}$

the statement A being elementary, since the application of rule 8 will introduce no fresh factors. It remains to find the limits of integration for the numerator from the statement AQ.

Now $Q = p'\,(4xz - y^2)$, and since x is positive, this

$$= p'\left(z - \frac{y^2}{4x}\right) = z_{2'}.$$

Hence $$AQ = x_{1'.0}y_{1'.0}z_{2'.1'.0}.$$

But $$z_{2'.1'} = z_{2'}y_{2'} + z_{1'}y_2, \quad \text{by Rule 6.}$$

Hence $$AQ = x_{1'.0}\,y_{1'.0}\,(z_{2'}\,y_{2'} + z_{1'}\,y_2)\,z_0$$
$$= x_{1'.0}\,(y_{2'.1'.0}\,z_{2'.0} + y_{1'.0.2}\,z_{1'.0}).$$

But $y_{2'.1'} = y_{2'}x_{2'} + y_{1'}x_2$ by Rule 6, and $y_{0.2} = y_2$ by Rule 7 or mere inspection of the table.

Hence $AQ = x_{2'.1'.0}\,y_{2'.0}\,z_{2'.0} + x_{1'.0.2}\,y_{1'.0}\,z_{2'.0} + x_{1'.0}\,y_{1'.2}\,z_{1'.0}.$

But by mere inspection of the table we get $x_{2'.1'} = x_{2'}$ and $x_{0.2} = x_2$; and in the third term we get (by Rule 8) $y_{1'.2} = y_{1'.2}\,x_{2'}$.

Hence, finally,

$$AQ = x_{2'.0}\,y_{2'.0}\,z_{2'.0} + x_{1'.2}\,y_{1'.0}\,z_{2'.0} + x_{2'.0}\,y_{1'.2}\,z_{1'.0}.$$

These three terms are elementary, since rule 8 will introduce no fresh factors. Hence

$$AQ\iiint dx\,dy\,dz = \int_0^{x_2} dx \int_0^{y_2} dy \int_0^{z_2} dz + \int_{x_2}^{x_1} dx \int_0^{y_1} dy \int_0^{z_2} dz + \int_0^{x_2} dx \int_{y_2}^{y_1} dy \int_0^{z_1} dz.$$

The integrations are easy, and the result is $\left(\frac{5}{36} + \frac{1}{6}\log_e 2\right) a^3$.

The required chance is therefore $\frac{5}{36} + \frac{1}{6}\log_e 2$.

With reference to the preceding solution, the referees of the Mathematical Society have kindly made the following valuable suggestion, which may also be extended to other problems:

"The process will be considerably abbreviated and simplified if from the outset the statements x_0, y_0, z_0, $x_{1'}$, $y_{1'}$, $z_{1'}$ are severally regarded as *unit*-factors, and therefore omitted when not wanted. Thus the whole working would be as follows:

$$\begin{aligned} AQ &= z_{2'} = z_{2'.1'} = z_{2'}\,y_{2'} + z_{1'}\,y_2 \\ &= z_{2'}\,y_{2'.1'} + y_2 = z_{2'}\,(y_{2'}\,x_{2'} + y_{1'}\,x_2) + y_2. \end{aligned}$$

And, by restoring or supplying the proper unit-factors, the final result is at once obtained."

In accordance with this suggestion I would propose the following convention:—

When we are analyzing any factor A of any compound statement $ABC\ldots$, the truth of its co-factors B, C, ... may for the time be taken for granted, so that as *unit-factors* they may be introduced or suppressed at pleasure. The equation $A = a$, according to this convention, will assert, *not* that a is *always* an equivalent for A, but *that a may replace A in the particular compound statement of which A is a factor.* We may then have such equations as $A = AB = aB = a = aC =$ &c. According to this convention, it is evident that $A = 1$ asserts either that A is *implied* in some co-factor (which co-factor may be true or false), or else that A is *true absolutely* and independently of any co-factor. Also $A = 0$ asserts either that A is *inconsistent* with some co-factor (which co-factor may be true or false), or else that A is *false absolutely* and independently of any co-factor.

These abbreviations, however, necessitate more caution in the working of the process: for, on the one hand, care is needed lest any factor, left temporarily understood for the sake of brevity, should inadvertently be left out of account altogether; and, on the other hand, lest factors which have been already taken into account should again be needlessly introduced when their services are no longer required. For these reasons I think the abbreviations cannot be employed with safety till some familiarity has been first acquired with the longer but easier method adopted in the text.

Appendix.

