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XIX. On the instability of cylindrical fluid surfaces

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XIX. *On the Instability of Cylindrical Fluid Surfaces.* *By* Lord Rayleigh, *Sec. R.S.**

In former papers† I have investigated the character of the equilibrium of a cylindrical fluid column under the action of capillary force. If the column become varicose with wave-length λ, the equilibrium is unstable, provided λ exceed the circumference ($2\pi a$) of the cylinder; and the degree of instability, as indicated by the value of q in the exponential e^{qt} to which the motion is proportional, depends upon the value of λ, reaching a maximum when $\lambda = 4{\cdot}51 \times 2a$. In these investigations the external pressure is supposed to be constant; and this is tantamount to neglecting the inertia of the surrounding fluid.

When a column of liquid is surrounded by air, the neglect of the inertia of the latter will be of small importance; but there are cases where the situation is reversed, and where it is the inertia of the fluid outside rather than of the fluid inside the cylinder which is important. The phenomenon of the disruption of a jet of air delivered under water, easily illustrated by instantaneous photography, suggests the consideration of the case where the inside inertia may be neglected; and to this the present paper is specially directed. For the sake of comparison the results of the former problem are also exhibited.

Since the fluid is supposed to be inviscid, there is a velocity-potential, proportional to e^{ikz} as well as to e^{qt}, and satisfying the usual equation

$$\frac{d^2\phi}{dr^2} + \frac{1}{r}\frac{d\phi}{dr} - k^2\phi = 0. \quad . \quad . \quad . \quad . \quad . \quad (1)$$

If the fluid under consideration is inside the cylinder, the appropriate solution of (1) is

$$\phi = \mathrm{J}_0(ikr) = \mathrm{I}_0(kr); \quad . \quad . \quad . \quad . \quad . \quad (2)$$

and the final result for q^2 is

$$q^2 = \frac{\mathrm{T}}{\rho a^3}\frac{(1-k^2a^2)\,ika\,.\,\mathrm{J}_0'(ika)}{\mathrm{J}_0(ika)}$$

$$= \frac{\mathrm{T}}{\rho a^3}\frac{(1-k^2a^2)\,ka\,\mathrm{I}_0'(ka)}{\mathrm{I}_0(ka)}, \quad . \quad . \quad . \quad . \quad (3)$$

* Communicated by the Author.

† (1) "On the Instability of Jets," Math. Soc. Proc. November 1878.
(2) "On the Capillary Phenomena of Jets," Proc. Roy. Soc. May 1879.
(3) "On the Instability of a Cylinder of Viscous Liquid under Capillary Force," *suprà*, p. 145.

in which T represents the capillary tension, ρ the density, and, as usual,

$$I_0(x)=J_0(ix)=1+\frac{x^2}{2^2}+\frac{x^4}{2^2.4^2}+\frac{x^6}{2^2.4^2.6^2}+\ldots, \quad (4)$$

$$I_1(x)=iJ_0'(ix)=\frac{x}{2}+\frac{x^3}{2^2.4}+\frac{x^5}{2^2.4^2.6}+\ldots \quad . \quad (5)$$

But if the fluid be outside the cylinder, we have to use that solution of (1) for which the motion remains finite when $r=\infty$. This may be expressed in two ways*. When r is great we have the semi-convergent form

$$\phi=-\left(\frac{\pi}{2kr}\right)^{\frac{1}{2}}e^{-kr}\left\{1-\frac{1^2}{1.8kr}+\frac{1^2.3^2}{1.2.(8kr)^2}\right.$$
$$\left.-\frac{1^2.3^2.5^2}{1.2.3\,(8kr)^3}+\ldots\right\}, \quad . \quad . \quad (6)$$

and for all values of r the fully convergent series

$$\phi=(\gamma+\log\tfrac{1}{2}kr)\,I_0(kr)-\frac{k^2r^2}{2^2}S_1-\frac{k^4r^4}{2^2.4^2}S_2-\ldots. \quad . \quad (7)$$

in which γ is Euler's constant, equal to $\cdot5772\ldots$, and

$$S_n=1+\frac{1}{2}+\frac{1}{3}+\ldots+\frac{1}{n}. \quad . \quad . \quad . \quad . \quad (8)$$

In this case the solution of the problem becomes

$$q^2=\frac{T}{\rho a^3}\,\frac{(k^2a^2-1)\,ka\,\phi'(ka)}{\phi(ka)}, \quad . \quad . \quad . \quad . \quad (9)$$

ϕ being defined by (7). In (9) ρ represents the inertia of the external fluid, that of the internal fluid being neglected, while in the corresponding formula (3) ρ is the inertia of the internal fluid, that of the external fluid being neglected. There would be no difficulty in writing down the analytical solution applicable to the more general case where both densities are regarded as finite.

