XXXIV.—*On the Vibrations of an Interrupted Medium.* By the Rev. PHILIP KELLAND, M.A., F.R.S.S.L. & E., *Professor of Mathematics in the University of Edinburgh.*

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IN certain investigations which I have presented to the Society, relative to the modifications which light undergoes when it meets with a medium more dense than that in which it is travelling, the law of force has been supposed to be that of the inverse square of the distance. On re-examination of this subject, I find that there is no necessity for restricting the computations by the hypothesis of any particular law. The conclusions are perfectly independent of the law, provided one of the equations of reduction, the value of which it is not possible to compute by any known methods of analysis, be admitted as an experimental result. The object of the present Memoir is twofold : 1st, To present the analysis of the general theorem of vibrations at the surface of an interrupted medium in its most simple form ; and, 2d, To apply the results to the case of reflection unaccompanied by refraction. To accomplish the former object, I have, after deducing the equations of motion, sought to determine the values of the different constants which enter them, by means of the condition of symmetry. This investigation has led me to the generalization of some of the remarkable results which M. Cauchy has given relative to this subject, besides presenting me with some other conclusions, one of which is remarkable for its simplicity and completeness. It is this : A particle in a medium substantially symmetrical, will, when displaced from its position of equilibrium, be acted on by no accelerating forces ; it will consequently not be urged farther from its position of rest, at least until the other particles shall have been displaced. This is an important theorem, for it removes one objection to the possibility of attractive particles forming a system of stable equilibrium. In reference to the latter part of my investigation, I have only to remark that the equations cannot be solved in all their generality, inasmuch as they contain two more unknown quantities than the number of equations. But by approximation (omitting all terms of high orders of small quantities), I find that, in general, when there is no refraction, the reflection is equal to the incidence, and the rays suffer no retardation. But when the medium is crystallized, that is, when the forces within it have a different value from those in air, the reflected light is less than the incident, and the vibrations in the plane of incidence suffer retardation.

In order to avoid repetition, and to be as brief as possible, I will adopt the figure at p. 40 of the Memoir "On the polarization of light reflected at the surface of a crystal."—Vol. xv. The notation may also remain similar to that of the article referred to, with the exception that $R_x, R_y, R_z; T_x, T_y, T_z$ stand for the normal (or lost) vibrations reflected and transmitted.

$$\therefore \quad \begin{aligned} a &= (I - R) \sin \phi + R_x \\ \&c. &= \&c. \end{aligned}$$

and
$$\delta \alpha = -2 I \sin^2 \frac{k i}{2} \sin \phi + 2 R \sin^2 \frac{k r}{2} \sin \phi + \frac{1}{e} \frac{d I}{d x} \sin k i \sin \phi$$

$$- \frac{1}{e} \frac{d R}{d x} \sin k r \sin \phi - R_x (1 - e^{-m \delta x} \cos f \delta y) + \frac{d R_x}{d y} \frac{1}{f} e^{-m \delta x} \sin f \delta y,$$

$\delta \beta$ differs from the same quantity in the article referred to, by

$$\delta R_y \text{ or } -R_y (1 - e^{-m \delta x} \cos f \delta y) + \frac{d R_y}{d y} \frac{1}{f} e^{-m \delta x} \sin f \delta y$$

$\delta \gamma$ differs from the like quantity, by

$$\delta R_z \text{ or } -R_z (1 - e^{-m \delta x} \cos f \delta y) + \frac{d R_z}{d y} \frac{1}{f} e^{-m \delta x} \sin f \delta y$$

$\delta \beta$, and $\delta \gamma$, differ in the same way from the same quantities in the previous Memoir.

Also (p. 44) the equation for the vibration parallel to the axis of x is

$$\frac{d^2 \alpha}{d t^2} = \Sigma (\phi r + \frac{\phi' r}{r} \delta x^2) \left\{ -2 I \sin \phi \sin^2 \frac{k i}{2} + 2 R \sin \phi \sin^2 \frac{k r}{2} \right.$$

$$+ \frac{1}{e} \frac{d I}{d x} \sin \phi \sin k i - \frac{1}{e} \frac{d R}{d x} \sin \phi \sin k r - R_x (1 - e^{-m \delta x} \cos f \delta y) + \frac{d R_x}{d y} \frac{1}{f} e^{-m \delta x} \sin f \delta y \left. \right\}$$

$$+ \&c.$$

Now, for the motion in an ordinary medium, we have

$$2 \Sigma (\phi r + \frac{\phi' r}{r} p^2) \sin^2 \frac{k i}{2} = \frac{c^2}{2};$$

where i is in the direction of incidence, and p perpendicular to it.

Hence, also, $\delta x = i \cos \phi + p \sin \phi$, $\delta y = -i \sin \phi + p \cos \phi$.

By substitution, we obtain

$$\Sigma (\phi r + \frac{\phi' r}{r} \delta x^2) \delta \alpha = \Sigma (\phi r + \frac{\phi' r}{r} \overline{i^2 \cos^2 \phi + p^2 \sin^2 \phi}) \times$$

$$\left\{ -2 I \sin \phi \sin^2 \frac{k i}{2} + 2 R \sin \phi \sin^2 \frac{k r}{2} + \frac{1}{e} \frac{d I}{d x} \sin \phi \sin k i - \frac{1}{e} \frac{d R}{d x} \sin \phi \sin k r \right.$$

$$\left. - R_x (1 - e^{-m \delta x} \cos f \delta y) + \frac{d R_x}{d y} \frac{1}{f} e^{-m \delta x} \sin f \delta y \right\},$$

$$\begin{aligned} \Sigma \frac{\phi' r}{r} \delta x \delta y \delta \beta &= \Sigma \frac{\phi' r}{r} (-i^2 + p^2) \sin \phi \cos^2 \phi \left\{ -2 I \sin^2 \frac{k i}{2} \right. \\ &\quad \left. + 2 R \sin^2 \frac{k r}{2} + \frac{1}{e} \frac{d I}{d x} \sin k i - \frac{1}{e} \frac{d R}{d x} \sin k r \right\} \\ &- \Sigma \frac{\phi' r}{r} \delta x \delta y R_y (1 - e^{-m \delta x} \cos f \delta y) + \Sigma \frac{\phi' r}{r} \delta x \delta y \frac{1}{f} \frac{d R_y}{d y} e^{-m \delta x} \sin f \delta y \end{aligned}$$

the sum of which is

$$\begin{aligned} &\Sigma (\phi r + \frac{\phi' r}{r} p^2) \sin \phi \left\{ -2 I \sin^2 \frac{k i}{2} + 2 R \sin^2 \frac{k r}{2} + \frac{1}{e} \frac{d I}{d x} \sin k i \right. \\ &\quad \left. - \frac{1}{e} \frac{d R}{d x} \sin k r \right\} - \Sigma (\phi r + \frac{\phi' r}{r} \delta x^2) \left\{ R_x \frac{1 - e^{-m \delta x} \cos f \delta y}{1 - e^{-m \delta x} \cos f \delta y} - \frac{d R_x}{d y} \frac{1}{f} e^{-m \delta x} \sin f \delta y \right\} \\ &- \Sigma \frac{\phi' r}{r} \delta x \delta y \left\{ R_y (1 - e^{-m \delta x} \sin f \delta y) - \frac{d R_y}{d y} \frac{1}{f} e^{-m \delta x} \sin f \delta y \right\} \\ &= \frac{c^2}{2} \sin \phi (-I + R) + \frac{M}{e} \sin \phi \left(\frac{d I}{d x} - \frac{d R}{d x} \right) \\ &- D R_x + \frac{F}{f} \frac{d R_y}{d y} \end{aligned}$$

$$\begin{aligned} \text{where } M &= \Sigma \left\{ \phi r + \frac{\phi' r}{r} \delta x^2 \pm \frac{\phi' r}{r} \delta x \delta y \frac{\cos \phi}{\sin \phi} \right\} \sin (e \delta x \pm f \delta y) \\ &= \Sigma (\phi r + \frac{\phi' r}{r} \delta x^2) \sin e \delta x \cos f \delta y + \frac{\cos \phi}{\sin \phi} \Sigma \frac{\phi' r}{r} \delta x \delta y \cos e \delta x \sin f \delta y \end{aligned}$$

for both the incident and the reflected wave.

$$D = \Sigma (\phi r + \frac{\phi' r}{r} \delta x^2) (1 - e^{-m \delta x} \cos f \delta y)$$

$$F = \Sigma \frac{\phi' r}{r} \delta x \delta y e^{-m \delta x} \sin f \delta y.$$

$$\begin{aligned} \text{Again, } \Sigma (\phi r' + \frac{\phi' r'}{r'} \delta x'^2) \delta \alpha &= \Sigma (\phi r' + \frac{\phi' r'}{r'} t^2 \cos^2 \bar{\phi} + p^2 \sin^2 \bar{\phi}) \\ &\left\{ -2 T \sin \phi, \cos \theta \sin^2 \frac{k e}{2} + 2 T' \sin \phi' \sin \theta' \sin^2 \frac{k o}{2} \right. \\ &\quad \left. + \frac{1}{e} \frac{d T}{d x} \sin \phi, \cos \theta \sin k, e - \frac{1}{e'} \frac{d T'}{d x} \sin \phi' \sin \theta' \sin k o \right\} \\ &+ \Sigma (\phi r' + \frac{\phi' r'}{r'} \delta x'^2) \left\{ -T_x (1 - e^{-m \delta x} \cos f \delta y) + \frac{1}{f} \frac{d T_x}{d y} e^{-m \delta x} \sin f \delta y \right\} \end{aligned}$$

where t stands for either e or o , as the case may be; and $\bar{\phi}$ for ϕ , or ϕ' :

$$\text{and } \Sigma \frac{\phi' r'}{r'} \delta x' \delta y' \delta \beta = \Sigma \frac{\phi' r'}{r'} \sin \phi, \cos \phi, (-e^2 + p^2) (-2 T \cos \phi, \cos \theta \sin^2 \frac{k e}{2}$$

$$+ \frac{1}{e} \frac{dT}{dx} \cos \phi, \cos \theta \sin k_e) + \Sigma \frac{\phi' r'}{r'} \sin \phi' \cos \phi' (-o^2 + p^2) (2 T' \cos \phi' \sin \theta' \sin^2 \frac{k_o}{2} \\ - \frac{1}{e'} \frac{dT'}{dx} \cos \phi' \sin \theta' \sin k'_o) + \Sigma \frac{\phi' r'}{r'} \delta x' \delta y' \frac{1}{f} \frac{dT_y}{dy} e^{-m, \delta x} \sin f dy ;$$

the sum of which is

$$\Sigma (\phi' r' + \frac{\phi' r'}{r'} p^2) (-2 T \sin \phi, \cos \theta \sin^2 \frac{k_e}{2} + 2 T' \sin \phi' \sin \theta' \sin^2 \frac{k_o}{2} \\ + \frac{1}{e} \frac{dT}{dx} \sin \phi, \cos \theta \sin k_e - \frac{1}{e'} \frac{dT'}{dx} \sin \phi' \sin \theta' \sin k'_o) \\ - D, T_x + F, \frac{1}{f} \frac{dT_y}{dy} \\ = -\frac{c^2}{2} T \sin \phi, \cos \theta + \frac{c^2}{2} T' \sin \phi' \sin \theta' + \frac{M}{e} \frac{dT}{dx} \sin \phi, \cos \theta \\ - \frac{M'}{e'} \frac{dT'}{dx} \sin \phi' \sin \theta' - D, T_x + \frac{F}{f} \frac{dT_y}{dy}$$

Lastly, $\Sigma (\phi' r' + \frac{\phi' r'}{r'} \delta x'^2) (\alpha, -\alpha) = Q_{x'} (\alpha, -\alpha).$

Hence, adding these terms together, we get

$$\frac{d^2 \alpha}{dt^2} = \frac{c^2}{2} \sin \phi (-I + R) + \frac{M}{e} \sin \phi \left(\frac{dI}{dx} - \frac{dR}{dx} \right) \\ - \frac{c^2}{2} T \sin \phi, \cos \theta + \frac{c^2}{2} T' \sin \phi' \sin \theta' + \frac{M}{e} \frac{dT}{dx} \sin \phi, \cos \theta \\ - \frac{M'}{e'} \frac{dT'}{dx} \sin \phi' \sin \theta' - D, T_x + \frac{F}{f} \frac{dT_y}{dy} - D R_x + \frac{F}{f} \frac{dR_y}{dy} \\ + Q_{x'} (\alpha, -\alpha).$$

