Proposed Extension of the Powers of Quaternion Differentiation. By **Alexander M'Aulay**, Ormond College, Melbourne. *Communicated* by Professor TAIT.

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It will, I think, be acknowledged that Quaternions, while providing for the physicist a machinery much more natural and graceful than the Cartesian, for all conceptions strictly geometrical, do not at present afford equal facilities for the consideration of questions involving differentiation. It is true that there is one well-known symbol of differentiation of great utility, which enables Quaternions to deal in a suitable manner with many such questions; but there are left whole classes of differentiations in which the symbol is of no avail.

This has led me to the consideration of other symbols of differentiation, and to a slight generalisation of the powers of the symbol already mentioned. I had thought it necessary only to define the extensions here referred to, and proceed to apply them. But Professor Tait, while kindly procuring me this opportunity of bringing forward my views, has given me fair warning that the repugnance of physicists to some of my notation may prove an insuperable obstacle to their paying any attention to investigations conducted in that notation. This personal reference will explain why, in the present short paper, an apology for, and explanation of, the methods are entered into, much more detailed than could otherwise be considered advisable or even justifiable.

In what follows, after an explanation of the proposed changes of, and additions to, quaternion differential notation, a brief account in the abstract is given of the reasons for and against what can be called innovation; and the rest of the paper is devoted to some examples of the application to the theories of elasticity and electrostatics. I may here remark that, in a short paper like the present, it is impossible to do full justice to the views enunciated, because for this purpose it would be necessary to go over a large part of the ground covered by mathematical physics; but if the slight variations from previous custom indicated below do not meet with

that studied discountenance from all authorities in the matter, which I have been assured they must, I could give in a future paper some few applications of the methods which would probably prove interesting to physicists, and which cannot be treated readily, if at all, by methods other than those in question.

As is well known, Hamilton's symbol ∇ may be defined by the equation

$$\nabla = i \frac{d}{dx} + j \frac{d}{dy} + k \frac{d}{dz}$$

where i, j, k, x, y, z have the usual meanings. Hamilton himself did not examine the utility of ∇ . This explains, most likely, why he did not state more exactly how to consider to what symbols the differentiations implied should refer. In the simplest applications of ∇ these differentiations will refer to that symbol, and only to it, which immediately follows the operator. Very little practice in the application of ∇ to physical questions serves to show that, if we are to be bound by this rule, at least seven-eighths of the potential utility of ∇ must be sacrificed. Professor Tait has recognised this in the 3rd edition of his *Quaternions*, where he freely separates the ∇ from the symbol or symbols it affects; and, according to a well-known custom, indicates the connection between the operator and the symbols affected by attaching the same suffix to both. This I had already done in a paper to be referred to immediately. We seem then to be led by a natural process to the following statement:—

The operator ∇ and its operand or operands may have any relative positions in a term which are convenient, the connection between them being indicated in the usual way by suffixes.

For the propriety of this view, I now wish to contend, although at the time of writing the 3rd ed. of *Quaternions* Professor Tait was not prepared to endorse it in full. Notwithstanding that it is immaterial how great, in a term, be the separation of a ∇ from the symbol affected, so long as the ∇ is on the *left* of the symbol, he* emphatically—observe his italics—lays down the law, that the ∇ must not be removed to the *right* of the symbol, even the immediate

* With regard to the use of ∇ Professor Tait (*Quaternions*, 3rd ed., § 149) says:—"The precautions necessary in such matters are twofold—(a) *The operator must never be placed after the operand*; (b) *its commutative or non-commutative character must be carefully kept in view.*"

right. Is it not akin to inconsistency to say in effect—"It is justifiable to violate, for convenience, the custom of writing an operator to the immediate left of the symbol affected, and to indicate the connection by some other method, but the justification only covers a *limited* violation of the custom. It matters not that the new method of indication will equally well serve *whatever* be the relative positions in a term of the operator and the symbol affected—we will strictly adhere to one part of the restriction hitherto imposed upon this relativity of position, though freeing ourselves from the other."

It may be—has been—urged that to place a ∇ to the right of the symbol affected is as criminally ridiculous as to write $X \frac{d}{dx}$ for $\frac{dX}{dx}$. It may be equally well urged that the removal of ∇ to the left from its primitive position is on a par with writing $\frac{d}{dx} XY$ for $X \frac{dY}{dx}$.

It must be conceded that both of these violations of custom are objectionable, and cannot be justified, unless some convenience accrues greater than the counterbalancing inconvenience of having to show by some method, other than that of juxtaposition, the connection between the operator and the symbol affected. Whether such convenience does exist must be left undecided till we come to the applications. What I want to point out here is, that it is extremely inconvenient, and there is absolutely no ground whatever for not utilising this new method of indication in all cases where it is applicable.

To take an instance. In considering Maxwell's electrostatic theory below, we are led to consider the very simple stress, whose corresponding linear vector function ϕ is given by the equation

$$\phi\omega = V\mathfrak{D}\omega\mathfrak{E},$$

where $\mathfrak{D}$ and $\mathfrak{E}$ are the electric displacement and electro-motive force respectively.* The force per unit volume due to this is

$$\frac{d\phi i}{dx} + \frac{d\phi j}{dy} + \frac{d\phi k}{dz} = \frac{dV\mathfrak{D}i\mathfrak{E}}{dx} + \frac{dV\mathfrak{D}j\mathfrak{E}}{dy} + \frac{dV\mathfrak{D}k\mathfrak{E}}{dz}.$$

* Maxwell's *Elect. and Mag.*, 2nd ed., § 68.

Now, what possible objection can there be to writing this last equation in the form

$$\phi_1 \nabla_1 = \nabla \mathfrak{D}_1 \nabla_1 \mathfrak{E}_1 \cdot \cdot \cdot \cdot \cdot \cdot (1).$$

On the left of this equation the ∇ , as indicated by the suffixes, refers to the symbol immediately preceding it. On the right it refers to both the symbol immediately preceding and also to that immediately following. Certainly the expression on the right can be written

$$\nabla \mathfrak{D} \nabla \mathfrak{E} + \nabla \mathfrak{E} \nabla \mathfrak{D};$$

but in this shape we at once lose the connection of form between the vector in question and the corresponding linear vector function $\nabla \mathfrak{D}(\) \mathfrak{E}$.

Professor Tait suggests in his treatise how such vectors as $\phi_1 \nabla_1$ above may be treated. He shows how we may prove properties of the vector, and even suggests a method of writing down the vector. Adopting that suggestion, we should have, instead of equation (1) above

$$-S \nabla \Delta . \phi \rho = -S \nabla \Delta . \nabla \mathfrak{D} \rho \mathfrak{E} . *$$

To explain my notation I will quote from a paper † on this subject, written in 1884:—

“If ϕ be any *linear quaternion function*, ϕ itself being a function of the position of a point

$$\phi \Delta = \frac{d\phi i}{dx} + \frac{d\phi j}{dy} + \frac{d\phi k}{dz} \cdot$$

It is necessary, as will be seen below, that this Δ should be distinguished from ∇ .