The accompanying Table gives the values of

$$\left\{\frac{(1-x^2)xI_1(x)}{I_0(x)}\right\}^{\frac{1}{2}}, \quad . \quad . \quad . \quad . \quad . \quad . \quad (10)$$

to which q in (3) is proportional, and of

$$\left\{\frac{(x^2-1)\,x\,\phi'(x)}{\phi(x)}\right\}^{\frac{1}{2}}, \quad . \quad . \quad . \quad . \quad . \quad . \quad (11)$$

* See the writings of Sir G. Stokes; or 'Theory of Sound,' § 341.

corresponding in a similar manner to (9). In the second case we have

$$\phi(x) = (\gamma + \log \tfrac{1}{2} x)\, \mathrm{I}_0(x)$$
$$- \frac{x^2}{2^2} \mathrm{S}_1 - \frac{x^4}{2^2 . 4^2} \mathrm{S}_2 - \frac{x^6}{2^2 . 4^2 . 6^2} \mathrm{S}_3 - \dots, \quad . \quad (12)$$

$$x\phi'(x) = \mathrm{I}_0(x) + (\gamma + \log \tfrac{1}{2} x)\, x\, \mathrm{I}_1(x)$$
$$- \frac{x^2}{2} \mathrm{S}_1 - \frac{x^4}{2^2 . 4} \mathrm{S}_2 - \frac{x^6}{2^2 . 4^2 . 6} \mathrm{S}_3 - \dots . \quad . \quad (13)$$

x.	$\mathrm{I}_0(x)$.	$x\,\mathrm{I}_1(x)$.	(10).	$-\phi(x)$.	$x\phi'(x)$.	(11).
0·0	1·0000	·0000	·0000	∞	1·0000	·0000
0·1	1·0025	·0050	·0703	2·4270	·9854	·6339
0·2	1·0100	·0201	·1382	1·7527	·9551	·7233
0·3	1·0226	·0455	·2012	1·3724	·9169	·7795
0·4	1·0404	·0816	·2567	1·1146	·8738	·8113
0·5	1·0635	·1289	·3015	·9244	·8283	·8198
0·6	1·0920	·1882	·3321	·7774	·7817	·8022
0·7	1·1264	·2603	·3433	·6607	·7353	·7535
0·8	1·1665	·3463	·3269	·5654	·6894	·6625
0·9	1·2130	·4474	·2647	·4869	·6449	·5017
1·0	......	......	·0000	......	......	·0000

On account of the factor $(1-x^2)$ both (10) and (11) vanish when $x=0$ and when $x=1$. Beyond $x=1$, (10), (11) become imaginary, indicating stability. It will be seen that when the fluid is internal the instability is a maximum between $x=·6$ and $x=·7$; and when the fluid is external, between $x=·4$ and $x=·5$. That the maximum instability would correspond to a longer wave-length in the case of the external fluid might have been expected, in view of the greater room available for the flow. The same consideration also explains the higher maximum attained by (11) than by (10).

In order the better to study the region of the maximum, the following additional values have been calculated by the usual bisection formula

$$\frac{q+r}{2} + \frac{q+r-(p+s)}{16}.$$

x.	(10).	x.	(11).
·65	·3406	·45	·8186
·70	·3433	·50	·8198
·75	·3397	·55	·8147

The value of x for which (10) is a maximum may now be found from Lagrange's interpolation formula. It is

$$x = \cdot 696,$$

corresponding to

$$\lambda = 2a \times \pi / x = 4\cdot 51 \times 2a, \quad . \quad . \quad . \quad . \quad (14)$$

and agreeing with the value formerly obtained by a different procedure.

In like manner we get for the value of x giving maximum instability in the case of the external fluid,

$$x = \cdot 485,$$

and

$$\lambda = 6\cdot 48 \times 2a. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (15)$$

Some numerical examples applicable to the case of water were given in a former paper. It appeared that for a diameter of one millimetre the disturbance of maximum instability is multiplied 1000-fold in about one-fortieth of a second of time. This is for the case of internal fluid. If the fluid were external, the amplification in the same time would be more than one-million-fold.

Terling Place, Witham,
July 2.

XX. *Breath Figures*.

By W. B. Croft, *M.A., Winchester College**.

FIFTY years back Prof. Karsten, of Berlin, placed a coin upon glass, and by electrifying it made a latent impression, which revealed itself when breathed upon. About the same time Mr. W. R. (now Sir W. R.) Grove made similar impressions with simple paper devices, and fixed them so as to be always visible. A discussion of Karsten's results occurs in several places, but I have not been able to find details of his method of performing the experiment. During my attempts to repeat it some effects have appeared which seem to be new and worthy of record.

After many trials I found the following method the most successful:—A glass plate, 6 inches square, is put on the table for insulation: in the middle lies a coin with a strip of tinfoil going from it to the edge of the glass: on this coin lies the glass to be impressed, 4 or 5 inches square, and above it a second coin. It is essential to polish the glass scrupulously

* Communicated by the Physical Society: read June 24, 1892.