By a precisely similar process we find

$$\frac{d^2 \beta}{dt^2} = -\frac{c^2}{2} \cos \phi (I + R) + \frac{M}{e} \left(\frac{dI}{dx} + \frac{dR}{dx} \right) \cos \phi \\ - \frac{c^2}{2} T \cos \phi, \cos \theta + \frac{c^2}{2} T' \cos \phi' \sin \theta' + \frac{M}{e} \frac{dT}{dx} \cos \phi, \cos \theta \\ - \frac{M'}{e'} \frac{dT'}{dx} \cos \phi' \sin \theta' + \frac{F}{f} \frac{dR_x}{dy} + \frac{F}{f} \frac{dT_y}{dy} - D R_y - D, T_y \\ + \frac{F}{f} \frac{dR_x}{dy} + \frac{F}{f} \frac{dT_x}{dy} + Q_{y'} (\beta, -\beta) \\ \frac{d^2 \gamma}{dt^2} = -\frac{c^2}{2} (I' - R') - \frac{c^2}{2} (T \sin \theta + T' \cos \theta') \\ \frac{M}{e} \left(\frac{dI'}{dx} - \frac{dR'}{dx} \right) + \frac{M}{e} \frac{dT}{dx} \sin \theta + \frac{M'}{e'} \frac{dT'}{dx} \cos \theta' - D R_z - D, T_z \\ + Q_{z'} (\gamma, -\gamma).$$

Moreover, $\frac{d^2 \alpha}{dt^2}$, $\frac{d^2 \beta}{dt^2}$, $\frac{d^2 \gamma}{dt^2}$ respectively differ from $\frac{d^2 \alpha}{dt^2}$, $\frac{d^2 \beta}{dt^2}$, $\frac{d^2 \gamma}{dt^2}$ only in having $Q_x (\alpha - \alpha_0)$ in place of $Q_x (\alpha, -\alpha)$, $Q_y (\beta - \beta_0)$ in place of $Q_y (\beta, -\beta)$, and $Q_z (\gamma - \gamma_0)$ in place of $Q_z (\gamma, -\gamma)$.

But $\frac{d^2 \alpha}{dt^2} = -c^2 \alpha$, &c. = &c.

Hence the following equations result:

$$\begin{aligned} & -\frac{c^2}{2} \sin \phi (I - R) + \frac{M}{e} \sin \phi \left(\frac{dI}{dx} - \frac{dR}{dx} \right) - \frac{c^2}{2} T \sin \phi, \cos \theta \\ & + \frac{c^2}{2} T' \sin \phi' \sin \theta' + \frac{M'}{e'} \frac{dT}{dx} \sin \phi, \cos \theta - \frac{M'}{e'} \frac{dT'}{dx} \sin \phi' \sin \theta' \\ & - D_x T_x - D R_x + \frac{F}{f} \frac{dR_y}{dx} + \frac{F'}{f'} \frac{dT_y}{dy} + Q_x (\alpha, -\alpha) \\ & = -c^2 (I - R) \sin \phi - c^2 R_x \quad \dots \dots \dots (1.) \end{aligned}$$

$$= -c^2 T \sin \phi, \cos \theta + c^2 T' \sin \phi' \sin \theta' - c^2 T_x + \overline{Q_x} + \overline{Q_x} (\alpha, -\alpha) \quad (2.)$$

By subtracting these equations, we get

$$-c^2 \alpha + c^2 \alpha_0 - (Q_x + \overline{Q_x}) (\alpha, -\alpha) = 0$$

or

$$(c^2 - Q_x - \overline{Q_x}) (\alpha, -\alpha) = 0$$

wherefore

$$\alpha, -\alpha = 0 \quad \dots \dots \dots (1.)$$

Similarly

$$\beta, -\beta = 0 \quad \dots \dots \dots (2.)$$

$$\gamma, -\gamma = 0 \quad \dots \dots \dots (3.)$$

And by adding the above equations, and striking out the parts which are common to both sides, we obtain

$$\begin{aligned} & \frac{M}{e} \left(\frac{dI}{dx} - \frac{dR}{dx} \right) \sin \phi + \frac{M'}{e'} \frac{dT}{dx} \sin \phi, \cos \theta - \frac{M'}{e'} \frac{dT'}{dx} \sin \phi' \sin \theta' \\ & + \left(\frac{c^2}{2} - D \right) R_x + \left(\frac{c^2}{2} - D' \right) T_x + \frac{F}{f} \frac{dR_y}{dy} + \frac{F'}{f'} \frac{dT_y}{dy} = 0 \quad \dots \dots \dots (4.) \end{aligned}$$

$$\begin{aligned} & \frac{M}{e} \left(\frac{dI}{dx} + \frac{dR}{dx} \right) \cos \phi + \frac{M'}{e'} \frac{dT}{dx} \cos \phi, \cos \theta - \frac{M'}{e'} \frac{dT'}{dx} \cos \phi' \sin \theta' \\ & + \frac{F}{f} \frac{dR_x}{dy} + \frac{F'}{f'} \frac{dT_x}{dy} + \left(\frac{c^2}{2} - D \right) R_y + \left(\frac{c^2}{2} - D' \right) T_y = 0 \quad \dots \dots \dots (5.) \end{aligned}$$

$$\begin{aligned} & \frac{M}{e} \left(\frac{dI'}{dx} - \frac{dR'}{dx} \right) + \frac{M'}{e'} \frac{dT}{dx} \sin \theta + \frac{M'}{e'} \frac{dT'}{dx} \cos \theta' \\ & + \left(\frac{c^2}{2} - D \right) R_z + \left(\frac{c^2}{2} - D' \right) T_z = 0 \quad \dots \dots \dots (6.) \end{aligned}$$

These equations differ from those which I gave in my Memoir on light reflected at the surface of a crystal, in having the terms R_y , R_z , T_y , T_z additional. Thus it

appears that the equations do not depend on the *law of force*, but are equally true in all cases.

Before I proceed with any further discussion of these equations, I desire to prove some important theorems relative to the values of the constants in symmetrical media.

THEOREM 1.—*Relation between the sums of powers of one co-ordinate, and products of powers of different ones.*

Let f, g, h be the co-ordinates of a particle; $x-f, y-g, z-h$ those of another measured from it. Then

$$\Sigma m \phi r = \Sigma m \phi \sqrt{(x-f)^2 + (y-g)^2 + (z-h)^2}$$

Suppose the particle whose co-ordinates are f, g, h , to be moved to a point $f+\alpha, g+\beta, h+\gamma$, then $\Sigma m \phi r$, becomes $\Sigma m \phi \sqrt{(x-f-\alpha)^2 + (y-g-\beta)^2 + (z-h-\gamma)^2}$.

$$\begin{aligned} \text{Also} \quad & (x-f-\alpha)^2 + (y-g-\beta)^2 + (z-h-\gamma)^2 = r^2 \\ & - 2\{(x-f)\alpha + (y-g)\beta + (z-h)\gamma\} + \alpha^2 + \beta^2 + \gamma^2. \end{aligned}$$

Let δ be the distance through which the particle is moved, and let $(x-f)\alpha + (y-g)\beta + (z-h)\gamma$ be denoted by ϵ ; then

$$\begin{aligned} \Sigma m \phi r &= \Sigma m \phi \sqrt{r^2 - 2\epsilon + \delta^2} \\ &= \Sigma m \phi r + \dots + \Sigma m f r \left(\frac{2\epsilon - \delta^2}{r^2} \right)^{2n} + \dots \end{aligned}$$

Now this must be a function of δ : consequently

$$\Sigma m f, r \epsilon^{2n} \text{ must equal } P \delta^{2n}$$

$$\text{or} \quad \Sigma m f, r \{\alpha(x-f) + \beta(y-g) + \gamma(z-h)\}^{2n} = P (\alpha^2 + \beta^2 + \gamma^2)^n$$

P being some function of r .

By expanding each side we get

$$\begin{aligned} & \Sigma m f, r \left\{ \alpha^{2n} (x-f)^{2n} + \frac{2n(2n-1)}{1 \cdot 2} \beta (y-g) + \gamma (z-h)^2 \alpha^{2n-2} (x-f)^{2n-2} + \&c. \right. \\ & \quad \left. + \frac{2n(2n-1) \dots (2n-2r+1)}{1 \cdot 2 \dots \dots \dots 2r} \beta (y-g) + \gamma (z-h)^{2r} \alpha^{2n-2r} (x-f)^{2n-2r} + \&c. \right\} \\ &= P \left\{ \alpha^{2n} + n(\beta^2 + \gamma^2) \alpha^{2n-2} + \dots + \frac{n(n-1) \dots (n-r+1)}{1 \cdot 2 \dots \dots \dots r} (\beta^2 + \gamma^2)^r \alpha^{2n-2r} + \&c. \right\} \end{aligned}$$

Hence we obtain

$$\Sigma m f, r (x-f)^{2n} = P$$

$$\Sigma m f, r (y-g)^2 (x-f)^{2n-2} = \frac{n P}{2n(2n-1)} \cdot \cdot \cdot \cdot (A(1))$$

$$\begin{aligned} \Sigma m f, r (x-f)^{2n-2r} (y-g)^{2r-2s} (z-h)^{2s} &= \frac{n(n-1)\dots(n-r+1)}{1 \cdot 2 \dots r} \cdot \frac{r(r-1)\dots(r-s+1)}{1 \cdot 2 \dots s} P \\ &= \frac{2n(2n-1)\dots(2n-2r+1)}{1 \cdot 2 \dots 2r} \cdot \frac{2r(2r-1)\dots(2r-2s+1)}{1 \cdot 2 \dots 2s} P \\ &= \frac{(n-r+1)\dots(2n-2r) \times (r-s+1)\dots(2r-2s)(s+1)\dots 2s}{(n+1)\dots 2n} P. \quad (2) \end{aligned}$$

THEOREM 2.—To find the relation between $\Sigma m f, r (x-f)^{2n}$ and $\Sigma m f, r r^{2n}$

This amounts to the determination of the value of P in the above expression.

Let θ be the angle between r and δ

then $\epsilon = r \delta \cos \theta$

and $\epsilon^{2n} = r^{2n} \delta^{2n} \cos^{2n} \theta$

$\therefore P = \Sigma m r^{2n} \cos^{2n} \theta f, r$

Now, the area of a spherical surface in the mass is $\iint r^2 \sin \theta d\theta d\phi$

Hence for such a surface

$$\begin{aligned} P &= \iint r^{2n+2} \cos^{2n} \theta \sin \theta d\theta d\phi f, r \\ &= - \int \frac{r^{2n+2} f, r \cos^{2n+1} \theta}{2n+1} d\phi \\ &= \int \frac{r^{2n+2} f, r d\phi}{2n+1} \\ &= \frac{1}{2n+1} \Sigma m r^{2n} f, r \end{aligned}$$

$$\text{or} \quad \Sigma m (x-f)^{2n} f, r = \frac{1}{2n+1} \Sigma m r^{2n} f, r \quad \dots \dots \dots (B)$$

This proposition might have been proved with little difficulty without having recourse to integrals, but the result is so obvious, that I do not think it necessary to add such a proof.

THEOREM 3.—A system of particles act on one another by forces which vary inversely as the square of the distance between them. One of the particles is removed from its position of equilibrium, to find the force put in play on it.

Let the co-ordinates of this particle be measured in such directions that the axis of x may be the line of motion.