Whenever numerical suffixes occur in this paper it will be to

* Tait's *Quaternions*, 3rd ed., § 508. The following words of Professor Tait seem to me to form a powerful argument in favour of my *natural* notation:—
“The highest art is the absence of artifice. . . . The difficulties of Physics are sufficiently great in themselves to tax the highest resources of the human intellect; to mix them up with avoidable mathematical difficulties is unreasonable little short of crime. . . . In Quaternions, a subject uniquely adapted to Euclidian space, this entire freedom from artifice and its inevitable complications is the chief feature. . . . What is required for Physics is, that we should be enabled at every step to feel instinctively what we are doing. Till we have banished artifice we are not entitled to hope for full success in such an undertaking” (Tait, “On the Importance of Quaternions in Physics,” *Phil. Mag.*, 5th series, vol. xxix. pp. 84–97).

† *Mess. of Math.*, vol. xiv. p. 26. I have substituted in the present paper Δ for the ∇' of the original paper.

show to what quantities the operator ∇ refers, both the ∇ and the quantities having, for this purpose, the same suffix. Thus below,

$$* \nabla \nabla_1 \sigma_2 S \sigma_1 \nabla_2 = \nabla_i \frac{d\sigma}{dx} S \frac{d\sigma}{dx} i + \nabla_j \frac{d\sigma}{dz} S \frac{d\sigma}{dy} k + \dots$$

One advantage of this notation is, that the ∇_1 's, ∇_2 's, &c., as well as the σ_1 's, σ_2 's, &c., can be treated as mere vectors."

It is this last statement which points to the most powerful argument for not restricting the position of the ∇ 's in a term in any way. Let them have all the freedom of vectors, and they will obey all the laws thereof. As soon as we allow this freedom, but not till then, can we draw upon the large storehouse of already accumulated knowledge of vectors.†

It will be observed that in the above definitions there is no absolute need for introducing the symbol Δ . It is, however, as in many cases of duplicate notation, extremely convenient. Its differentiations refer to *all* the symbols in the term in which it occurs. Thus, for instance, the above equation (1) can be written

$$\phi \Delta = V \mathfrak{D} \Delta \mathfrak{E},$$

which still more clearly shows the connection between the stress $V \mathfrak{D}(\) \mathfrak{E}$ and the resulting force per unit volume $V \mathfrak{D} \Delta \mathfrak{E}$.

With these definitions, Professor Tait's well-known theorems in integration can be thrown into a rather more general form.‡ If ϕ be any linear quaternion function of a quaternion, then with Professor Tait's (*Quaternions*, 3rd ed., § 482) notation for integration—

* According to Professor Tait this *must* be written in some such form as the following:— $V \nabla_1 (S \nabla_2 \sigma_1) \sigma_2$ or $-S \nabla_3 \nabla_1 \cdot V \nabla_2 \sigma_1 S \rho_3 \sigma_2$ or $-S \nabla_3 \nabla_1 S \nabla_2 \sigma_1 \cdot V \rho_3 \sigma_2$. Even if the first of these is chosen, it may be pertinently asked why symbols which will, in the nature of things, obey all the laws of vectors, should be restricted in a purely arbitrary manner in obeying those laws. Why should we not be allowed to ring all the possible changes on the form of $V \nabla_1 \sigma_2 S \sigma_1 \nabla_2$ as on that of $V a \beta S \gamma \delta$, and reap the corresponding advantages?

† In the *Trans. Roy. Soc. Edin.*, xxvii. p. 251 (1874), Dr Plarr suggests the notation $\triangleleft r = i \frac{dr}{dx} + \dots$, $\triangleright r = \frac{dr}{dx} i + \dots$ where r is any quaternion. He suggests no notation for $\frac{d\bar{\phi}i}{dx} + \dots$ where ϕ is a general linear quaternion function. His symbols $\triangleleft$ and $\triangleright$ do not obey the laws of vectors, the first only because it is not allowed the freedom of a vector.

‡ The proof is given in the paper already quoted from.

$$\int \phi(d\rho) = \iint \phi(\nabla U v \Delta) ds \quad . \quad . \quad . \quad . \quad . \quad (2),$$

$$\iint \phi U v ds = \iiint \phi \Delta ds \quad . \quad . \quad . \quad . \quad . \quad (3).$$

The last equation shows at once, as pointed out in the paper already referred to, that the stress ϕ (when ϕ has the particular form of a vector function of a vector) causes a force per unit volume $\phi \Delta$.

Before considering the properties and applications of ∇ in this extended form, I will now introduce all the innovations that I propose.

A large class of differentiations can be included under a symbol somewhat analogous to ∇ . Suppose we have a linear vector function, ϕ , of a vector depending on the nine scalars, $a, b, c, a', b', c', a'', b'', c''$, by means of the equations—

$$\phi i = ai + bj + ck$$

$$\phi j = a'i + b'j + c'k$$

$$\phi k = a''i + b''j + c''k$$

Then $a, b, c, a', \&c.$, can be called the coordinates of ϕ . Again, P, Q, R, L, M, N may be called the coordinates of the self-conjugate linear vector function, ϖ , of a vector, if

$$\varpi i = Pi + Nj + Mk$$

$$\varpi j = Ni + Qj + Lk$$

$$\varpi k = Mi + Lj + Rk.$$

Now, just as, if u, v, w are the coordinates of an independent vector σ , $\sigma \nabla$ may be defined as a symbolic vector, whose coordinates are $\frac{d}{du}, \frac{d}{dv}, \frac{d}{dw}$; so if $a, b, c, a', b', c', a'', b'', c''$ be the coordinates of an independent linear vector function, ϕ , of a vector, $\phi \nabla$ may be* defined as a symbolic linear vector function, whose coordinates are—

$$\frac{d}{da}, \quad \frac{d}{db}, \quad \frac{d}{dc}, \quad \frac{d}{da'}, \quad \frac{d}{db'}, \quad \frac{d}{dc'}, \quad \frac{d}{da''}, \quad \frac{d}{db''}, \quad \frac{d}{dc''}.$$

If ϖ be an independent *self-conjugate* linear vector function of a

* I use the inverted D to suggest the analogy to Hamilton's inverted Δ . It is advisable to write $\phi \nabla$, $\&c.$, instead of $\nabla \phi$, $\nabla \sigma$, $\&c.$, as the numerical suffixes must be much more frequently introduced than the symbols ϕ or σ , which may be very frequently understood, and it is not advisable that both the literal and numerical suffixes should be on the same side of the ∇ or ∇ .

vector, whose coordinates are P, Q, R, L, M, N, α is defined as meaning the symbolic self-conjugate linear vector function of a vector, whose coordinates are—

$$\frac{d}{dP}, \quad \frac{d}{dQ}, \quad \frac{d}{dR}, \quad \frac{1}{2} \frac{d}{dL}, \quad \frac{1}{2} \frac{d}{dM}, \quad \frac{1}{2} \frac{d}{dN}.$$

This twofold meaning for α leads to no confusion, and is convenient, as all the formulæ which are true of one α are true of the other also. To α may be attached a system of suffixes for the same objects and with the same meanings, as in the case of ∇ .

It is convenient, though not absolutely necessary, to introduce one other symbol. Suppose $Q(\alpha, \beta)$ is a function of two vectors, linear in each. Then we have

$$Q(\nabla_1, \rho_1) = Q(i, i) + Q(j, j) + Q(k, k),$$

so that ∇_1 and ρ_1 may be interchanged. This particular use of ∇ is so frequent that it is advisable to use a symbol which will call attention to the symmetry. I therefore define ζ by the equation

$$Q(\nabla_1, \rho_1) = Q(\zeta_1, \zeta_1).$$

Similarly, if $Q(\alpha, \beta, \gamma, \delta)$ be a function of four vectors, $\alpha, \beta, \gamma, \delta$, linear in each, we may put

$$Q(\nabla_1, \rho_1, \nabla_2, \rho_2) = Q(\zeta_1, \zeta_1, \zeta_2, \zeta_2),$$

and so to any number of pairs of ζ 's. Of course if there be only one pair the suffix may be dropped entirely.