Note *a*.—This is obtained as follows:—Multiplying the last two indeterminate factors, namely,

$$x_1 y_{0'.1'} + x_2 y_{0.2} + x_3 y_{2'.1},$$

and

$$x_{4'.0} z_0 + x_0 z_{0'} + x_{4'} z_{0'},$$

we get for our product

$$\begin{aligned} & x_{4'.1.0}\, y_{0'.1'}\, z_0 + x_{4'.2.0}\, y_{0.2}\, z_0 + \underline{x_{4'.3.0}\, y_{2'.1}\, z_0} \\ & + x_{1.0}\, y_{0'.1'}\, z_{0'} + x_{2.0}\, y_{0.2}\, z_{0'} + x_{3.0}\, y_{2'.1}\, z_{0'} \\ & + x_{4'.1}\, y_{0'.1'}\, z_{0'} + x_{4'.2}\, y_{0.2}\, z_{0'} + x_{4'.3}\, y_{2'.1}\, z_{0'}. \end{aligned}$$

But the third term (the one underlined) is zero, since it contains the inconsistent compound factor $y_{2'.1}\, z_0$; for (by Rule 8)

$$y_{2'.1} = y_{2'.1}\, p(y_2 - y_1) = y_{2'.1}\, p(-z - \tfrac{1}{2}z) = y_{2'.1}\, z_{0'},$$

and $z_{0'}$ is inconsistent with z_0.

Omitting the underlined term therefore, and multiplying the terms left by $u_3 x_{0'} + u_4 x_0$, we get (omitting the terms which contain the inconsistent factor $x_{0'.0}$)

$$\begin{aligned} & u_3 x_{0'} (\underline{x_{4'.1}\, y_{0'.1'}} + \underline{x_{4'.2}\, y_{0.2}} + \underline{x_{4'.3}\, y_{2'.1}})\, z_{0'} \\ & + u_4 x_0 (x_{4'.1}\, y_{0'.1'}\, z_0 + x_{4'.2}\, y_{0.2}\, z_0 + x_1\, y_{0'.1'}\, z_{0'} \\ & \qquad + x_2\, y_{0.2}\, z_{0'} + x_3\, y_{2'.1}\, z_{0'} + \underline{x_{4'.1}\, y_{0'.1'}\, z_{0'}} \\ & \qquad\qquad + \underline{x_{4'.2}\, y_{0.2}\, z_{0'}} + \underline{x_{4'.3}\, y_{2'.1}\, z_{0'}}). \end{aligned}$$

But each of the compound factors $x_{0'.1}\, y_{0'}$, $x_{0'.2}\, y_0$, $x_{0'.3}\, z_{0'}$, $x_{4'.0}\, z_{0'}$ is *zero*, as may be seen by application of Rule 8 or mere inspection of the Table of Limits. Hence the terms underlined vanish. Substituting the terms left, we get

$$\begin{aligned} A = u_{2'.4} (& x_{4'.1.\underset{\cdot}{0}}\, y_{0'.\underset{\cdot}{1'}}\, z_0 + x_{4'.2.\underset{\cdot}{0}}\, y_{\underset{\cdot}{2}.0}\, z_0 + x_{1.\underset{\cdot}{0}}\, y_{\underset{\cdot}{0'}.1'}\, z_{0'} \\ & + x_{2.\underset{\cdot}{0}}\, y_{2.\underset{\cdot}{0}}\, z_{0'} + x_{3.\underset{\cdot}{0}}\, y_{2'.1}\, z_{0'}). \end{aligned}$$

But the factors *dotted underneath* in the respective terms may be omitted, for they are implied in their co-factors *of the same variable*, as may be seen by mere inspection of the table, or by application of Rules 6 and 7. Taking, for example, the factors of the first term, we

have $$x_{1.0} = x_1\alpha + x_0\beta,$$

in which $$\alpha = p\,(x_1 - x_0) = p\,(-y) = y_{0'},$$

and $$\beta = p\,(x_0 - x_1) = y_0 = 0,$$

because of the co-factor $y_{0'}$;

and $$y_{0'.1'} = y_{0'}\alpha + y_{1'}\beta,$$

in which $$\alpha = p'(y_0 - y_1) = p'(-\tfrac{1}{2}z) = z_0,$$

and $$\beta = p'(y_1 - y_0) = p'(\tfrac{1}{2}z) = z_{0'} = 0.$$

The cancelling in the other terms may be similarly verified by Rules 6 and 7.

Note β.—Substituting this value of $u_{3'.4}$, we get

$$A = u_{3'.4}\,(x_{7'.4'.1}\,y_{0'} + x_{7'.4'.3}\,y_0)\,z_0$$
$$+ u_{3'.4}\,(x_{7'.1}\,y_{1'} + x_{7'.3}\,y_3 + x_{7'.3}\,y_{3'.1})\,z_{0'}.$$