Now, it is evident that, since the medium is one of symmetry, the force put in play is in the direction of the motion; and is represented by

$$\begin{aligned} & \Sigma m \frac{x-f-a}{\{(x-f-a)^2 + (y-g)^2 + (z-h)^2\}^{\frac{3}{2}}} \\ &= \Sigma m \frac{x-f-a}{\{r^2 - 2(x-f)a + a^2\}^{\frac{3}{2}}} \\ &= \Sigma m \frac{(x-f-a)}{r^3} \left\{ 1 + \frac{3}{2} \left(\frac{2(x-f)a - a^2}{r^2} \right) + \frac{3 \cdot 5}{2 \cdot 4} \left(\frac{2(x-f)a - a^2}{r^2} \right)^2 \right. \\ & \quad \left. + \&c. + \frac{3 \cdot 5 \dots (2n+1)}{2 \cdot 4 \dots 2n} \left(\frac{2(x-f)a - a^2}{r^2} \right)^n + \&c. \right\} \end{aligned}$$

The co-efficient of a^{2n} in this expression is as follows:

$$\begin{aligned} & + \Sigma m \left\{ \frac{3 \cdot 5 \dots (2n+1)}{2 \cdot 4 \dots 2n} \frac{n}{r^{2n+3}} 2(x-f) - \frac{3 \cdot 5 \dots (2n+3)}{2 \cdot 4 \dots (2n+2)} \frac{(n+1)n(n-1)}{1 \cdot 2 \cdot 3} \frac{2^3(x-f)^3}{r^{2n+5}} \right. \\ & \quad \left. + \&c. \right\} \end{aligned}$$

every term of which involves $(x-f)$ as a factor. Hence this quantity is zero. Again, the co-efficient of a^{2n+1} is

$$\begin{aligned} & + \Sigma m \left\{ \frac{3 \cdot 5 \dots (2n+1)}{2 \cdot 4 \dots 2n} \frac{1}{r^{2n+3}} - \frac{3 \cdot 5 \dots (2n+3)}{2 \cdot 4 \dots (2n+2)} \frac{(n+1)n}{1 \cdot 2} \frac{2^2(x+f)^2}{r^{2n+5}} \right. \\ & + \frac{3 \cdot 5 \dots (2n+5)}{2 \cdot 4 \dots (2n+4)} \frac{(n+2)(n+1)n(n-1)}{1 \cdot 2 \cdot 3 \cdot 4} \frac{2^4(x-f)^4}{r^{2n+7}} - \&c. \\ & - \frac{3 \cdot 5 \dots (2n+3)}{2 \cdot 4 \dots (2n+2)} (n+1) \frac{2(x-f)^2}{r^{2n+5}} + \frac{3 \cdot 5 \dots (2n+5)}{2 \cdot 4 \dots (2n+4)} \frac{(n+2)(n+1)n}{1 \cdot 2 \cdot 3} \frac{2^3(x-f)^4}{r^{2n+7}} \\ & \left. - \frac{3 \cdot 5 \dots (2n+7)}{2 \cdot 4 \dots (2n+6)} \frac{(n+3)(n+2)(n+1)n(n-1)}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5} \frac{2^5(x-f)^6}{r^{2n+9}} + \&c. \right\} \end{aligned}$$

It remains that we find the sum of this series. To effect this, we remark that the co-efficients can all be reduced by the result of Theorem 2. Applying this theorem, the co-efficient is reduced to

$$\begin{aligned} & + \Sigma \frac{m}{r^{2n+3}} \left\{ \frac{3 \cdot 5 \dots (2n+1)}{2 \cdot 4 \dots 2n} - \frac{3 \cdot 5 \dots (2n+3)}{2 \cdot 4 \dots (2n+2)} \frac{(n+1)n}{1 \cdot 2} \frac{2^2}{3} \right. \\ & + \frac{3 \cdot 5 \dots (2n+5)}{2 \cdot 4 \dots (2n+4)} \frac{(n+2)(n+1)n(n-1)}{1 \cdot 2 \cdot 3 \cdot 4} \frac{2^4}{5} - \&c. \\ & \left. - \frac{3 \cdot 5 \dots (2n+3)}{2 \cdot 4 \dots (2n+2)} (n+1) \frac{2}{3} + \frac{3 \cdot 5 \dots (2n+5)}{2 \cdot 4 \dots (2n+4)} \frac{(n+2)(n+1)n}{1 \cdot 2 \cdot 3} \frac{2^3}{5} - \&c. \right\} \end{aligned}$$

Now, we observe that the co-efficient of a^{2n+1} in

$$\frac{p-a}{(1-2p a + a^2)^{\frac{3}{2}}} \text{ is } + \left\{ \frac{3 \cdot 5 \dots (2n+1)}{2 \cdot 4 \dots 2n} - \frac{3 \cdot 5 \dots (2n+3)}{2 \cdot 4 \dots (2n+2)} \frac{(n+1)n}{1 \cdot 2} \cdot 2^2 p^2 \right.$$

$$+ \quad \&c. \quad - \frac{3 \cdot 5 \dots (2n+3)}{2 \cdot 4 \dots (2n+2)} (n+1) 2p^2 + \frac{3 \cdot 5 \dots (2n+5)}{2 \cdot 4 \dots (2n+4)} \frac{(n+2)(n+1)n}{1 \cdot 2 \cdot 3} 2^3 p^4 \\ - \quad \&c. \quad \}$$

Hence the part of the required co-efficient which lies within brackets is evidently the same as that of a^{2n+1} in $\int_0^1 \frac{p-a}{(1-2pa+a^2)^{\frac{3}{2}}} dp$.

$$\text{But} \quad \int \frac{p-a}{(1-2pa+a^2)^{\frac{3}{2}}} dp = \frac{\sqrt{1-2pa+a^2}}{2a^2} + \frac{1-a^2}{2a^2} \frac{1}{\sqrt{1-2pa+a^2}}$$

$\therefore$ The above co-efficient is the same thing as the co-efficient of a^{2n+1} in the expansion of

$$\pm \Sigma \frac{m}{r^{2n+3}} \left\{ \frac{1-a}{2a^2} + \frac{1-a^2}{2a^2(1-a)} - \frac{\sqrt{1+a^2}}{2a^2} - \frac{1-a^2}{2a^2} \frac{1}{\sqrt{1+a^2}} \right\}$$

$$\text{or of} \quad \pm \Sigma \frac{m}{r^{2n+3}} \left\{ \frac{1}{a^2} - \frac{1}{a^2 \sqrt{1+a^2}} \right\}.$$

But this co-efficient is evidently zero. Hence, every term in the expression of

$$\Sigma m \frac{x-f-a}{\{(x-f-a)^2 + (y-g)^2 + (z-h)^2\}^{\frac{3}{2}}} \text{ is zero.}$$

We have consequently proved, that a particle of a system exerting forces which vary inversely as the square of the distance, will not tend to move *at all* by the action of the other particles of the system on it when out of its position of equilibrium.

Let us next proceed to find the values of c^2 , M, &c. For the first of them, we will take the sum *throughout* an uniform medium, whereby its value will become

$$c^2 = \Sigma m \left(\phi r + \frac{\phi' r}{r} \delta y^2 \right) 2 \sin^2 \frac{k \delta x}{2} \quad (1)$$

where δx , δy , δz are the co-ordinates of any particle m , measured from that under consideration, r the distance between the two, and $r \phi r$ the law of force.

$$\text{Now,} \quad 2 \sin^2 \frac{k \delta x}{2} = 1 - \cos k \delta x = 1 - \left(1 - \frac{(k \delta x)^2}{1 \cdot 2} + \&c. \right)$$

$$\therefore c^2 = \Sigma m \left(\phi r + \frac{\phi' r}{r} \delta y^2 \right) - \Sigma m \left(\phi r + \frac{\phi' r}{r} \delta y^2 \right) \left(1 - \frac{(k \delta x)^2}{1 \cdot 2} + \&c. \right)$$

This expression can be reduced by means of the Theorems just obtained.

$$\text{Equation A (1) gives} \quad \Sigma m \delta x^2 \delta y^{2n-2} \frac{\phi' r}{r} = \frac{1}{2n-1} \Sigma m \delta x^{2n} \frac{\phi' r}{r}$$

which, by applying equation (B), is reduced to

$$\Sigma m \delta x^2 \delta y^{2n-2} \frac{\phi' r}{r} = \frac{1}{(2n-1)(2n+1)} \Sigma m r^{2n} \frac{\phi' r}{r}$$

The same equations give likewise,

$$\begin{aligned}\Sigma m \frac{\phi' r}{r} \delta y^2 &= \frac{1}{3} \Sigma m \frac{\phi' r}{r} r^2 \\ \Sigma m \phi r x^{2n} &= \frac{1}{2_{n+1}} \Sigma m \phi r r^{2n} \text{ \&c. \&c.}\end{aligned}$$

By substituting these results in the value of c^2 it becomes

$$\begin{aligned}c^2 &= \Sigma m \left(\phi r + \frac{\phi' r}{r} \frac{r^2}{3} \right) \\ &- \Sigma m \phi r \left(1 - \frac{k^2 r^2}{3 \cdot 1 \cdot 2} + \frac{k^4 r^4}{5 \cdot 1 \cdot 2 \cdot 3 \cdot 4} - \text{\&c.} \right) \\ &- \Sigma m \frac{\phi' r}{r} \left(\frac{r^2}{3} - \frac{k^2 r^4}{3 \cdot 5 \cdot 1 \cdot 2} + \frac{k^4 r^6}{5 \cdot 7 \cdot 1 \cdot 2 \cdot 3 \cdot 4} - \text{\&c.} \right)\end{aligned}$$

Of this expression the first line is not dependent on expansion, the second and third are. It is desirable to keep them distinct.

To sum the series which multiply ϕr and $\frac{\phi' r}{r}$ respectively under the symbol Σm .

$$\begin{aligned}1. \quad 1 - \frac{k^2 r^2}{3 \cdot 1 \cdot 2} + \frac{k^4 r^4}{5 \cdot 1 \cdot 2 \cdot 3} - \text{\&c.} &= \frac{1}{k r} \left\{ k r - \frac{k^3 r^3}{1 \cdot 2 \cdot 3} + \text{\&c.} \right\} \\ &= \frac{\sin k r}{k r}\end{aligned}$$

which is the multiplier of ϕr .

$$2. \quad \text{Let } \frac{r^2}{3} - \frac{k^2 r^4}{3 \cdot 5 \cdot 1 \cdot 2} + \frac{k^4 r^6}{5 \cdot 7 \cdot 1 \cdot 2 \cdot 3 \cdot 4} - \text{\&c.} = u ;$$

$$\text{then } \int_0^r \frac{u}{r} d r = \frac{r^2}{1 \cdot 2 \cdot 3} - \frac{k^2 r^4}{1 \cdot 5} + \frac{k^4 r^6}{1 \cdot 5 \cdot 7} - \text{\&c.}$$

$$= \frac{1}{k^3 r} \left(\frac{k^3 r^3}{1 \cdot 2 \cdot 3} - \frac{k^5 r^5}{1 \cdot 5} + \text{\&c.} \right)$$

$$= \frac{k r - \sin k r}{k^3 r}$$

$$\therefore u = - \frac{r}{k^3} \frac{d}{d r} \cdot \frac{\sin k r}{r}$$

$$= \frac{1}{k^3} \left(\frac{\sin k r}{r} - k \cos k r \right)$$

which is the co-efficient of $\frac{\phi' r}{r}$.

By substituting these values in the expression for c^2 it is reduced to

$$\begin{aligned}c^2 &= \Sigma m \left(\phi r + \frac{\phi' r}{r} \frac{r^2}{3} \right) \\ &- \Sigma m \phi r \frac{\sin k r}{k r} - \Sigma m \frac{\phi' r}{r} \left(\frac{\sin k r}{k^3 r} - \frac{\cos k r}{k^2} \right) \quad (a).\end{aligned}$$

This is a remarkably simple expression for c^2 .

By combining the two portions, it may be written thus :

$$c^2 = \Sigma \frac{m}{r^2} \left\{ \phi r \left(r^2 - \frac{r \sin k r}{k} \right) + \phi' r \left(\frac{r^3}{3} - \frac{\sin k r}{k^3} + \frac{r \cos k r}{k^2} \right) \right\}$$

$$= \Sigma \frac{m}{r^2} \frac{d}{dr} \left\{ \phi r \left(\frac{r^3}{3} - \frac{\sin k r}{k^3} + \frac{r \cos k r}{k^2} \right) \right\}.$$

If we write $f(r)$ for $r \phi(r)$, the force at distance r , we get

$$c^2 = \Sigma \frac{m}{k^2 r^2} \frac{d}{dr} \left\{ \left(\frac{k^2 r^2}{3} - \frac{\sin k r}{k r} + \cos k r \right) f r \right\} \quad (b),$$

which is identical with M. Cauchy's equation (15), *Nouveaux Exercices* (Prague), p. 187, and leads immediately to his equation (28), *Exercices d'Analyse*, &c., p. 299. The facility with which I have deduced this equation, is a proof of the utility of the method of proceeding which I adopted in the Memoirs from which the original value of c^2 is extracted.