I now state a number of theorems in pure mathematics which will be useful, for brevity leaving them unproved.

If ϕ be any linear vector function of a vector, and ϕ' be its conjugate,

$$\phi' \omega = -\zeta S \omega \phi \zeta \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4),$$

$$(\phi - \phi') \omega = V V \zeta \phi \zeta \omega \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5).$$

If $Q(\alpha, \beta)$ be any quaternion function of two vectors, α and β , which is linear in each,

$$Q(\zeta, \phi \zeta) = Q(\phi' \zeta, \zeta) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6).$$

To apply this in practice, it is convenient to remember it in words:—
In any term in which ζ and $\phi \zeta$ occur, we may, without altering the value of that term, substitute for them $\phi' \zeta$ and ζ respectively. (It is

not necessary to say that the term is linear in each of the symbols ζ and $\phi\zeta$, for from the definition of ζ it must be so.) If

$$\left. \begin{aligned} \phi\omega &= -\Sigma\beta S\omega a \\ Q(\zeta, \phi\zeta) &= \Sigma Q(a, \beta) \end{aligned} \right\} \dots \dots \dots (7).$$

If Q be symmetrical in its constituents, and if

$$\left. \begin{aligned} \omega\omega &= -\frac{1}{2}\Sigma(\beta S\omega a + a S\omega\beta) \\ Q(\zeta, \omega\zeta) &= \Sigma Q(a, \beta) \end{aligned} \right\} \dots \dots \dots (8).$$

If m have the usual* meaning with reference to the linear vector function ϕ

$$6m = S\zeta_1\zeta_2\zeta_3 S\phi\zeta_1\phi\zeta_2\phi\zeta_3 \dots \dots \dots (9).$$

$$\phi^{-1}\omega = -\frac{3V\zeta_1\zeta_2 S\omega\phi\zeta_1\phi\zeta_2}{S\zeta_1\zeta_2\zeta_3 S\phi\zeta_1\phi\zeta_2\phi\zeta_3} \dots \dots \dots (10).$$

From these we deduce that if

$$\left. \begin{aligned} \phi\omega &= -S\omega \nabla \cdot \sigma \\ \phi^{-1}\omega &= -\frac{3V \nabla_1 \nabla_2 S\omega \sigma_1 \sigma_2}{S \nabla_1 \nabla_2 \nabla_3 S\sigma_1 \sigma_2 \sigma_3} \\ \phi'^{-1}\omega &= -\frac{3V \sigma_1 \sigma_2 S\omega \nabla_1 \nabla_2}{S \nabla_1 \nabla_2 \nabla_3 S\sigma_1 \sigma_2 \sigma_3} \end{aligned} \right\} \dots \dots \dots (11).$$

If ϕ and ψ , two linear vector functions of a vector, be connected by the equation

$$S\phi\zeta\chi\zeta = S\psi\zeta\chi\zeta,$$

where χ is a perfectly arbitrary linear vector function of a vector,† it follows that

$$\phi \equiv \psi,$$

or, again, if χ be self-conjugate but otherwise perfectly arbitrary and the above equation hold, it follows that

$$\bar{\phi} \equiv \bar{\psi}$$

where

$$2\bar{\phi} = \phi + \phi'$$

and similarly for $\bar{\psi}$, i.e., $\bar{\phi}, \bar{\psi}$ are the pure parts of ϕ and ψ .

* Tait's *Quaternions*, 3rd ed., §§ 158 *et seq.*

† This frequently-recurring and cumbersome phrase is very annoying. Might I suggest the term *Hamiltonian*. Thus, in the present case, we should say—"If ϕ and ψ be two Hamiltonians connected by the equation $S\phi\zeta\chi\zeta = S\psi\zeta\chi\zeta$ where χ is a perfectly arbitrary Hamiltonian," &c.

If Q be any function of an independent variable vector σ , we already know (Tait's *Quaternions*, 3rd ed., § 480) that

$$dQ = -Sd\sigma \nabla \cdot Q \quad . \quad . \quad . \quad (12).$$

A similar theorem in Ω holds. If Q be any function of an independent variable linear vector function of a vector ϕ (whether general or self-conjugate)

$$dQ = -Q_1 Sd\phi \zeta_1 \Omega_1 \zeta \quad . \quad . \quad . \quad (13).$$

From this it is not hard to show that Ω is an invariant. That is, it is a symbol, independent in meaning of the three mutually perpendicular unit vectors, i, j, k , used in defining it.

I now proceed to a few examples of the use, in the subject of *Elasticity*, of the symbols introduced.

Assuming uniformity of temperature throughout, let us investigate the general equations connected with the state of an elastic body. The two usual assumptions will not be made (1) that the strain is small, and (2) that there is no molecular couple. Let $d\varsigma$ be the volume of an element in some standard state—say the initial state. We assume that the pot. en. of any finite volume of the body is of the form $\iiint w d\varsigma$, where w depends only on the state as to strain of the body at the point. (Notice that it is not here assumed, as I believe is invariably done, that w depends only on the *coordinates* of pure strain at the point. It may, *e.g.*, so far as we know at present, involve also the space-derivatives of those coordinates. We do, however, assume that it is not a function of the state of the body at a point that is at a finite distance from the point under consideration.)

Our first object will be to get, not the equations of motion, but the equations connecting stress and strain. As will be seen, these two problems are not necessarily identical. The plan adopted is to assume the body to be strained in the most general manner. Then impose the most general small additional strain, and use the principle that for any finite portion of the body

$$\begin{aligned} \iiint \delta w d\varsigma &= (\text{work done by stresses on boundary of portion}) \\ &- (\text{work done by stresses throughout volume of portion}) \end{aligned} \quad (14)$$

Let ρ be the coordinate vector of the element $d\varsigma$ in its standard

the variation of the pot. en., ϕ does not enter in all its generality. A linear vector function, such as ϕ , can only be split up in one way into what may be called a pure part, $\bar{\phi}$, and a rotational part, $V\epsilon$ (). For the first part 6, and for the last 3, independent scalars must be assigned, and it is only the first 6 that have anything to do with the pot. en. Expressed in physical language:—*A stress can only be split up in one way, into two stresses, of which the first is an ordinary stress, producing no couple per unit volume, and the second is a couple-stress. The latter is quite independent of the pot. en.* Here, by a “couple-stress,” is meant a stress which produces on any interface, whatever be the direction of its normal, a tangential force. Shortly stated, the above may be put:—*The part of the stress to which the couple per unit of volume is due, is independent of the pot. en.* This is strictly analogous to the well-known corresponding strain theorem that the rotation of an element is independent of the pot. en.*