In the *first term* of the first bracket,

$$x_{7'.4'} = x_{7'}\alpha + x_{4'}\beta,$$

in which $$\alpha = p'(x_7 - x_4) = p'\left(4a - \frac{y^2}{2z}\right)$$
$$= p\,(y^2 - 8az),$$

for $z_{0'}$ is inadmissible because of the co-factor z_0 outside the bracket,

thus, $$\alpha = p\,\{(y - \sqrt{8az})\,(y + \sqrt{8az})\}$$
$$= p'(y + \sqrt{8az}),$$

for positive values of y are inadmissible because of the co-factor $y_{0'}$;

also $$\beta = p'(x_4 - x_7) = p'(y^2 - 8az)$$
$$= p'\,\{(y - \sqrt{8az})\,(y + \sqrt{8az})\}$$
$$= p\,(y + \sqrt{8az}).$$

Thus, $$\alpha = y_{4'} \quad\text{and}\quad \beta = y_4.$$

Again, in the *second term* of the first bracket, we have

$$x_{7'.4'} = x_{7'}\alpha + x_{4'}\beta,$$

in which (as before)

$$\alpha = p\,(y^2 - 8az) = p\,\{(y - \sqrt{8az})\,(y + \sqrt{8az})\}$$
$$= p\,(y - \sqrt{8az}),$$

for *negative* values of y are inadmissible this time because of the co-factor y_0. Hence $\alpha = y_3$.

Similarly, we get $\beta = y_{3'}$.

Substituting these values of $x_{7'.4'}$ in the first and second terms respectively, we get

$$A = u_{2'.4}\{(x_{7'.1}\,y_{4'.0'} + x_{4'.1}\,y_{0'.4} + x_{7'.2}\,y_{3.0} + x_{4'.2}\,y_{3'.0})\,z_0 + (x_{7'.1}\,y_{1'} + x_{7'.2}\,y_2 + x_{7'.3}\,y_{2'.1})\,z_{0'}\}.$$

Applying Rule 8 to the x-statements in each term, we get

in the first term, $x_{7'.1} = x_{7'.1}\,y_6$;

in the second term, $x_{4'.1} = x_{4'.1}\,y_{5'}$,

because of the co-factors z_0 and $y_{0'}$;

in the third term, $x_{7'.2} = x_{7'.2}\,y_{8'}$;

in the fourth term, $x_{4'.2} = x_{4'.2}\,y_7$,

because of the co-factors z_0 and y_0.

Taking next the second bracket, we have

in the fifth term, $x_{7'.1} = x_{7'.1}\,y_6$;

in the sixth term, $x_{7'.2} = x_{7'.2}\,y_{8'}$;

in the seventh term, $x_{7'.3} = x_{7'.3}\,z_1$.

Substituting in every term, we get

$$A = u_{2'.4}\{(x_{7'.1}\,y_{4'.\underset{\cdot}{0'}.6} + x_{4'.1}\,y_{\underset{\cdot}{0'}.5'.4} + x_{7'.2}\,y_{8'.3.\underset{\cdot}{0}} + x_{4'.2}\,y_{3'.0.7})\,z_0 + (x_{7'.1}\,y_{1'.6} + x_{7'.2}\,y_{8'.2} + x_{7'.3}\,y_{2'.1}\,z_1)\,z_{0'}\}.$$

But by inspection of the table, or by application of Rules 6 and 7, we have $y_{4'.0'} = y_{4'}$, $y_{0'.5'} = y_{5'}$, $y_{3.0} = y_3$, and $y_{0.7} = y_7$, so that the factors dotted underneath may be cancelled. Hence we get

$$A = u_{2'.4}\{(x_{7'.1}\,y_{4'.6} + x_{4'.1}\,y_{5'.4} + x_{7'.2}\,y_{8'.3} + x_{4'.2}\,y_{3'.7})\,z_0 + (x_{7'.1}\,y_{1'.6} + x_{7'.2}\,y_{8'.2} + x_{7'.3}\,y_{2'.1}\,z_1)\,z_{0'}\}.$$

Applying Rule 8 to the y-statements in each term, we get

in the *first* term, $y_{4'.6} = y_{4'.6}\,z_{2'}$;

in the *second* term, $y_{5'.4} = y_{5'.4}\,z_{2'}$;

in the *third* term, $y_{8'.3} = y_{8'.3}\,z_{3'}$;

in the *fourth* term, $y_{3'.7} = y_{3'.7}\,z_{3'}$;

in the *fifth* term, $y_{1'.6} = y_{1'.6}\,z_1$;

in the *sixth* term, $y_{8'.2} = y_{8'.2}\,z_1$;

in the *seventh* term, $y_{2'.1} = y_{2'.1}\,z_{0'}$.

Substituting in every term, we get

$$A = \{z_{2'.0}\,(y_{4'.6}\,x_{7'.1} + y_{5'.4}\,x_{4'.1}) + z_{3'.0}\,(y_{8'.3}\,x_{7'.2} + y_{3'.7}\,x_{4'.2}) + z_{0'.1}\,(y_{1'.6}\,x_{7'.1} + y_{8'.2}\,x_{7'.2} + y_{2'.1}\,x_{7'.3})\}\,u_{2'.4},$$

which, except in the arrangement of the terms, agrees with the result in the text.