It is worthy of remark, that, by converting sums into integrals *at once*, we obtain the same result as by the process we have followed. We shall prove this as follows :

Let r be the distance of a particle from the origin, θ the angle between r and the axis of x , ϕ the angle between the planes of x and r and x and z : then the mass of an element is $\rho r^2 \sin \theta d\theta d\phi dr$

$$\therefore \Sigma m \phi r \cos k \delta x = \rho \iiint dr d\theta d\phi r^2 \phi r \sin \theta \cos k \delta x$$

But

$$\delta x = r \cos \theta, \delta y = r \sin \theta \sin \phi$$

$$\therefore \Sigma m \phi r \cos k x = 2\pi \rho \int_0^\pi dr d\theta r^2 \phi r \sin \theta \cos (k r \cos \theta)$$

$$= 2\pi \rho \int dr r \phi r \left(-\frac{\sin k r \cos \theta}{k} + C \right)$$

$$= 4\pi \rho \int dr r \phi r \frac{\sin k r}{k}$$

and

$$\Sigma m \frac{\phi' r}{r} \delta y^2 = \iint_0^\pi \int_0^{2\pi} dr d\theta d\phi \frac{\phi' r}{r} r^4 \sin^3 \theta \sin^2 \phi$$

$$= \pi \rho \iint_0^\pi dr d\theta \phi' r \cdot r^3 \sin^3 \theta$$

$$= \pi \rho \int dr \left(-\cos \theta + \frac{\cos^3 \theta}{3} + C \right) \phi' r r^3$$

$$= \frac{4\pi \rho}{3} \int dr \phi' r r^3$$

$$\Sigma m \frac{\phi' r}{r} \delta y^2 \cos (k \delta x) = \pi \rho \int_0^\pi dr d\theta \phi' r r^3 \sin^3 \theta \cos (k r \cos \theta)$$

$$= \pi \rho \int_{-kr}^{kr} \frac{r^3 \phi' r dr dv}{k r} \left(1 - \frac{v^2}{k^2 r^2} \right) \cos v$$

Now

$$\int v^2 \cos v \, dv = v^2 \sin v + 2 v \cos v - 2 \sin v$$

$$\therefore \sum m \frac{\phi' r}{r} \delta y^2 \cos(k \delta x) = 4 \pi \rho \int \frac{r^2 \phi' r \, dr}{k} \left(-\frac{\cos k r}{k r} + \frac{\sin k r}{k^2 r^2} \right)$$

Substituting all these results in equation (1), it gives

$$c^2 = 4 \pi \rho \int_0^\infty dr \left\{ r^2 \phi r + \frac{\phi' r r^3}{3} - r \phi r \frac{\sin k r}{k} + r \phi' r \frac{\cos k r}{k^2} - \phi' r \frac{\sin k r}{k^3} \right\} \quad (c).$$

This is evidently the same as equation (a).

Finally let us sum this expression for r .

If we integrate, the result is

$$c^2 = \frac{4 \pi \rho}{k^2} \left\{ \frac{k^2 r^2}{3} - \frac{\sin k r}{k r} + \cos k r \right\} f r$$

to be taken between the limits $r=0$, $r=\infty$. For the latter limit the expression *generally* vanishes. Hence

$$\begin{aligned} c^2 &= -\frac{4 \pi \rho}{k^2} \left\{ \frac{k^4 r^4}{1 \cdot 2 \cdot 3 \cdot 4} - \frac{k^4 r^4}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5} \right\} f r_{r=0} \\ &= -\frac{2 \pi \rho}{15} k^2 (r^4 f r)_{r=0} \end{aligned}$$

For one value of $f r$ this process fails, viz. when $f r = r^2$. In this case it is evident that the above expression does not vanish when $r = \infty$. The reason why this is so, is that $r^2 f r$ is a finite quantity. Now, if we return to the original expression for c^2 , we shall find that it is the representation of the *difference* between two terms, which in the operation have assumed different forms. To obtain the correct result, I would suggest that the terms retain both the same form, which they will do, if we write $\cos a \delta x$ instead of 1, and, finally, put $a=0$. We shall then have,

$$c^2 = \sum m \left(\phi r + \frac{\phi' r}{r} \delta y^2 \right) (\cos a \delta x - \cos k \delta x)$$

which, when the law of force is that of the inverse square of the distance becomes,

$$\begin{aligned} c^2 &= \sum \frac{m}{r^3} \left\{ \frac{\sin a r}{a r} - \frac{3 \sin a r}{a^3 r^3} + \frac{3 \cos a r}{a^2 r^2} \right. \\ &\quad \left. - \left(\frac{\sin k r}{k r} - \frac{3 \sin k r}{k^3 r^3} + \frac{3 \cos k r}{k^2 r^2} \right) \right\} \quad (d). \end{aligned}$$

If we now substitute integrals in place of sums, we obtain

$$\begin{aligned} c^2 &= 4 \pi \rho \int_0^\infty dr \left(\frac{\sin a r}{a r^2} - \frac{3 \sin a r}{a^3 r^4} + \frac{3 \cos a r}{a^2 r^3} \right. \\ &\quad \left. - \frac{\sin k r}{k r^2} + \frac{3 \sin k r}{k^3 r^4} - \frac{3 \cos k r}{k^2 r^3} \right) \end{aligned}$$

$$\begin{aligned}
&= 4 \pi \rho \left\{ \left(-\frac{\cos a r}{a^2 r^2} + \frac{\sin a r}{a^3 r^3} + \frac{\cos k r}{k^2 r^2} - \frac{\sin k r}{k^3 r^3} \right)_{r=\infty}^{a=0} \right. \\
&\quad \left. \left(-\frac{\cos a r}{a^2 r^2} + \frac{\sin a r}{a^3 r^3} + \frac{\cos k r}{k^2 r^2} - \frac{\sin k r}{k^3 r^3} \right)_{r=0}^{a=0} \right\} \\
&= 0
\end{aligned}$$

Thus, that portion of the value of c^2 , which we obtain by substituting integrals in place of sums, is zero. This conclusion I arrived at previously, and it appears to be, in every way, conformable with the nature of the function.

Let us now recur to equation (d), and endeavour to obtain the approximate form of c^2 by *summation*. We will suppose (as an approximation merely) that the particles may be regarded as aggregated in spherical surfaces about the molecule under consideration. Let ϵ be the distance between two consecutive particles, $r = n \epsilon$, where n is a number; then $4 \pi n^2$ is the number of particles in a spherical surface, of which the radius is r . Putting, therefore, instead of Σm , $S 4 \pi m n^2$, where S refers to the number n : we get

$$\begin{aligned}
c^2 &= 4 \pi m S \frac{1}{n \epsilon^3} \left\{ \frac{\sin a n \epsilon}{a n \epsilon} - \frac{3 \sin a n \epsilon}{a^3 n^3 \epsilon^3} + \frac{3 \cos a n \epsilon}{a^2 n^2 \epsilon^2} \right. \\
&\quad \left. - \left(\frac{\sin k n \epsilon}{k n \epsilon} - \frac{3 \sin k n \epsilon}{k^3 n^3 \epsilon^3} + \frac{3 \cos k n \epsilon}{k^2 n^2 \epsilon^2} \right) \right\} \\
&= \frac{4 \pi m}{\epsilon^3} S \left\{ \frac{1}{\beta} \left(\frac{\sin \beta n}{n^2} + \frac{3 \sin \beta n}{\beta^2 n^4} + \frac{3 \cos \beta n}{\beta n^3} \right) \right. \\
&\quad \left. - \frac{1}{\alpha} \left(\frac{\sin \alpha n}{n^2} - \frac{3 \sin \alpha n}{\alpha^2 n^4} + \frac{3 \cos \alpha n}{\alpha n^3} \right) \right\}
\end{aligned}$$

where β is written for $a \epsilon$, and α for $k \epsilon$.

$$\begin{aligned}
\therefore c^2 &= \frac{4 \pi m}{\epsilon^3} S \left\{ \beta \frac{d}{d \beta} \left(\frac{\sin \beta n}{\beta^3 n^4} - \frac{\cos \beta n}{\beta^2 n^3} \right) - \alpha \frac{d}{d \alpha} \left(\frac{\sin \alpha n}{\alpha^3 n^4} - \frac{\cos \alpha n}{\alpha^2 n^3} \right) \right\} \\
&= \frac{4 \pi m}{\epsilon^3} S \left\{ \beta \frac{d}{d \beta} \frac{1}{\beta} \frac{d}{d \beta} \frac{-\sin \beta n}{\beta n^4} - \alpha \frac{d}{d \alpha} \frac{1}{\alpha} \frac{d}{d \alpha} \frac{-\sin \alpha n}{\alpha n^4} \right\} \\
&= \frac{4 \pi m}{\epsilon^3} \left\{ \alpha \frac{d}{d \alpha} \left(\frac{1}{\alpha} \frac{d}{d \alpha} \left(\frac{1}{\alpha} S \frac{\sin \alpha n}{n^4} \right) \right) - \beta \frac{d}{d \beta} \left(\frac{1}{\beta} \frac{d}{d \beta} \left(\frac{1}{\beta} S \frac{\sin \beta n}{n^4} \right) \right) \right\}
\end{aligned}$$

The value of this expression depends on the summation of the series $S \frac{\sin \alpha n}{n^4}$ from $n=1$, to $n=\infty$. There is no known process, so far as I am aware, by which this summation can be effected. We can, however, obtain that part of the series which is required for our present purpose by the following process.

$$\begin{aligned}
\sin \alpha + \sin 2 \alpha + \sin 3 \alpha + \&c. &= \frac{1}{2} \cot \frac{\alpha}{2} \\
&= \frac{1}{\alpha} - \frac{\alpha}{12} + \&c. \\
\therefore -\frac{\cos \alpha}{1} - \frac{\cos 2 \alpha}{2} - \frac{\cos 3 \alpha}{3} - \&c. &= \log \alpha - \frac{\alpha^2}{24} + \&c. + C.
\end{aligned}$$

by integration.

$$\text{and} \quad -\frac{\sin \alpha}{1^2} - \frac{\sin 2 \alpha}{2^2} - \frac{\sin 3 \alpha}{3^2} - \&c. = \alpha \log \alpha - \alpha + C \alpha - \frac{\alpha^3}{72} + \&c.$$

which vanishes when $\alpha=0$.

$$\frac{\cos \alpha}{1^3} + \frac{\cos 2 \alpha}{2^3} + \frac{\cos 3 \alpha}{3^3} + \&c. = \frac{\alpha^2}{2} \log \alpha - \frac{\alpha^2}{4} - \frac{\alpha^2}{2} + C \frac{\alpha^2}{2} + D - \frac{\alpha^4}{288} + \&c.$$

$$\text{Finally,} \quad \frac{\sin \alpha}{1^4} + \frac{\sin 2 \alpha}{2^4} + \frac{\sin 3 \alpha}{3^4} + \&c. = \frac{\alpha^3}{6} \log \alpha + A \alpha^3 + D \alpha - \frac{\alpha^5}{1440} + \&c.$$

where A is a constant.

$$\therefore \quad \frac{1}{\alpha} S \frac{\sin \alpha n}{n^4} = \frac{\alpha^2}{6} \log \alpha + A \alpha^2 + D - \frac{\alpha^4}{1440} + \&c.$$

$$\frac{d}{d \alpha} \left(\frac{1}{\alpha} S \frac{\sin \alpha n}{n^4} \right) = \frac{\alpha}{6} + \frac{\alpha}{3} \log \alpha + 2 A \alpha - \frac{\alpha^3}{360} + \&c.$$

$$\frac{d}{d \alpha} \left(\frac{1}{\alpha} \frac{d}{d \alpha} \left(\frac{1}{\alpha} S \frac{\sin \alpha n}{n^4} \right) \right) = \frac{1}{3 \alpha} - \frac{\alpha}{160} + \&c.$$

By substitution, we obtain,

$$\begin{aligned} c^2 &= \frac{4 \pi m}{\epsilon^3} \left\{ \frac{1}{3} - \frac{\alpha^2}{180} - \&c. - \left(\frac{1}{3} - \frac{\beta^2}{180} + \&c. \right) \right\} \\ &= \frac{4 \pi m}{\epsilon^3} \left\{ \frac{\alpha^2}{180} - \&c. \right\}, \text{ for } \beta=0. \end{aligned}$$

$$\text{Hence in value,} \quad c^2 = - \frac{4 \pi m}{\epsilon} \frac{k^2}{180}.$$

$$\text{and} \quad v^2 = - \frac{4 \pi m}{180 \epsilon} = - \frac{\pi m}{45 \epsilon}.$$

This is precisely of the *form* which experiment requires. We learn from the result, too, that m is negative, that is, that the force is *repulsive*.