We may anticipate here by saying that this part of the stress is therefore also independent of the strain, though of course other means enable us to determine it. In fact, in order that an element should not have infinite angular acceleration, it is easy to see that on the whole it must be subject to zero couple per unit volume. In other words, if $\mathfrak{M}$ be the given external couple per unit volume of the unstrained body

$$\mathfrak{M} + 2m\epsilon = 0 \quad . \quad . \quad . \quad . \quad . \quad (17)$$

is the equation which gives the part of ϕ we have called ϵ . Here m stands for ds'/ds , so that by equations (9) and (7) above

$$\left. \begin{aligned} 6m &= S\zeta_1\zeta_2\zeta_3 S\chi\zeta_1\chi\zeta_2\chi\zeta_3 = S\zeta_1\zeta_2\zeta_3 S\chi'\zeta_1\chi'\zeta_2\chi'\zeta_3 \\ &= S\zeta_1\zeta_2\zeta_3 S\psi\zeta_1\psi\zeta_2\psi\zeta_3 \quad . \quad . \quad . \quad . \quad . \\ &= S\nabla_1\nabla_2\nabla_3 S\rho'_1\rho'_2\rho'_3 \quad . \quad . \quad . \quad . \quad . \end{aligned} \right\} \quad . \quad . \quad (18).$$

If $\mathfrak{F}$ be the given external force per unit volume of the unstrained body, we must have for equilibrium

$$\mathfrak{F} + m\phi_1\nabla'_1 = 0 \quad . \quad . \quad . \quad . \quad . \quad (19),$$

or

$$\mathfrak{F} + m(\bar{\phi}_1\nabla'_1 - V\nabla'\epsilon) = 0 \quad . \quad . \quad . \quad . \quad . \quad (20).$$

* That there is at least very grave doubt whether any such thing as a molecular couple, and therefore that a stress-couple exists (even in the case of magnetism), I hope to show on some future occasion. It is well to learn, however, the nature of the phenomenon, if it *possibly* exist.

Now

$$\delta\chi\omega = -\delta\rho'_1 S\omega \nabla_1 \dots \dots \dots (29).$$

Hence we see from equation (7) that in any expression, linear both in ∇_1 and $\delta\rho'_1$, such as the last expression for δw , we may substitute instead of these two ζ , $\delta\chi\zeta$ respectively. Hence

$$\delta w = -mS\delta\chi\bar{\zeta}\phi\chi'^{-1}\zeta \dots \dots \dots (30).$$

This expresses δw in terms of the variation of strain. It is convenient to modify the last equation by means of equation (6). Applying the rule given in connection with that equation, and changing $\zeta, \chi'^{-1}\zeta$ into $\chi^{-1}\zeta, \zeta$ respectively, we get

$$\delta w = -mS\delta\chi\chi^{-1}\bar{\zeta}\phi\zeta \dots \dots \dots (31).$$

Suppose, now * that the strain χ is made up of a pure strain ψ , followed by a rotation of $q(q)^{-1}$, so that

$$\chi\omega = q\psi\omega q^{-1} \dots \dots \dots (32).$$

Then it is easy to prove that

$$\delta\chi\omega = 2VV\delta qq^{-1}\chi\omega + q\delta\psi\omega q^{-1}.$$

Substituting in the last expression for δw , we get

$$\delta w/m = -2S.V\delta qq^{-1}.\bar{\zeta}\phi\zeta - S q\delta\psi\chi^{-1}\bar{\zeta}q^{-1}\phi\zeta.$$

The first term on the right is zero, $\because V\bar{\zeta}\phi\zeta = 0$. That δq should disappear from the expression for δw is, of course, what we should expect. It is, however, well to show that this follows as a mathematical consequence of our fundamental assumptions. We now have

$$\begin{aligned} \delta w &= -mS\delta\psi\chi^{-1}\bar{\zeta}q^{-1}\phi\zeta q \\ &= -mS\delta\psi\psi^{-1}(q^{-1}\bar{\zeta}q)q^{-1}\phi\zeta q \\ &\quad [\text{by substituting for } \chi^{-1} \text{ in terms of } \psi \text{ and } q] \\ &= -mS\delta\psi\psi^{-1}\bar{\zeta}q^{-1}\phi(q\zeta q^{-1})q \\ &\quad [\text{by substituting } \zeta, q\zeta q^{-1} \text{ for } q^{-1}\bar{\zeta}q, \zeta \text{ respectively}] \\ &= -mS\delta\psi\psi^{-1}\bar{\zeta}\phi\zeta \end{aligned}$$

* See Tait's *Quaternions*, 3rd ed., § 381, where it is shown how to determine both ψ and q in terms of χ .

where ϖ is a self-conjugate linear vector function of a vector defined by the equation

$$\varpi\omega = q^{-1}\bar{\phi}(q\omega q^{-1})q \quad \dots \quad (33).$$

Substituting $\zeta, \psi^{-1}\zeta$ for $\psi^{-1}\zeta, \zeta$ respectively in the last expression for δw , we finally get

$$\delta w = -mS\delta\psi\zeta\varpi\psi^{-1}\zeta \quad \dots \quad (34).$$

Thus the only infinitesimal appearing in δw is $\delta\psi$. It follows that w can only be a function of the pure strain ψ . This is not an obvious truth, as I have before remarked. Assuming it now, we have by equation (13)

$$\delta w = -S\delta\psi\zeta_{\psi}\varpi w\zeta.$$

Comparing this with the last equation, and noticing that $\delta\psi$ is a perfectly arbitrary self-conjugate linear vector function of a vector, we see (p. 103 above) that

$$2_{\psi}\varpi w = m(\varpi\psi^{-1} + \psi^{-1}\varpi) \quad \dots \quad (35),$$

$\therefore \frac{1}{2}(\varpi\psi^{-1} + \psi^{-1}\varpi)$ is the pure part of $\varpi\psi^{-1}$. The last equation can be easily shown to lead to

$$\left. \begin{aligned} m\varpi\omega &= \psi\varpi\psi\omega + V\theta\psi\omega \quad \dots \quad \dots \\ \text{where} \quad \theta &= (\psi + S\zeta\psi\zeta)^{-1}V\zeta_{\psi}\varpi\psi\zeta \quad \dots \end{aligned} \right\} \quad \dots \quad (36).$$

This, with equation (33) above, completes the present problem of expressing the stress $\bar{\phi}$ explicitly in terms of the strain q and ψ .

ϖ can be easily shown to be that stress which $\bar{\phi}$ becomes when the body is rotated by the operator $q^{-1}()$ q , without altering the force exerted across any interface of the body. We thus see why it is ϖ and not $\bar{\phi}$ which bears the simplest relation to ψ .

$\phi\omega$ or $\bar{\phi}\omega + V\epsilon\omega$, where ϵ is perfectly arbitrary so far as the strain is concerned, is the force exerted on the *strained* area ω . This last would more properly be denoted by ω' . Calling it ω' , and denoting by ω the same vector area before strain, we have *

$$\omega' = m\chi'^{-1}\omega \quad \dots \quad (37).$$

* By Tait's *Quaternions*, 3rd ed., §§ 157, 158, we have $V\chi\mu\chi\nu = m\chi'^{-1}V\mu\nu$. If μ, ν be taken as the conterminous edges of a small parallelogram in the unstrained state $V\mu\nu$ will be its vector area; $\chi\mu, \chi\nu$ will be the edges of the strained parallelogram, and $V\chi\mu\chi\nu$ its vector area.