It will be remarked that whilst approximation, on the hypothesis that the principal effect is due to the particles in the immediate vicinity of that under consideration, gives the same *order* to the first term in c^2 as the process of summation does, it gives it a different *sign*.

This will appear by expanding either equation (1) or equation (a) in terms of k .

$$\begin{aligned} \text{From the former} \quad c^2 &= 2 \sum m \left(\frac{1}{r^3} - \frac{3 \delta y^3}{r^5} \right) \sin^2 \frac{k \delta x}{2} \\ &= 2 \sum m \left(\frac{1}{r^3} - \frac{3 \delta y^2}{r^5} \right) \frac{k^2 \delta x^2}{4} \\ &= \frac{1}{2} \sum m \left(\frac{1}{3 r} - \frac{3}{3 \cdot 5 r} \right) k^2 \\ &= \frac{k^2}{15} \sum \frac{m}{r} \end{aligned}$$

$$\begin{aligned}
\text{From the latter} \quad c^2 &= \Sigma \left(-\frac{\sin k r}{k r^4} + \frac{3}{r^5} \left(\frac{\sin k r}{k^3 r} - \frac{\cos k r}{k^2} \right) \right) \\
&= \Sigma m \left\{ \frac{k^2}{6 r} + \frac{3 k^2}{r} \left(\frac{1}{120} - \frac{1}{24} \right) \right\} \\
&= \Sigma \frac{m}{r} k^2 \left(\frac{1}{6} - \frac{1}{10} \right) \\
&= \frac{k^2}{15} \Sigma \frac{m}{r}.
\end{aligned}$$

It is clear that an approximation of this kind is illusory, inasmuch as $\frac{\Sigma m}{r}$ is infinite. We are not justified, therefore, in arguing any thing relative to the sign of m from this process.

To find the value of M .

$$M = \Sigma m \left(\phi r + \frac{\phi' r}{r} \delta x^2 \right) \sin e \delta x \cos f \delta y + \frac{\cos \phi}{\sin \phi} \Sigma m \frac{\phi' r}{r} \delta x \delta y \cos e \delta x \sin f \delta y$$

taken for *half* the medium.

$$\text{Now} \quad m = \rho r^2 \sin \theta d\theta d\phi dr, \quad \delta x = r \cos \theta, \quad \delta y = r \sin \theta \sin \phi.$$

Instead of $\cos f \delta y$ write 1, and instead of $\cos e \delta x$ write 1, for the other parts of these expressions respectively are very small compared with these, as we have just proved.

$$\text{Now} \quad \Sigma m \phi r \sin e \delta x = \rho \iiint r^2 \phi r \sin \theta \sin (\rho r \cos \theta) dr d\phi d\theta$$

$$= \rho \iint r^2 \phi r \frac{1 - \cos e r}{e r} dr d\phi$$

$$= 2\pi \rho \int r^2 \phi r \frac{1 - \cos e r}{e r} dr$$

$$= \pi \rho \int e r^3 \phi r dr$$

$$\Sigma m \frac{\phi' r}{r} \delta x^2 \sin e \delta x = \iiint \rho r^2 \frac{\phi' r}{r} r^2 \cos^2 \theta \sin \theta \sin (e r \cos \theta) dr d\phi d\theta$$

$$= 2\pi \rho \iint r^3 \phi' r \frac{p^2}{e^3 r^3} \sin p dp dr$$

from $p=0$ to $p=er$

$$= 2\pi \rho \int \frac{r^3 \phi' r dr}{e^3 r^3} \left\{ -2 + 2 \cos e r + 2 e r \sin e r - e^2 r^2 \cos e r \right\}$$

$$= \frac{\pi \rho e}{2} \int r^4 \phi' r dr$$

$$\Sigma m \frac{\phi' r}{r} \delta x \delta y \sin f \delta y = \rho \iiint \frac{\phi' r}{r} r^4 \sin^2 \theta \cos \theta \sin \phi \sin (f r \sin \theta \sin \phi) dr d\theta d\phi$$

Denote $fy + ct$ by θ , and when $x=0$

$$I = i \cos \theta, R = r \cos (\theta + \alpha), R_x = \rho \cos (\theta + \beta)$$

$$T_x = t \cos (\theta + \beta'), R_y = \sigma \cos (\theta + \gamma), T_y = \tau \cos (\theta + \gamma')$$

The equations (1), (2), (4) and (5) become

$$(i \cos \theta - r \cos \overline{\theta + \alpha}) \sin \phi = t \cos (\theta + \beta') - \rho \cos (\theta + \beta) \quad . \quad . \quad . \quad (1)$$

$$(i \cos \theta + r \cos \overline{\theta + \alpha}) \cos \phi = \tau \cos \overline{\theta + \gamma'} - \sigma \cos \overline{\theta + \gamma} \quad . \quad . \quad . \quad (2)$$

$$a \rho \cos \overline{\theta + \beta} + b t \cos \overline{\theta + \beta'} - (i \sin \theta + r \sin \overline{\theta + \alpha}) \sin \phi + \sigma \sin \overline{\theta + \gamma} \tan \phi - \sigma \tau \sin \overline{\theta + \gamma'} \tan \phi = 0 \quad . \quad (3)$$

$$a \sigma \cos \overline{\theta + \gamma} + b \tau \cos \overline{\theta + \gamma'} - (i \sin \theta - r \sin \overline{\theta + \alpha}) \cos \phi + \rho \sin \overline{\theta + \beta} \tan \phi - s t \sin (\theta + \beta') \tan \phi = 0 \quad . \quad (4)$$

Now θ is indeterminate; hence, equating to zero the co-efficients respectively of $\cos \theta$ and of $\sin \theta$ we obtain:

$$(i - r \cos \alpha) \sin \phi = t \cos \beta' - \rho \cos \beta \quad . \quad . \quad . \quad (1)$$

$$r \sin \alpha \sin \phi = -t \sin \beta' + \rho \sin \beta \quad . \quad . \quad . \quad (2)$$

$$(i + r \cos \alpha) \cos \phi = \tau \cos \gamma' - \sigma \cos \gamma \quad . \quad . \quad . \quad (3)$$

$$-r \sin \alpha \cos \phi = -\tau \sin \gamma' + \sigma \sin \gamma \quad . \quad . \quad . \quad (4)$$

$$a \rho \cos \beta + b t \cos \beta' - r \sin \alpha \sin \phi + \sigma \sin \gamma \tan \phi - s \tau \sin \gamma' \tan \phi = 0 \quad . \quad (5)$$

$$-a \rho \sin \beta - b t \sin \beta' - (i + r \cos \alpha) \sin \phi + \sigma \cos \gamma \tan \phi - s \tau \cos \gamma' \tan \phi = 0 \quad (6)$$

$$a \sigma \cos \gamma + b \tau \cos \gamma' + r \sin \alpha \cos \phi + \rho \sin \beta \tan \phi - s t \sin \beta' \tan \phi = 0 \quad . \quad (7)$$

$$-a \sigma \sin \gamma - b \tau \sin \gamma' - (i - r \cos \alpha) \cos \phi + \rho \cos \beta \tan \phi - s t \cos \beta' \tan \phi = 0 \quad (8)$$

Also of i, r' represent the co-efficients of vibration of the incident and reflected wave parallel to the axis of z ; and if the actual vibrations make respectively the angles ω, δ with the plane of xy , $i' = i \tan \omega, r' = r \tan \delta$. If, however, the reflected vibration parallel to z suffer a different retardation from that which is in the place of xy , we must simply denote it by $r' \cos (\theta + \alpha')$. Let also $R_z = \pi \cos (\theta + \epsilon)$, $T_z = \psi \cos (\theta + \epsilon')$. Then we have the following equations:

$$i' - r' \cos \alpha' = \psi \cos \epsilon' - \pi \cos \epsilon \quad . \quad . \quad . \quad (9)$$

$$r' \sin \alpha' = -\psi \sin \epsilon' + \pi \sin \epsilon \quad . \quad . \quad . \quad (10)$$

$$a \pi \cos \epsilon + b \psi \cos \epsilon' - r' \sin \alpha' = 0 \quad . \quad . \quad . \quad (11)$$

$$-a \pi \sin \epsilon - b \psi \sin \epsilon' - (i' + r' \cos \alpha') = 0 \quad . \quad . \quad . \quad (12)$$

These are the twelve equations of motion: and the first eight of them contain ten unknown quantities; the last four, six; which have no apparent dependence on the others. We will first solve them separately by means of particular hypotheses.

We commence with waves in the plane of incidence. 1. Let us suppose that the lost vibrations in the same direction have the same phase, *i. e.*, that $\beta = \beta', \gamma = \gamma'$.

Our equations become

$$\begin{aligned}
 (i-r \cos \alpha) \sin \phi &= (t-\rho) \cos \beta; \quad r \sin \alpha \sin \phi = -(t-\rho) \sin \beta \\
 (i+r \cos \alpha) \cos \phi &= (\tau-\sigma) \cos \gamma; \quad r \sin \alpha \cos \phi = (\tau-\sigma) \sin \gamma \\
 r \sin \alpha \sin \phi &= (a\rho + b t) \cos \beta + (\sigma - s t) \tan \phi \sin \gamma \\
 (i+r \cos \alpha) \sin \phi &= -(a\rho + b t) \sin \beta + (\sigma - s \tau) \tan \phi \cos \gamma \\
 r \sin \alpha \cos \phi &= -(a\sigma + b \tau) \cos \gamma - (\rho - s t) \tan \phi \sin \beta \\
 (i-r \cos \alpha) \cos \phi &= -(a\sigma + b \tau) \sin \gamma + (\rho - s t) \tan \phi \cos \beta
 \end{aligned}$$

By substituting in the last four equations the values of $\cos \beta$, $\cos \gamma$, &c, from the first four, we obtain

$$\begin{aligned}
 r \sin \alpha &= \frac{a\rho + b t}{t-\rho} (i-r \cos \alpha) + \frac{\sigma - s \tau}{\tau - \sigma} r \sin \alpha \\
 (i+r \cos \alpha) &= \frac{a\rho + b t}{t-\rho} r \sin \alpha + \frac{\sigma - s \tau}{\tau - \sigma} (i+r \cos \alpha) \\
 r \sin \alpha &= -\frac{a\sigma + b \tau}{\tau - \sigma} (i+r \cos \alpha) + \frac{\rho - s t}{\tau - \rho} \tan^2 \phi r \sin \alpha \\
 i-r \cos \alpha &= -\frac{a\sigma + b \tau}{\tau - \sigma} r \sin \alpha + \frac{\rho - s t}{t-\rho} \tan^2 \phi (i-r \cos \alpha)
 \end{aligned}$$

Hence

$$\begin{aligned}
 \left(1 - \frac{\sigma - s \tau}{\tau - \sigma}\right) r \sin \alpha &= \frac{a\rho + b t}{t-\rho} (i-r \cos \alpha) \\
 \left(1 - \frac{\sigma - s \tau}{\tau - \sigma}\right) (i+r \cos \alpha) &= \frac{a\rho + b t}{t-\rho} r \sin \alpha
 \end{aligned}$$

$$\therefore i^2 - r^2 \cos^2 \alpha = r^2 \sin^2 \alpha$$

or $i = r$

Moreover, by means of the first four equations we get

$$(i^2 - r^2) \sin \phi \cos \phi = (t-\rho) (\tau-\sigma) \cos (\beta - \gamma)$$

$$\therefore t = \rho \text{ or } \tau = \sigma \text{ or } \beta - \gamma = \frac{\pi}{2}$$

a. If $t = \rho$, $\alpha = 0$

b. If $\tau = \sigma$, $\alpha = \pi$

c. If $\beta - \gamma = \frac{\pi}{2}$, $\sin \beta = \cos \gamma$, $\cos \beta = -\sin \gamma$

and $\frac{\sin \alpha}{1 - \cos \alpha} = -\tan \beta = \cot \gamma = \frac{1 + \cos \alpha}{\sin \alpha}$

$$\therefore \gamma = \frac{\alpha}{2} \text{ and } \beta = \frac{\pi}{2} + \frac{\alpha}{2}$$

For (a.) we get $r=i, \alpha=0, \beta=\frac{\pi}{2}, \gamma=0$

$$\rho = -\frac{4i}{a+b} \sin \phi, \sigma = -\frac{2bi}{a+b} \cos \phi$$

$$t = -\frac{4i}{a+b} \sin \phi, \tau = \frac{2ai}{a+b} \cos \phi$$

For (b.) we get $r=i, \alpha=\pi, \beta=0, \gamma=\frac{\pi}{2}$

$$\sigma = -\frac{2i}{(a+b) \cos \phi}, \rho = -\frac{2bi}{a+b} \sin \phi$$

$$\tau = -\frac{2i}{(a+b) \cos \phi}, t = \frac{2ai}{a+b} \sin \phi$$

For (c.) we get $i=r, \gamma=\frac{\alpha}{2}, \beta=\frac{\pi}{2}+\frac{\alpha}{2}$

which includes both the other cases.

The problem which we have now solved may be considered as the simplest form in which reflection can be produced without refraction. Perhaps, in the application of this theory to light, no instance might be found exactly to correspond with this, but it has the advantage of serving as a first approximation, by means of which the more complete solution of the problem can be arrived at by successive steps. It will appear in the sequel that the solution of the problem is too laborious to invite us to any thing beyond a second approximation. The results of that approximation will, however, we think, be found sufficiently important in themselves to warrant us in proceeding thus far with the solution of our problem.

We now proceed to the discussion of our equations. Let

$$\beta = \beta' + \delta, \gamma = \gamma' + \delta'$$

$$\cos \beta = \cos \beta' + h, \sin \beta = \sin \beta' + e$$

$$\cos \gamma = \cos \gamma' + k, \sin \gamma = \sin \gamma' + l$$

$$\therefore (i - r \cos \alpha) \sin \phi = t \cos \beta' - \rho \cos \beta' - \rho h \quad \text{by (1)}$$

and
$$\cos \beta' = \frac{(i - r \cos \alpha) \sin \phi}{t - \rho} + \frac{\rho h}{t - \rho}$$

Similarly
$$\sin \beta' = -\frac{r \sin \alpha \sin \phi}{t - \rho} + \frac{\rho e}{t - \rho} \quad \text{by (2)}$$

$$\therefore \cos \beta = \frac{(i - r \cos \alpha) \sin \phi}{t - \rho} + \frac{t h}{t - \rho}$$

$$\sin \beta = -\frac{r \sin \alpha \sin \phi}{t - \rho} + \frac{t e}{t - \rho}$$

Also $\cos \gamma = \frac{(i + r \cos \alpha) \cos \phi}{r - \sigma} + \frac{\sigma k}{r - \sigma} \quad \dots \dots \text{by (3)}$

$$\sin \gamma = \frac{r \sin \alpha \cos \phi}{r - \sigma} + \frac{\sigma l}{r - \sigma} \quad \text{by (4)}$$

$$\therefore \cos \gamma = \frac{(i+r \cos \alpha) \cos \rho}{r-\sigma} + \frac{\tau k}{r-\sigma}$$

$$\sin \gamma = \frac{r \sin \alpha \cos \phi}{r - \sigma} + \frac{\tau l}{r - \sigma}$$

From these equations we obtain

$$\rho^2 (e^2 + h^2) + 2 \rho \sin \phi (h i - r \cos \alpha - e r \sin \alpha) =$$

$$t^2 (e^2 + h^2) + 2 t \sin \phi (\overline{h i - r \cos \alpha - e r \sin \alpha})$$

$$\text{or} \quad (t^2 - \rho^2) (e^2 + h^2) + 2 (t - \rho) \sin \phi (\overline{hi - r \cos \alpha} - e r \sin \alpha) = 0$$

and $(\tau^2 - \sigma^2) (l^2 + k^2) + 2 (\tau - \sigma) \cos \phi (\overline{k i + r \cos \alpha} + l r \sin \alpha) = 0$

$$\text{or } \begin{cases} (t+\rho) (e^2+h^2)+2 \sin \phi (\overline{h i-r \cos \alpha}-e r \sin \alpha)=0 & \text{. . . (1')} \\ (\tau+\sigma) (l^2+k^2)+2 \cos \phi (\overline{h i+r \cos \alpha}+l r \sin \alpha)=0 & \text{. . . (2')} \end{cases}$$

$$(\tau + \sigma) (l^2 + k^2) + 2 \cos \phi (\overline{h i + r \cos \alpha} + l r \sin \alpha) = 0 \quad . \quad . \quad (2')$$

By substituting the values of $\cos \beta$, $\cos \beta'$, &c. given above, in the remaining four equations, they become

[illegible]

$$(a \rho + b t) \frac{r \sin \alpha \sin \phi}{t - \rho} - (i + r \cos \alpha) \sin \phi + \frac{(\sigma - s \tau) \sin \phi (i + r \cos \alpha)}{\tau - \sigma} - \frac{a \rho t e + b \rho t e}{t - \rho} + \tan \phi \frac{\sigma \tau k - s \tau \sigma k}{\tau - \sigma} = 0 \quad \dots \dots \dots (6)$$

$$(a \sigma + b \tau) \frac{(i + r \cos \alpha) \cos \phi}{\tau - \sigma} + r \sin \alpha \cos \phi - \frac{(\rho - s t) r \sin \alpha}{t - \rho} \cdot \frac{\sin^2 \phi}{\cos \phi} \\ + \frac{a \sigma \tau k + b \tau \sigma k}{\tau - \sigma} + \tan \phi \frac{\rho t e - s t \rho e}{t - \rho} = 0 \quad \dots \dots \dots (7)$$

[illegible]

From (5) and (6) by eliminating $a \varrho + b t$ we get

$$\begin{aligned} & -r^2 \sin^2 \alpha \sin \phi \frac{(\overline{1+s\tau-2\sigma})}{\tau-\sigma} + (i^2 - r^2 \cos^2 \alpha) \sin \phi \frac{(\overline{1+s\tau-2s})}{\tau-\sigma} \\ & + \frac{(a+b)\rho^\tau}{\tau-\rho} (h r \sin \alpha + e \overline{i-r \cos \alpha}) \\ & + \tan \phi \frac{(1-s)\sigma^\tau}{\tau-\sigma} (l r \sin \alpha - k \overline{i-r \cos \alpha}) = 0 \end{aligned}$$

$$\text{or } (i^2 - r^2) \sin \phi \frac{\overline{1+s\tau-2\sigma}}{\tau-\sigma} + \frac{(a+b)\rho t (h r \sin \alpha + e \overline{i-r \cos \alpha})}{t-\rho} \\ + \tan \phi \frac{(1-s)\sigma\tau}{\tau-\sigma} (l r \sin \alpha - k \overline{i-r \cos \alpha}) = 0 \quad \dots \quad \text{(I)}$$

And by eliminating the second term, we get

$$\frac{a\rho + b t}{t-\rho} (i^2 - r^2) \sin \phi + \frac{(a+b)\rho t}{t-\rho} (h \overline{i+r \cos \alpha} + e r \sin \alpha) \\ + \tan \phi \frac{\sigma\tau}{\tau-\sigma} (1-s) (l \overline{i+r \cos \alpha} - k r \sin \alpha) = 0 \quad \dots \quad \text{(II)}$$

By treating (7) and (8) in a similar manner, we obtain

$$- (i^2 - r^2) \frac{(\overline{t-\rho} \cos^2 \phi + s \overline{t-\rho} \sin^2 \phi)}{(t-\rho) \cos \phi} + \frac{(a+b)\sigma\tau}{\tau-\sigma} (k r \sin \alpha - l \overline{i+r \cos \alpha}) \\ + \tan \phi (1-s) \frac{\rho t}{t-\rho} (e r \sin \alpha + h \overline{i+r \cos \alpha}) = 0 \quad \dots \quad \text{(III)}$$

$$\frac{a\sigma + b\tau}{\tau-\sigma} (i^2 - r^2) \cos \phi + \frac{(a+b)\sigma\tau}{\tau-\sigma} (k \overline{i-r \cos \alpha} - l r \sin \alpha) \\ + \tan \phi (1-s) \frac{\rho t}{t-\rho} (e \overline{i-r \cos \alpha} + h r \sin \alpha) = 0 \quad \dots \quad \text{(IV)}$$

$$\text{From I. } (i^2 - r^2) \sin \phi \frac{(1+s)\tau-2\sigma}{\tau-\sigma} + \frac{(a+b)\rho t (h r \sin \alpha + e \overline{i-r \cos \alpha})}{t-\rho} = 0$$

$$\text{II. } (a\rho + b t) (i^2 - r^2) \sin \phi + (a+b)\rho t (h \overline{i+r \cos \alpha} + e r \sin \alpha) = 0$$

Now, our first approximation gave $\beta = \frac{\pi}{2}$.

Let β then be equal to $\frac{\pi}{2} + \theta$ where θ is a small quantity: $\beta' = \frac{\pi}{2} + \psi$

Hence $\cos \beta = -\sin \theta$, $\cos \beta' = -\sin \psi$

$$\sin \beta = \cos \theta, \sin \beta' = \cos \psi$$

$$\therefore h = -(\sin \theta - \sin \psi) = -(\theta - \psi) + \frac{1}{6} (\theta^3 - \psi^3) \text{ approximately}$$

$$e = \cos \theta - \cos \psi = -\frac{\theta^2 - \psi^2}{2} +$$

$$k = -\frac{\gamma^2 - \gamma'^2}{2} + \dots$$

$$l = (\gamma - \gamma') - \frac{1}{6} (\gamma^3 - \gamma'^3)$$

We return now to our original equations, and determine the values of t , ρ , &c. from them.

By (1) and (2) we obtain

$$t \sin (\beta - \beta') = (i - r \cos \alpha) \sin \phi \sin \beta + r \sin \alpha \sin \phi \cos \beta$$

$$\begin{aligned}
t &= \frac{(i-r \cos \alpha) \cos \theta - r \sin \alpha \sin \theta}{\sin (\theta - \psi)} \sin \phi \\
&= \frac{i \cos \theta - r \cos (\alpha - \theta)}{\sin (\theta - \psi)} \sin \phi \\
&= \frac{i - \frac{i \theta^2}{2} - r + \frac{r (\alpha - \theta)^2}{2}}{\theta - \psi} \left(1 + \frac{\overline{\theta - \psi^2}}{6}\right) \sin \phi \\
&= \sin \phi \frac{i - r - \frac{1}{2} (i \theta^2 - r \overline{\alpha - \theta^2})}{\theta - \psi}
\end{aligned}$$

omitting the product $(i-r) (\theta - \psi)^2$ of three dimensions.

Similarly we obtain

$$\begin{aligned}
\rho &= \frac{i \cos \psi - r \cos (\alpha - \psi)}{\sin (\theta - \psi)} \sin \phi \\
&= \sin \phi \frac{i - r - \frac{1}{2} (i \psi^2 - r \overline{\alpha - \psi^2})}{\theta - \psi} \\
t - \rho &= -\frac{1}{2} \sin \phi (i \overline{\theta + \psi} + r \overline{2 \alpha - \theta - \psi}) \\
&= -\frac{1}{2} \sin \phi (2 \alpha r + i - r \cdot \overline{\theta + \psi}) \\
\frac{\rho t}{t - \rho} &= -\frac{2 \sin \phi}{(\theta - \psi)^2} \frac{(i-r)^2 + (i-r) r \cdot \overline{\alpha^2 - \alpha (\theta + \psi)}}{i (\theta + \psi) + r (2 \alpha - \theta - \psi)} \\
&= -\frac{\sin \phi}{(\theta - \psi)^2} \frac{(i-r)}{\alpha r} \left\{ i - r + i (\alpha^2 - \alpha \overline{\theta + \psi}) - (i-r)^2 \frac{\theta + \psi}{2 \alpha i} \right\}
\end{aligned}$$

to no dimensions,

$$\begin{aligned}
a \rho + b t &= \frac{\sin \phi}{\theta - \psi} \left\{ (a+b) (i-r) + \frac{1}{2} r (\overline{a+b} \alpha^2 - 2 \alpha (\overline{a \psi + b \theta})) \right\} \\
\tau &= \cos \phi \frac{i \gamma - r \overline{\alpha - \gamma} - \frac{1}{6} (i \gamma^3 - r \overline{\alpha - \gamma^3}) + (i \gamma - r \overline{\alpha - \gamma}) \frac{\overline{\gamma - \gamma^2}}{6}}{\gamma - \gamma'} \\
\sigma &= \cos \phi \frac{i \gamma' - r \overline{\alpha - \gamma'} - \frac{1}{6} (i \gamma'^3 - r \overline{\alpha - \gamma'^3}) + (i \gamma' - r \overline{\alpha - \gamma'}) \frac{\overline{\gamma - \gamma'^2}}{6}}{\gamma - \gamma'} \\
\tau - \sigma &= \cos \phi \left\{ i + r - \frac{1}{6} (i \overline{\gamma^2 + \gamma \gamma' + \gamma'^2} + r \overline{3 \alpha^2 - 3 (\gamma + \gamma') \alpha + \gamma^2 + \gamma \gamma' + \gamma'^2}) \right. \\
&\quad \left. + \frac{1}{6} (i+r) (\gamma - \gamma')^2 \right\} \\
\frac{\sigma \tau}{\tau - \sigma} \text{ to no dimensions} &= \cos \phi \frac{(i \gamma - r \overline{\alpha - \gamma}) (i \gamma' - r \overline{\alpha - \gamma'})}{(i+r) (\gamma - \gamma')^2} \\
\frac{(1+s) \tau - 2 \sigma}{\tau - \sigma} &= 2 - (1-s) \frac{\tau}{\tau - \sigma} \\
&= 2 - (1-s) \frac{i \gamma - r (\alpha - \gamma)}{(i+r) (\gamma - \gamma')} \text{ to one dimension.}
\end{aligned}$$