Hence the force $\phi\omega'$ becomes $m\phi\chi'^{-1}\omega = m\bar{\phi}\chi'^{-1}\omega + mV\epsilon\chi'^{-1}\omega$. Thus we see that the force on an elementary area, which before strain is ω , is a linear vector function of ω , but even in the case when there is no molecular couple, this function is not in general (large strain) self-conjugate. If there be no rotation, this force

$$\begin{aligned} &= m\varpi\psi^{-1}\omega + mV\epsilon\psi^{-1}\omega \\ &= \psi\Delta w\omega + V\theta\omega + mV\epsilon\psi^{-1}\omega. \end{aligned}$$

If now the rotation take place this force merely rotates with the body, and we get for the force $\tau\omega$ on the area, due to the strain $q\psi(\)q^{-1}$,

$$\tau\omega = q(\psi\Delta w\omega + V\theta\omega + mV\epsilon\psi^{-1}\omega)q^{-1}. \quad (38).$$

It is not hard to see from the above that the couple per unit volume of the unstrained body is $2mq\epsilon q^{-1}$. The force per unit volume of the unstrained body can be shown by means of equation (3) to be $\tau\Delta$. From these we can write down equations of motion in which ψ occurs. To obtain equations of motion in which only ρ' and its derivatives occur explicitly we must adopt a different method.

Consider w a given function, not of ψ as above, but of ψ^2 or $\chi'\chi$ (Tait's *Quaternions*, 3rd ed., § 381). Let

$$\Psi\omega = \psi^2\omega = \chi'\chi\omega = \nabla_1 S\omega \nabla_2 S\rho_1\rho'_2. \quad (39).$$

In assuming that w is given as an explicit function of the coordinates of Ψ , we are following Thomson and Tait, for these coordinates will be found to be the A, B, C, a, b, c of Appendix C of their *Nat. Phil.*

To see the enormous advantage of employing quaternions in such questions as the present, it is only necessary to compare the processes and results of the present investigation with theirs. The results below are considerably more general than theirs, and yet how much less cumbrous.

By the methods already so often applied, it is easy to prove both the following identities:—

$$\begin{aligned} S\delta\chi\zeta\phi\chi'^{-1}\zeta &= S\chi'\delta\chi\zeta\chi^{-1}\bar{\phi}\chi'^{-1}\zeta \\ &= S\delta\chi'\chi\zeta\chi^{-1}\bar{\phi}\chi'^{-1}\zeta. \end{aligned}$$

Thus we see from equation (30) that

$$\begin{aligned}\frac{\delta w}{m} &= -\frac{1}{2}S(\chi'\delta\chi + \delta\chi'\chi)\zeta\chi^{-1}\bar{\phi}\chi'^{-1}\zeta \\ &= -\frac{1}{2}S\delta\Psi\zeta\chi^{-1}\bar{\phi}\chi'^{-1}\zeta.\end{aligned}$$

But we also have by equation (13) above

$$\delta w = -S\delta\Psi\zeta_{\Psi}\Omega w\zeta.$$

Hence from p. 104 above we see that

$$2_{\Psi}\Omega w = m\chi^{-1}\bar{\phi}\chi'^{-1},$$

or

$$\bar{\phi} = \frac{2}{m}\chi_{\Psi}\Omega w\chi', \quad . \quad . \quad . \quad . \quad . \quad (40).$$

(It is easy from here to go back and prove all our previous results over again. Perhaps this, in fact, would be the shortest method, but it would not be the most natural. To do so it is only necessary to notice that since $S\delta\psi\zeta_{\Psi}\Omega\zeta = S\delta\Psi\zeta_{\Psi}\Omega\zeta$, it can be proved that $_{\Psi}\Omega = _{\Psi}\Omega\psi + \psi_{\Psi}\Omega$ where the differentiations on the right are not to act upon ψ .)

Substitute from equation (27) for ∇' in the equations (17) and (20) of equilibrium. Thus

$$\mathfrak{F} - \frac{mm_1^{-1}}{2}V\mathfrak{M}_1\chi'^{-1}\nabla_1 + m\bar{\phi}_1\chi^{-1}\nabla_1 = 0.$$

Noticing that since by equation (11)

$$2m\chi'^{-1}\omega = -V\rho'_1\rho'_2S\omega\nabla_1\nabla_2,$$

we have

$$m\chi'^{-1}\Delta = 0,$$

this last equation can be written

$$\mathfrak{F} - \frac{1}{2}V\mathfrak{M}\chi'^{-1}\Delta + m\bar{\phi}\chi'^{-1}\Delta = 0.$$

Substituting now for $\bar{\phi}$ from equation (40) we have

$$\mathfrak{F} - \frac{1}{2}V\mathfrak{M}\chi'^{-1}\Delta + 2\chi_{\Psi}\Omega w\Delta = 0 \quad . \quad . \quad . \quad . \quad . \quad (41),$$

or

$$\mathfrak{F} + \frac{3}{2}\left(\frac{V\mathfrak{M}V\rho'_1\rho'_2}{S\nabla_3\nabla_4\nabla_5S\rho'_3\rho'_4\rho'_5}\right)S\Delta\nabla_1\nabla_2 - 2\rho'_1S\nabla_1_{\Psi}\Omega w\Delta = 0 \quad (42).$$

This is the equation of equilibrium of an elastic body (solid or fluid) subjected to an external force and couple per unit volume of the unstrained body $\mathfrak{F}$ and $\mathfrak{M}$ respectively. For the corresponding equation of motion we have to equate the left member of this equation to $D\ddot{p}'$, where D is the density of the body at the point in the standard state.

In the case considered by Thomson and Tait (*Nat. Phil.*, App. C.), both $\mathfrak{F}$ and $\mathfrak{M}$ are zero, and our equation takes the very simple form

$$\rho_1'S \nabla_1 \Psi \Delta w \Delta = 0 \quad . \quad . \quad . \quad (43)$$

the *exact* quaternion equivalent of their three equations (7).*

In connection with equations (41) and (42), it should be observed that

$$\tau \omega = -\frac{1}{2} V \mathfrak{M} \chi' \omega + 2 \chi_\Psi \Delta w \omega \quad . \quad . \quad . \quad (44)$$

where τ has the same meaning as in equation (38).

The next few applications will be of an extremely simple nature, and will be confined to work already well known in its Cartesian form.

We now assume, as is usual, that the strains are small, and that there is no external couple, and therefore no stress couple. We deduce our equations as particular cases of the above, though, of course, if the present were our sole object, we could adopt a much simpler process. We may put $q=1$, $\phi=\bar{\phi}=\varpi$, $2\psi=\chi+\chi'$, and $\omega'=\omega$. Also $V \xi_\psi \Delta w \psi \xi = V \xi_\psi \Delta w \xi = 0$, and $\therefore \theta$ of equations (36) = 0. Thus equation (36) gives

$$\varpi = \psi \Delta w \quad . \quad . \quad . \quad (45).$$

It is worth while giving one of these for comparison. I may remark that many of the results arrived at above, although quite simple enough in their quaternion form to be manageable, and therefore useful, become so extraordinarily complicated when translated into Cartesian notation as to be utterly unmanageable and useless. The equation referred to is

$$\begin{aligned} & \frac{d}{dx} \left\{ 2 \frac{dw}{dA} \left(\frac{da}{dx} + 1 \right) + \frac{dw}{db} \frac{da}{dz} + \frac{dw}{dc} \frac{da}{dy} \right\} \\ & + \frac{d}{dy} \left\{ 2 \frac{dw}{dB} \frac{da}{dy} + \frac{dw}{da} \frac{da}{dz} + \frac{dw}{dc} \left(\frac{da}{dx} + 1 \right) \right\} \\ & + \frac{d}{dz} \left\{ 2 \frac{dw}{dC} \frac{da}{dz} + \frac{dw}{da} \frac{da}{dy} + \frac{dw}{db} \left(\frac{da}{dx} + 1 \right) \right\} = 0. \end{aligned}$$