If we substitute these approximations in 1' we obtain, to two dimensions

$$2 \sin \phi \frac{i-r}{\theta-\psi} (\theta-\psi)^2 + 2 \sin \phi \{ -(\theta-\psi)(i-r) \} = 0 \quad \text{an identical equation :}$$

and in (2')

$$\cos \phi \left\{ \frac{i+r \overline{\gamma+\gamma'}-2r\alpha}{\gamma-\gamma'} \right\} (\gamma-\gamma')^2 + 2 \cos \phi \left(-\frac{\gamma^2-\gamma'^2}{2} \overline{i+r} + \overline{\gamma-\gamma'} r \alpha \right) = 0$$

also an identical equation.

If we proceed one dimension higher we get

$$- \sin \phi \frac{i \overline{\theta^2+\psi^2}-r(\overline{\alpha-\theta^2}+\overline{\alpha-\psi^2})}{2(\theta-\psi)} (\theta-\psi)^2 + 2 \sin \phi \left\{ -\frac{r\alpha^2}{2} \overline{\theta-\psi} + \frac{\theta^2-\psi^2}{2} r \alpha \right\} = 0$$

$$\text{or} \quad -\frac{(i-r)(\theta^2+\psi^2)-2r\alpha^2+2ra\overline{\theta+\psi}}{2} - r\alpha^2 + r\alpha(\theta+\psi) = 0$$

which (omitting higher dimensions) is again an identical equation.

Also (2') has no term of the next dimension.

We proceed to make the substitutions in I. II. III. IV.

$$\text{I.} \quad hr \sin \alpha + e(i-r \cos \alpha) = -(\theta-\psi) r \alpha - \frac{\theta^2-\psi^2}{2} (i-r)$$

Hence the equation to two dimensions is

$$(i^2-r^2) \sin \phi \left(2-\overline{1-s} \frac{i\gamma-r\alpha-\gamma}{(i+r)(\gamma-\gamma')} \right) + (a+b) \frac{2 \sin \phi}{\theta-\psi} \left(r\alpha + \frac{i-r}{2} \overline{\theta+\psi} \right) \frac{(i-r)^2-r(i-r)(\alpha^2-\alpha\overline{\theta+\psi})}{i(\theta+\psi)+r(2\alpha-\theta-\psi)} = 0$$

Dividing, we get

$$2(i+r)-(1-s) \frac{i\gamma-r(\alpha-\gamma)}{\gamma-\gamma'} + \frac{a+b}{\theta-\psi} \times \frac{i-r+r\alpha(\alpha-\overline{\theta+\psi}) + \frac{i-r^2(\theta+\psi)}{2ra}}{1 + \frac{i-r}{2ra}(\theta+\psi)} = 0$$

$$\text{or} \quad 2(i+r)-(1-s) \frac{i\gamma-r(\alpha-\gamma)}{\gamma-\gamma'} + \frac{a+b}{\theta-\psi} \left(i-r+r\alpha(\alpha-\overline{\theta+\psi}) \right) = 0$$

But

$$t = -\frac{4i}{a+b} \sin \phi \quad \therefore \quad \frac{4i}{a+b} = -\frac{i-r}{\theta-\psi} \text{ approx.}$$

and

$$r = \frac{2ai}{a+b} \cos \phi \quad \therefore \quad \frac{2a}{a+b} = \frac{2\gamma-\alpha}{\gamma-\gamma'}$$

$$s = -\frac{2bi}{a+b} \cos \phi \quad \therefore \quad -\frac{2b}{a+b} = \frac{2\gamma'-\alpha}{\gamma-\gamma'}$$

If we suppose $b=a$, as is most probably the case, we have

$$0 = 2\gamma + 2\gamma' - 2\alpha$$

$$\therefore \alpha = \gamma + \gamma' \quad . \quad . \quad . \quad . \quad . \quad (a)$$

$$\text{and} \quad i - r = -\frac{2i}{a}(\theta - \psi) \quad . \quad . \quad . \quad (b)$$

$$\text{II.} \quad h(i + r \cos \alpha) + e r \sin \alpha = -\overline{\theta - \psi}(i + r).$$

Hence, to no dimensions we have

$$\begin{aligned} & \frac{i^2 - r^2}{\theta - \psi} \{ \overline{a + b} \overline{i - r} + \frac{1}{2} r (\overline{a + b} \alpha^2 - 2\alpha \overline{a \psi + b \theta}) \} \\ & - \frac{\overline{a + b}}{(\theta - \psi)} \{ \overline{i - r^2} + \overline{i - r} r (\alpha^2 - \alpha \overline{\theta + \psi}) \} \overline{i + r} = 0 \end{aligned}$$

or

$$\begin{aligned} & \overline{a + b} \overline{i - r} + \frac{1}{2} r (\overline{a + b} \alpha^2 - 2\alpha \overline{a \psi + b \theta}) \\ & - \overline{a + b} \{ \overline{i - r} + r \overline{\alpha^2 - \alpha(\theta + \psi)} \} = 0 \\ \therefore \overline{a + b} \alpha^2 - 2\alpha \overline{a \psi + b \theta} &= 2(\alpha^2 - \alpha \overline{\theta + \psi})(a + b) \\ (a + b) \alpha - 2\overline{a \psi + b \theta} &= (2\alpha - 2\overline{\theta + \psi})(a + b) \\ (a + b) \alpha &= 2(-a \psi - b \theta + \overline{a + b} \overline{\psi + \theta}) \\ \therefore (a + b) \alpha &= 2(a \theta + b \psi) \end{aligned}$$

$$\text{If} \quad a = b; \quad \alpha = (\psi + \theta) \quad . \quad . \quad . \quad . \quad . \quad (c).$$

Approximately, then, $\alpha = \gamma + \gamma'$, and $\beta + \beta' - \pi$

$$\text{III.} \quad \frac{(t - \rho) \cos^2 \phi + (s t - \rho) \sin^2 \phi}{t - \rho} = 1 - \frac{(1 - s) t \sin^2 \phi}{t - \rho}.$$

$$\begin{aligned} & i - r + \frac{1}{2} i (\alpha^2 - 2\alpha \theta) \\ -1 &= (1 - s) \sin^2 \phi \frac{\overline{i - r + \frac{1}{2} i (\alpha^2 - 2\alpha \theta)}}{a r (\theta - \psi) \left(1 + \frac{(i - r)(\theta + \psi)}{2 a r} \right)} \\ &= 1 + (1 - s) \sin^2 \phi \frac{i - r + \frac{i}{2} (\alpha^2 - 2\alpha \theta) - \frac{(i - r)^2 (\theta - \psi)}{2 a r}}{a r (\theta - \psi)} \end{aligned}$$

Hence, to one dimension, we have

$$\begin{aligned} & -\frac{(i^2 - r^2)}{\cos \phi} \left\{ 1 + (1 - s) \sin^2 \phi \frac{i - r + \frac{i}{2} (\alpha^2 - 2\alpha \theta) - \frac{(i - r)^2 (\theta + \psi)}{2 a r}}{a r (\theta - \psi)} \right\} \\ & - (a + b) \cos \theta \frac{i^2 (2\gamma - \alpha) (2\gamma' - \alpha)}{\gamma - \gamma'} + \frac{\sin^2 \phi}{\cos \phi} (1 - s) \frac{(i^2 - r^2)}{a r (\theta - \psi)} \times \\ & \left\{ i - r + i \alpha^2 - \alpha \overline{\theta + \psi} - (i - r)^2 \frac{\theta + \psi}{2 a i} \right\} = 0 \end{aligned}$$

$$\begin{aligned}
& \text{or} \quad -\frac{i^2-r^2}{\cos \phi} - (1-s)(i^2-r^2) \frac{\sin^2 \phi}{\cos \phi} \left\{ \frac{\frac{i}{2}(a^2-2a\theta) - \frac{(i-r)^2(\theta+\psi)}{2ai}}{ai(\theta-\psi)} \right\} \\
& \quad - (a+b) \cos \phi \frac{i^2(2\gamma-\alpha)(2\gamma'-\alpha)}{\gamma-\gamma'} + (1-s) \frac{\sin^2 \phi}{\cos \phi} \frac{i^2-r^2}{ai(\theta-\psi)} \\
& \quad \left\{ i(a^2 - a\overline{\theta+\psi}) - (i-r)^2 \frac{\theta+\psi}{2ai} \right\} = 0 \\
& \text{or} \quad -\frac{i^2-r^2}{\cos \phi} + (1-s) \frac{\sin^2 \phi}{\cos \phi} \frac{i^2-r^2}{ai(\theta-\psi)} \left\{ \frac{i}{2} a^2 - i a \psi \right\} + \frac{4ab}{a+b} \cos \phi (\gamma-\gamma') i^2 = 0 \\
& \text{or} \quad \frac{i^2-r^2}{\cos \phi} + (1-s) \frac{\sin^2 \phi}{\cos \phi} \frac{4i^2}{a+b} (a-2\psi) - \frac{4ab}{a+b} \cos \phi (\gamma-\gamma') i^2 = 0
\end{aligned}$$

IV. To two dimensions :

$$\begin{aligned}
& \frac{a(2\gamma - r\overline{a-\gamma'}) + b(i\gamma - r\overline{a-\gamma})}{(\gamma-\gamma')(i+r)} \cos \phi (i^2-r^2) \\
& - (a+b) \cos \phi \frac{(i\gamma - r\overline{a-\gamma})(2\gamma' - r\overline{a-\gamma})}{(i+r)(\gamma-\gamma')} r a \\
& + (1-s) \tan \phi \frac{\sin \phi (i-r)}{a r (\theta-\psi)^2} \left\{ i-r + i(a^2 - a\overline{\theta+\psi}) - (i-r)^2 \frac{\theta+\psi}{2ai} \right\} \\
& \left\{ \frac{(\theta^2 - \psi^2)}{2} (i-r) + (\theta-\psi) r a \right\} = 0
\end{aligned}$$

If $a = b$; $a = \gamma + \gamma'$, $a = \psi + \theta$ (by II.)