Since we are dealing with small strains, it is convenient to alter the notation slightly. Instead of our previous χ and ψ we shall now write $1 + \chi$ and $1 + \psi$ respectively, so that both χ and ψ are small, and ψ is the pure part of χ . And, further, we shall write $\rho' = \rho + \eta$, so that η is the small displacement. Thus

$$\chi\omega = -S\omega \nabla . \eta \quad . \quad . \quad . \quad . \quad . \quad . \quad (46)$$

$$\psi\omega = -\frac{1}{2}(\eta_1 S\omega \nabla_1 + \nabla_1 S\omega \eta_1) \quad . \quad . \quad . \quad . \quad (47).$$

Since the strain ψ is small, the stress π , *i.e.*, ∇w is linear in ψ , and $\therefore w$ is quadratic. Now, for any such quadratic expression as can be easily proved

$$w = -\frac{1}{2}S\psi\zeta_\psi \nabla w \zeta \quad . \quad . \quad . \quad . \quad . \quad . \quad (48),$$

$\therefore$ from equation (45) we have

$$w = -\frac{1}{2}S\psi\zeta_\pi \pi \zeta \quad . \quad . \quad . \quad . \quad . \quad . \quad (49).$$

Also w , being quadratic in ψ , is, if regarded as a function of π , quadratic in it. Hence

$$w = -\frac{1}{2}S\pi\zeta_\pi \nabla w \zeta \quad . \quad . \quad . \quad . \quad . \quad . \quad (50),$$

and $\therefore$ by the last equation and p. 104 above,

$$\psi = \pi \nabla w \quad . \quad . \quad . \quad . \quad . \quad . \quad (51).$$

Instead of regarding w as a function of ψ or of π , it is perhaps simpler, from the mathematical point of view, to start with assuming it a given function of the first space derivatives of η . Let, in fact,

$$w = w(\nabla_1, \eta_1, \nabla_2, \eta_2) \quad . \quad . \quad . \quad . \quad . \quad . \quad (52)$$

where $w(\alpha, \beta, \gamma, \delta)$ is a scalar function of the vectors $\alpha, \beta, \gamma, \delta$, which is (1) linear in *each* of its constituents; (2) symmetrical in α and β , and also in γ and δ ; (3) such that the pair α, β may be interchanged with the pair γ, δ without altering the value of the function. (If this last is not true in the first form of w chosen, it may be made so by writing $\frac{1}{2}w(\alpha, \beta, \gamma, \delta) + \frac{1}{2}w(\gamma, \delta, \alpha, \beta)$ instead of $w(\alpha, \beta, \gamma, \delta)$, as this does not affect equation (52).) Such a function can be proved to involve twenty-one independent scalar constants, which is the number also required to determine an arbitrary homogeneous quadratic function of the six coordinates of ψ .

We may from this form of w at once go back to that in terms of ψ by means of equation (8) above. Thus,

$$w = w(\xi_1, \psi \xi_1, \xi_2, \psi \xi_2) \quad \dots \quad (53),$$

or, again, by equation (7),

$$w = w(\xi_1, \chi \xi_1, \xi_2, \chi \xi_2) \quad \dots \quad (54).$$

Hence by substituting $-\xi S \xi_1 \psi \xi$ for $\psi \xi_1$, we get

$$-\frac{1}{2} S \psi \xi \varpi \xi = w = -S \psi \xi \{ \xi_1 w(\xi_1, \xi, \xi_2, \psi \xi_2) \},$$

and $\therefore$ by p. 104

$$\varpi \omega = 2 \xi w(\xi, \omega, \xi_1, \psi \xi_1) \quad \dots \quad (55),$$

or, again, by equation (8),

$$\varpi \omega = 2 \xi w(\xi, \omega, \nabla_1 \eta_1) \quad \dots \quad (56).$$

The equation (23) of equilibrium thus becomes in this case

$$\mathfrak{F} + 2 \xi w(\xi, \Delta, \nabla_1 \eta_1) = 0 \quad \dots \quad (57)$$

Thus in the case of isotropic bodies in which c is the compressibility and n the rigidity, we have (*Mess. of Math.*, vol. xiv. pp. 30, 31)

$$\varpi \omega = 2n \psi \omega - \left(c - \frac{2}{3}n \right) \omega S \xi \psi \xi \quad \dots \quad (58),$$

$$\psi \omega = \frac{1}{2n} \varpi \omega + \left(\frac{1}{6n} - \frac{1}{9c} \right) \omega S \xi \varpi \xi \quad \dots \quad (59).$$

Thus from these equations and equation (49)

$$\left. \begin{aligned} w &= -n \psi \xi \psi \xi + \left(\frac{c}{2} - \frac{n}{3} \right) S^2 \xi \psi \xi \\ &= -\frac{1}{4n} \varpi \xi \varpi \xi - \left(\frac{1}{12n} - \frac{1}{18c} \right) S^2 \xi \varpi \xi \end{aligned} \right\} \quad \dots \quad (60).$$

Again, from the first of these equations and equation (8) we have

$$w = -n S \nabla_1 \psi \eta_1 + \left(\frac{c}{2} - \frac{n}{3} \right) S^2 \nabla \eta,$$

or

$$2w = n S \nabla_1 \eta_2 S \nabla_2 \eta_1 + n S \nabla_1 \nabla_2 S \eta_1 \eta_2 + \left(c - \frac{2}{3}n \right) S^2 \nabla \eta \quad (61),$$

and from equation (56)

$$\varpi \omega = -n S \omega \nabla \cdot \eta - n \nabla_1 S \omega \eta_1 - (m - n) \omega S \nabla \eta \quad \dots \quad (62),$$

where m stands for $c + \frac{n}{3}$. This last may be derived in a great many other ways still more simple.

The equation of equilibrium thus becomes

$$\mathfrak{F} = n \nabla^2 \eta + m \nabla S \nabla \eta \quad . \quad . \quad . \quad (63).$$

As a final application in the theory of elasticity let us consider St Venant's torsion problem from a quaternion point of view. Let r, ϕ, z be ordinary columnar coordinates whose axis is parallel to the generating lines of the cylinder. Let λ, μ, ν be unit vectors in the directions of $dr, d\phi, dz$ respectively, so that

$$\nabla = \lambda \frac{d}{dr} + \frac{\mu}{r} \frac{d}{d\phi} + \nu \frac{d}{dz}.$$

It is required to determine v a scalar function of r and ϕ , so that η given by the equation

$$\eta = \tau(zr\mu + v\nu) \quad . \quad . \quad . \quad (64),$$

where τ is a given small scalar constant, may satisfy (1) equation (63) of equilibrium, and (2) the equation

$$0 = \varpi\omega = -nS\omega \nabla \cdot \eta - n \nabla_1 S\omega\eta_1 - (m-n)\omega S \nabla \eta$$

at every point of the curved surface; ω in this case standing for the normal at the point. Q having the same general meaning as on p. 104 above, we have

$$Q(\nabla_1 \eta_1) = \tau\{z[Q(\lambda, \mu) - Q(\mu, \lambda)] + rQ(\nu, \mu) + Q(\nabla v, \nu)\} \quad (65).$$

In the case when Q is symmetrical in its constituents this takes the simple form

$$Q(\nabla_1 \eta_1) = \tau\{rQ(\nu, \mu) + Q(\nabla v, \nu)\} \quad . \quad . \quad . \quad (66).$$