$$\theta - \psi = -\frac{a}{2i}(i-r)$$

$$\therefore 2\psi = a + \frac{a}{2i}(i-r)$$

$$a - 2\psi = -\frac{a}{2i}(i-r)$$

$$\text{III. gives} \quad \frac{i^2-r^2}{\cos \phi} - (1-s) \frac{\sin^2 \phi}{\cos \phi} \frac{4i^2}{a-b} \frac{a}{2i}(i-r) - \frac{4ab}{a+b} \cos \phi (\gamma-\gamma') i^2 = 0$$

$$\text{or} \quad \left(i - (1-s) \sin^2 \phi \frac{i}{2} \right) (i-r) = a \cos^2 \phi (\gamma-\gamma')$$

$$\therefore \gamma - \gamma' = \frac{(i-r)}{i a \cos^2 \phi} \left(1 - \frac{1-s}{2} \sin^2 \phi \right)$$

$$\text{or} \quad \gamma - \gamma' = \frac{(i-r)}{i a \cos^2 \phi} \text{ nearly;}$$

$$\gamma + \gamma' = a$$

$$\therefore \gamma = \frac{1}{2} \left(a + \frac{(i-r)}{ai \cos^2 \phi} \right)$$

$$\gamma' = \frac{1}{2} \left(a - \frac{(i-r)}{ai \cos^2 \phi} \right)$$

$$\text{IV. gives } \frac{a(i\gamma' - r\gamma) + i\gamma - r\gamma'}{(\gamma - \gamma')(i + r)} \cos \phi (i^2 - r^2) = a(i - r)^2 \cos \phi \frac{a}{\gamma - \gamma'}$$

$$\frac{(a + b) \cos \phi}{(i + r)(\gamma - \gamma')} (i\gamma - r) \overline{a - \gamma} (i\gamma' - r\overline{a - \gamma'}) r a = -2 a \cos \phi (i + r) r a \frac{(\gamma - \gamma')}{4}$$

$$\therefore a(i - r)^2 \cos^2 \phi \frac{a}{\gamma - \gamma'} + a i^2 a (\gamma - \gamma') \cos^2 \phi - (1 - s) \sin^2 \phi (i - r) \frac{2i}{a} = 0$$

$$\text{or } a^2 i (i - r) \cos^4 \phi . a + i (i - r) a - (1 - s) \frac{2i}{a} (i - r) \sin^2 \phi = 0$$

$$\therefore a = \frac{\frac{2(1-s)}{a} \sin^2 \phi}{a^2 \cos^4 \phi + 1}$$

Finally, I. gives to two dimensions

$$\begin{aligned} & (i^2 - r^2) \left\{ 2 - (1 - s) \frac{i\gamma - r\gamma'}{(i + r)(\gamma - \gamma')} \right\} \\ & + (a + b) \frac{(i - r)}{(\theta - \psi)^2 a r} \left\{ i - r - \frac{(i - r)^2}{2i} \right\} \left\{ (\theta - \psi) r a + \frac{\theta^2 - \psi^2}{2} (i - r) \right\} \\ & + (1 - s) \frac{(i\gamma - r\gamma')(i\gamma' - r\gamma)}{(i + r)(\gamma - \gamma')} r a = 0 \\ & 2(i^2 - r^2) - (1 - s)(i^2 - r^2) \left(\frac{1}{2} + \frac{a a \cos^2 \phi}{4} \right) \\ & - \frac{(a + b)(i - r)}{a + b} 4i \left\{ 1 - \frac{i - r}{2i} \right\} \left\{ 1 + \frac{i - r}{2r} \right\} \\ & - \frac{(1 - s)}{2} i^2 a \frac{(i - r)}{a i \cos^2 \phi} = 0 \end{aligned}$$

$$\text{or } 2(i + r) - \frac{(i + r)^2}{r} - (1 - s) \frac{i + r}{2} - (1 - s) a \left\{ \frac{a i \cos^2 \phi}{2} + \frac{i}{2 a \cos^2 \phi} \right\} = 0$$

$$-2(i - r) - (1 - s)i = (1 - s) a \left\{ \frac{a i \cos^2 \theta}{2} + \frac{i}{2 a \cos^2} \right\}$$

$$i - r = -\frac{1}{2}(1 - s)i - \frac{1 - s}{2} i a \frac{a^2 \cos^4 \phi + 1}{2 a \cos^2 \phi}$$

$$= -\frac{1}{2}(1 - s)i - \frac{(1 - s)^2}{2 a^2} i \tan^2 \phi$$

$$= \frac{1}{2} m i + \frac{m^2}{2 a^2} i \tan^2 \phi \quad (m = s - 1)$$

$$r = i \left\{ 1 - \frac{1}{2} m + \frac{m^2}{2 a^2} \tan^2 \phi \right\}$$

But if a be not equal to b , we have the following equations:

$$\theta - \psi = -\frac{a + b}{4i} (i - r) \quad . \quad . \quad . \quad . \quad (1.)$$

$$\frac{2\gamma - a}{\gamma - \gamma'} = \frac{2a}{a + b} \quad . \quad . \quad . \quad . \quad (2.)$$

$$\frac{2\gamma - \alpha}{\gamma - \gamma'} = -\frac{2b}{a + b} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3.)$$

$$2(i+r) - (1-s) \frac{i\gamma - r(\alpha - \gamma)}{\gamma - \gamma'} + \frac{a+b}{\theta + \psi} \left(i - r + r\alpha(\alpha - \overline{\theta + \psi}) \right) = 0 \quad (4.)$$

$$(a+b) \alpha = 2(a \theta + b \psi) \quad (5.)$$

$$i^2 - r^2 + (1-s) \sin^2 \phi \frac{4 i^2}{a+b} (\alpha - 2 \psi) - \frac{4 a b}{a+b} \cos^2 \phi i^2 (\gamma - \gamma') = 0 \quad . \quad (6.)$$

and IV. . . . . (7.)

From (1.) and (5.),

$$2(a+b)\theta = (a+b)\alpha - \frac{(a+b)}{2i}b(i-r)$$

$$\theta = \frac{\alpha}{2} - \frac{b}{4i}(i-r)$$

$$\psi = \frac{a}{2} + \frac{a}{4i}(i-r)$$

From (2.) and (3.), $b\gamma + a\gamma' = \frac{a+b}{2}a$

$$\text{From (6.),} \quad \gamma - \gamma' = \frac{i^2 - r^2 - (1-s) \frac{4i^2}{a+b} \frac{a}{2i} (i-r) \sin^2 \phi}{\frac{4ab}{a+b} \cos^2 \phi i^2}$$

$$= \frac{a+b}{4ab} \frac{i^2 - r^2}{i^2 \cos^2 \phi} \text{ nearly}$$

$$= \frac{(a+b)(i-r)}{2ab i \cos^2 \phi} \text{ nearly}$$

$$\therefore \gamma = \frac{a}{2} + \frac{a}{2abi \cos^2 \phi} (i-r) = \frac{a}{2} + \frac{i-r}{2bi \cos^2 \phi}$$

$$\gamma = \frac{a}{2} - \frac{b}{2abi\cos^2\phi}(i-r) - \frac{a}{2} - \frac{i-r}{2ai\cos^2\phi}$$

$$\text{IV. gives } \frac{\frac{a+b}{2} \alpha (i+r) - (a+b) r \alpha}{\frac{(a+b)(i^2-r^2)}{2 a b i \cos^2 \phi}} \cos \phi (i^2-r^2)$$

$$+ \frac{(a+b) \cos \phi \frac{i^2 r}{i+r} a \frac{(i-r)^2}{a b i^2 \cos^4 \phi}}{\frac{(a+b)(i-r)}{2 a b i \cos^2 \phi}} - (1-s) \frac{\sin^2 \phi}{\cos \phi} \frac{4 i}{a+b} = 0$$

or
$$a b i \cos^2 \phi \alpha (i-r) + \frac{\alpha (i-r) i}{\cos^2 \phi} - (1-s) \frac{\sin^2 \phi}{\cos^2 \phi} \frac{4 i}{a+b} (i-r) = 0$$

$$a = \frac{(1-s) \frac{4}{a+b} \sin^2 \phi}{ab \cos^4 \phi + 1}$$

Equation I. gives

$$\begin{aligned} & (i^2 - r^2) \left\{ 2 - (1-s) \frac{i\gamma - r(\alpha - \gamma)}{(i+r)(\gamma - \gamma')} \right\} \\ & + (a+b) \frac{i-r}{(\theta - \psi)^2} \alpha r \left\{ i - r + i\alpha(\alpha - \theta - \psi) - \frac{(i-r)^2(\theta + \psi)}{2\alpha i} \right\} \left\{ (\theta + \psi)ra + \frac{\theta^2 - \psi^2}{2}(i-r) \right\} \\ & + (1-s) \frac{\{i\gamma - r(\alpha - \gamma)\} \{i\gamma' - r(\alpha - \gamma')\}}{(i+r)(\gamma - \gamma')^2} (\gamma - \gamma') r \alpha = 0 \end{aligned}$$

Now

$$\begin{aligned} \frac{i\gamma - r(\alpha - \gamma)}{(i+r)(\gamma - \gamma')} &= \frac{(i-r)\frac{\alpha}{2} + (i+r)\frac{i-r}{2bi\cos^2\phi}}{(i+r)(a+b)\frac{i-r}{2abi\cos^2\phi}} \\ &= \frac{\alpha abi\cos^2\phi}{(a+b)(i+r)} + \frac{a}{a+b} \\ \frac{i\gamma' - r(\alpha - \gamma')}{(i+r)(\gamma - \gamma')} &= \frac{\alpha abi\cos^2\phi}{(a+b)(i+r)} - \frac{b}{a+b} \\ \therefore \frac{\{i\gamma - r(\alpha - \gamma)\} \{i\gamma' - r(\alpha - \gamma')\}}{(i+r)^2(\gamma - \gamma')^2} &= -\frac{ab}{a+b} + \frac{a-b}{(a+b)^2} \frac{\alpha abi\cos^2\phi}{i+r} \end{aligned}$$

Substituting, we obtain

$$\begin{aligned} & 2(i^2 - r^2) - (1-s) \frac{a}{a+b} (i^2 - r^2) - (1-s) \frac{ab}{a+b} i(i-r) \alpha \cos^2\phi \\ & - (a+b) \frac{i-r}{\frac{a+b}{4i}(i-r)} \left\{ i - r - i\alpha \frac{a-b}{4i}(i-r) - \frac{(i-r)^2}{2i} - \frac{(a-b)(i-r)^3}{8i^2\alpha} \right\} \\ & \quad \times \left\{ \frac{i+r}{2r} + \frac{(a-b)}{8ir}(i-r)^2 \right\} \\ & + (1-s) \frac{(a+b)(i^2 - r^2)}{2abi\cos^2\phi} r \alpha \left\{ -\frac{ab}{(a+b)^2} + \frac{a-b}{(a+b)^2} \frac{\alpha abi\cos^2\phi}{i+r} \right\} = 0 \end{aligned}$$

Taking this to two dimensions, to which alone it is correct, we have

$$\begin{aligned} & 2(i^2 - r^2) - (1-s) \frac{a}{a+b} (i^2 - r^2) - (1-s) \frac{ab}{a+b} i(i-r) \alpha \cos^2\phi \\ & - \frac{2i}{r} (i^2 - r^2) + (i-r)^2 \frac{i+r}{r} \\ & - \frac{(1-s)}{2} \frac{(i^2 - r^2) r \alpha}{(a+b)i\cos^2\phi} = 0 \end{aligned}$$

or

$$\begin{aligned} & -2(i-r) - (1-s) \frac{ab}{a+b} 2i - (1-s) \frac{ab}{a+b} i \alpha \cos^2\phi \\ & - \frac{1-s}{(a+b)} \frac{i\alpha}{\cos^2\phi} = 0 \end{aligned}$$

$$\begin{aligned}
i - r &= - (1-s) \frac{a i}{a+b} - \frac{(1-s) a i}{2} \left\{ \frac{a b}{a+b} \cos^2 \phi + \frac{1}{(a+b) \cos^2 \phi} \right\} \\
r &= i \left\{ 1 - (s-1) \frac{a}{a+b} + \frac{(s-1)^2}{a(a+b)} \tan^2 \phi \right\} \\
&= i \left\{ 1 - \frac{m a}{a+b} + \frac{m^2}{a(a+b)} \tan^2 \phi \right\}
\end{aligned}$$

If $a-b$ be not supposed small, there is another term, which is introduced thus :

$$\begin{aligned}
&2(i^2 - r^2) + \text{\&c.} + \left\{ i a (i - r) + \frac{(i-r)^3}{2 i a} \right\} (a-b) \\
\text{or} \quad &-2(i-r) + \text{\&c.} + (a-b) \left\{ i \frac{4 \sin^2 \phi}{a+b} (1-s) + \frac{(i-r)^2 (a b \cos^4 \phi + 1)}{8(1-s) \sin^2 \phi} \right\} \\
r &= i \left\{ 1 - \frac{m a}{a+b} + \frac{m^2}{a(a+b)} \tan^2 \phi + (a-b) \left(\frac{m}{a b \cos^4 \phi + 1} + \frac{m a^2 (a b \cos^4 \phi + 1)}{(a+b) 16 \sin^2 \phi} \right) \right\}
\end{aligned}$$

Thus the reflection increases with the angle of incidence, so far as this approximation proceeds.

It is not difficult to work out the solution for the vibrations parallel to the surface.

Should the conclusions to be derived from the comparison of the two results appear to give the law of elliptic polarization, I shall recur to the subject at some future time.