From the first of these we have

$$\begin{aligned} \nabla \eta &= \tau\{(2\nu - r\lambda) + \nabla v\nu\} \\ &= \tau\{\nabla(z^2 - \frac{1}{2}r^2) + \nabla v\nu\} \end{aligned}$$

so that $S \nabla \eta = 0$ and $\nabla^2 \eta = \nu \nabla^2 v$. Thus equation (63) becomes

$$\nabla^2 v = 0 \quad . \quad . \quad . \quad (67).$$

From equations (62) and (66) above, we see that in the present case the stress is given by

$$\varpi\omega = -n\tau\{r(\mu S\omega\nu + \nu S\omega\mu) + (\nu S\omega \nabla v + \nabla v S\omega\nu)\} \quad . \quad (68),$$

and therefore consists of two shears; the first of magnitude $n\tau r$, with faces perpendicular to μ and ν , and the second of magnitude $n\tau T \nabla v$, with faces perpendicular to ν and ∇v . Assuming ω to be the normal at a point of the curved surface, and therefore perpendicular to ν , we have

$$\begin{aligned} 0 &= rS\omega\mu + S\omega \nabla v \\ &= -\frac{1}{2}S\nu\omega \nabla(r^2) + S\omega \nabla v. \end{aligned}$$

Thus

$$\frac{dv}{dn} = \frac{d}{dl} \left(\frac{r^2}{2} \right) \cdot \cdot \cdot \cdot \cdot \cdot (69),$$

where d/dn denotes differentiation along the normal outwards, and d/dl differentiation in the positive direction round the boundary.

The surface traction on the plane end

$$= \omega\nu = n\tau(r\mu + \nabla v).$$

Hence the total moment round the axis

$$\begin{aligned} &= n\tau \iint (r^2 - rS\mu \nabla v) dA \\ &= n\tau \left(I + \iint \frac{dv}{d\phi} dA \right), \end{aligned}$$

where dA is an element of area of the cross-section, and I is the moment of inertia of the cross-section round the axis. Thus the torsional rigidity is, as usual, $n \left(I + \iint \frac{dv}{d\phi} dA \right)$. We leave the problem here to the theories of complex variables and Fourier's theorem.

It is in the general theories of electrostatics and electromagnetism that I have found the methods now being defended the most powerful. I have been led to believe that there is in all the accepted theories which are based on general dynamical reasoning an error of a somewhat serious character. I have also been led to a considerable modification and extension of Poynting's theories. There are two reasons against giving these here. I have been told that this preliminary apology, as it may be termed, for my methods should be as short and simple as possible. Moreover, the greater part of my notes on this subject are at present inaccessible. I therefore limit myself in this branch to a single example.

Maxwell has not investigated what are the general mechanical results of his electrostatic theory for crystallised dielectrics.

According to him, the properties of the medium depend on six independent constants for each point, called the coefficients of spec. ind. cap. These coefficients will themselves be functions of the state of the medium, and therefore in particular of its strain. Assume with Maxwell that

$$\mathfrak{E} = -\nabla v, \quad \mathfrak{D} = \frac{K}{4\pi} \mathfrak{E} \quad \dots \quad (70),$$

where K is not a mere scalar, but a *self-conjugate linear vector function of a vector*. K is itself a function of the position of a point and also of the strain at the point. Here v is for obvious reasons put for Maxwell's V . Assume further, that if W be the pot. en. of the field; D, σ be the volume and surface density of electricity; and the rest of the notation be identical with Maxwell's,

$$\left. \begin{aligned} 2W &= \iint v \sigma ds + \iiint v D ds \\ &= -\iiint S \mathfrak{D} \mathfrak{E} ds \end{aligned} \right\} \quad \dots \quad (71),$$

and * $\therefore$

$$W = \iint v \sigma ds + \iiint v D ds + \frac{1}{2} \iiint S \mathfrak{D} \mathfrak{E} ds \quad \dots \quad (72),$$

$$D = -S \nabla \mathfrak{D} \quad \sigma = [S \mathfrak{D} U \nu]_a + [S \mathfrak{D} U \nu]_b \quad \dots \quad (73),$$

the last occurring only at a surface of discontinuity in $\mathfrak{D}$, $U \nu$ pointing *away* from the region of the corresponding $\mathfrak{D}$, and the two regions bounded by the surface being denoted by the suffixes a and b . In future, such expressions as $[]_a + []_b$ will, for brevity, be written $[]_{a+b}$. All our integrals are supposed to extend throughout all space; though, as $K = 0$ for conductors, these may be excluded. The boundaries of space are the surface at infinity and all surfaces of discontinuity in $\mathfrak{D}$ or $\mathfrak{E}$.

To find the mechanical results flowing from the above assumptions, let ρ be the *present* (whether strained or not) vector coordinate of any point, and let the medium be (additionally) strained by a small displacement $\delta \eta$, vanishing at infinity. Let δW be the increment in W . Then if δW can be expressed in the form

$$\delta W = -\iiint S \delta \eta_1 \phi \nabla_1 ds \quad \dots \quad (74),$$

* In taking this for the form of W , and operating upon it as follows, we are following Helmholtz for the particular case when K is a mere scalar. See *Wiss. Abh.*, equation (2d), p. 805. For the various assumptions above see Maxwell's *Electricity and Magnetism*, part i.

we shall have the following expressions for $\mathfrak{F}$ and $\mathfrak{F}_s$, the forces per unit volume and surface respectively due to the electric system.

$$\mathfrak{F} = \phi \Delta \quad \mathfrak{F}_s = -[\phi U_\nu]_{a+b} \quad \dots \quad (75).$$

And, further, if ϕ be self-conjugate, both these forces will be explained by a stress ϕ , as can be seen by the above work on stress. For proof we have by equation (3) above

$$\delta W = -\iiint S \delta \eta_1 \phi \nabla_1 ds = -\iint S \delta \eta \phi U_\nu ds + \iiint S \delta \eta \phi_1 \nabla_1 ds$$

where of course the element ds is taken twice, namely, once for the region on each side. But

$$\begin{aligned} \delta W &= -(\text{work done by the system } \mathfrak{F}, \mathfrak{F}_s \text{ of forces}) \\ &= \iint S \mathfrak{F}_s \delta \eta ds + \iiint S \mathfrak{F} \delta \eta ds, \end{aligned}$$

where the element ds is now only taken once. Equating coefficients of the arbitrary vector $\delta \eta$ we get the required equations.

To avoid difficulty at surfaces of discontinuity, δ when applied to a function of the position of a point must be thus defined. Suppose that by means of the displacement $\delta \eta$, any point is moved from P to P' . Then Q being the value of any function at P before the displacement, $Q + \delta Q$ will be the value at P' after the displacement. δQ is thus in all cases a small quantity of the same order as $\delta \eta$. With this meaning of δ , $\delta \nabla$ is not $= 0$. To find it observe that

$$S(d\rho + \delta d\rho)(\nabla + \delta \nabla) = S d\rho \nabla$$

but

$$\delta d\rho = -S d\rho \nabla \cdot \delta \eta$$

$\therefore$

$$S d\rho \delta \nabla = S d\rho \nabla_1 S \delta \eta_1 \nabla$$

whence

$$\delta \nabla = \nabla_1 S \delta \eta_1 \nabla \quad \dots \quad (76),$$

which might have been derived from the equation (27) $\nabla' = \chi'^{-1} \nabla$.

Assuming that the strain due to $\delta \eta$ does not alter the charge of any portion of matter,

$$0 = \delta(D ds) = \delta(\sigma ds) \quad \dots \quad (77).$$

To find δW , notice that

$$\delta ds = -ds S \nabla \delta \eta$$

and

$$4\pi \delta \mathfrak{D} = K \delta \mathfrak{E} + \delta K \cdot \mathfrak{E},$$

$$\text{and } \therefore \delta(S\mathfrak{D}\mathfrak{E}ds) = \left(2S\mathfrak{D}\delta\mathfrak{E} - S\mathfrak{D}\mathfrak{E}S\nabla\delta\eta + \frac{1}{4\pi}S\mathfrak{E}\delta K\mathfrak{E} \right) ds.$$

$$\begin{aligned} \text{Also} \quad & \iiint \delta v.Dds + \iiint \delta v.\sigma ds \\ &= -\iiint \delta v S \nabla \mathfrak{D} ds + \iiint \delta v S U \nabla \mathfrak{D} ds \quad [\text{equations (73)}] \\ &= \iiint S \mathfrak{D} \nabla \delta v ds \quad [\text{equation (3)}]. \end{aligned}$$

Hence

$$\delta W = \iiint ds \left\{ S\mathfrak{D}(\nabla\delta v + \delta\mathfrak{E}) - \frac{1}{2}S\mathfrak{D}\mathfrak{E}S\nabla\delta\eta + \frac{1}{8\pi}S\mathfrak{E}\delta K\mathfrak{E} \right\}.$$

But

$$\nabla\delta v + \delta\mathfrak{E} = -\delta\nabla.v = -\nabla_1 S\delta\eta_1 \nabla v.$$

Hence

$$\delta W = -\iiint S\delta\eta_1 \left(\frac{1}{2}\nabla_1 S\mathfrak{D}\mathfrak{E} - \mathfrak{E}S\nabla_1 \mathfrak{D} \right) ds + \frac{1}{8\pi} \iiint S\mathfrak{E}\delta K\mathfrak{E} ds.$$

The increment δK is conveniently considered as consisting of two parts—first, δK_r , due to the rotation of the body; and second, δK_s , due to the change of shape, *i.e.*, the pure strain. If the rotation vector due to $\delta\eta$ be ϵ , *i.e.*, if any vector ω become $\omega + V\epsilon\omega$, the result of operating on $\omega + V\epsilon\omega$ by $K + \delta K_r$ must be the same as first operating on ω by K and then rotating. In symbols

$$(K + \delta K_r)(\omega + V\epsilon\omega) = K\omega + V\epsilon K\omega,$$

whence

$$\delta K_r \omega = V\epsilon K\omega - KV\epsilon\omega.$$

Thus

$$S\mathfrak{E}\delta K_r \mathfrak{E} = S\mathfrak{E}\epsilon K\mathfrak{E} - S\mathfrak{E}KV\epsilon\mathfrak{E} = 8\pi S\epsilon\mathfrak{D}\mathfrak{E},$$

whence giving ϵ its value, $\frac{1}{2}V\nabla\delta\eta$,

$$\frac{1}{8\pi}S\mathfrak{E}\delta K_r \mathfrak{E} = \frac{1}{2}S\nabla\delta\eta V\mathfrak{D}\mathfrak{E}.$$

If we put w for the pot. en. per unit volume we have

$$w = -\frac{1}{2}S\mathfrak{D}\mathfrak{E} = -\frac{1}{8\pi}S\mathfrak{E}K\mathfrak{E} \quad \dots \quad (78).$$

K is a function of the independent variables ϵ and ψ , where ψ is the pure strain given by

$$\psi\omega = -\frac{1}{2}(\eta_1 S\omega \nabla_1 + \nabla_1 S\omega \eta_1).$$

Thus w may be regarded as a function of the independent variables $\mathfrak{E}$, ϵ , ψ , and we shall have

$$\frac{1}{8\pi} S \mathfrak{E} \delta K_1 \mathfrak{E} = S \delta \psi \zeta_\psi \mathfrak{A} w \zeta = S \delta \eta_1 \psi \mathfrak{A} w \nabla_1$$

by equations (13) and (8) above.

Thus

$$\delta W = - \iiint S \delta \eta_1 (\tfrac{1}{2} \nabla_1 S \mathfrak{D} \mathfrak{E} - \mathfrak{E} S \nabla_1 \mathfrak{D} + \tfrac{1}{2} V \nabla_1 V \mathfrak{D} \mathfrak{E} - \psi \mathfrak{A} w \nabla_1) ds$$

or

$$\delta W = \iiint S \delta \eta_1 (\tfrac{1}{2} V \mathfrak{D} \nabla_1 \mathfrak{E} + \psi \mathfrak{A} w \nabla_1) ds \quad \dots \quad (79).$$

Hence we see that

$$\mathfrak{F} = -\tfrac{1}{2} V \mathfrak{D} \Delta \mathfrak{E} - \psi \mathfrak{A} w \Delta \quad \dots \quad (80)$$

$$\mathfrak{F}_s = [\tfrac{1}{2} V \mathfrak{D} U_\nu \mathfrak{E} + \psi \mathfrak{A} w U_\nu]_{a+b} \quad \dots \quad (81),$$

and that these forces per unit volume and surface respectively can be supposed due to a stress ϕ given by

$$\phi \omega = -\tfrac{1}{2} V \mathfrak{D} \omega \mathfrak{E} - \psi \mathfrak{A} w \omega \quad \dots \quad (82).$$

From this we see that the stresses in the electric field can only be determined when for the particular strain existing at any point forty-two scalar functions of that strain are known, viz., the six coefficients of specific inductive capacity and their thirty-six differential coefficients with respect to the six coefficients of pure strain.

The two parts of the stress ϕ (1) that which is independent of the variation of specific inductive capacity with strain, and (2) that which depends on this variation are conveniently considered separately.

The first is more general than Maxwell's, because we have not, as he does in this connection, assumed that $\mathfrak{D}$ is parallel to $\mathfrak{E}$. It consists of a tension in the direction bisecting the positive directions (or negative directions) of both $\mathfrak{D}$ and $\mathfrak{E}$, of magnitude $\tfrac{1}{2} T \mathfrak{D} T \mathfrak{E}$, an equal pressure in the direction bisecting the positive direction of either and the negative direction of the other, and a pressure at right angles to both, of magnitude $-\tfrac{1}{2} S \mathfrak{D} \mathfrak{E}$ or w . This stress, of course, reduces to Maxwell's when $\mathfrak{D}$ is parallel to $\mathfrak{E}$.

The other part of the stress $-\psi \mathfrak{A} w = \frac{1}{8\pi\psi} \mathfrak{A}_1 S \mathfrak{E} K_1 \mathfrak{E}$ can be only

usefully considered when certain simplifying assumptions are made. In Professor J. J. Thomson's consideration of this subject (*Applications of Dynamics to Physics and Chemistry*, §§ 35 and 39), he not only assumes K to be a mere scalar both before and after strain, but he also does not consider the most general case of strain which involves 6 instead of 3 independent coefficients. If this last restriction had not been made, our 42 constants would have reduced to 7, and in his case they reduce to 4. In either of these cases the stress is simple enough, viz., $\frac{\mathfrak{E}^2}{8\pi\psi} \alpha K$.*

* For other particular cases of the stress under consideration, see Helmholtz, *Wiss. Abh.*, i. 798; Korteweg, *Wied. Ann.*, ix. 48; Lorberg, *Wied. Ann.*, xxi. 300; Kirchhoff, *Wied. Ann.*, xxiv. 52, xxv. 601.