184. The Third Approximation to the n^{th} root of a Number

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assume too much, and neither makes it very clear precisely what he is assuming. On the other hand both authors handle the troublesome (though of course not really difficult) subject of "differentials" in a lucid and unobjectionable way. G. H. HARDY.

MATHEMATICAL NOTES.

183. [P. 3. b.] The name of the author of the note on Inversion in the *Gazette*, No. 47, p. 88, was accidentally omitted. The note was by Mr. V. Ramaswami Aiyar, of Arni, India. For want of space we were compelled to omit certain applications and a further development of the theorem.

184. [I. 1.] *The third approximation to the n^{th} root of a number.*

Mr. C. S. Jackson and Mr. F. J. W. Whipple have given investigations (in the May and December numbers of the *Gazette*) for the approximation*

$$(1+x)^{\frac{1}{n}} = (1+px)/(1+qx),$$

where x is small and $p = \frac{1}{2}(1+1/n)$, $q = \frac{1}{2}(1-1/n)$.

The following method follows the lines which I have suggested (*Gazette*, October, 1904) for dealing with some of the elementary power-series.

$$\text{Write } y = (1+x) - [(1+px)/(1+qx)]^n,$$

$$\text{then } \frac{dy}{dx} = 1 - n(p-q) \frac{(1+px)^{n-1}}{(1+qx)^{n+1}};$$

so that $\frac{dy}{dx}$ vanishes with x , if $n(p-q) = 1$.

$$\text{Then } \frac{d^2y}{dx^2} = \frac{(1+px)^{n-1}}{(1+qx)^{n+1}} \left[\frac{(n+1)q}{1+qx} - \frac{(n-1)p}{1+px} \right],$$

which also vanishes with x , if $(n-1)p = (n+1)q$.

Thus $p = \frac{1}{2}(1+1/n)$, $q = \frac{1}{2}(1-1/n)$, as given above; with these values we find

$$\frac{d^2y}{dx^2} = \frac{n^2-1}{2n^2} x \frac{(1+px)^{n-2}}{(1+qx)^{n+2}}.$$

If we differentiate $(1+px)^{n-2}(1+qx)^{-(n+2)}$, it will be found that the result is $-(1+px)^{n-3}(1+qx)^{-(n+3)}(1+4pqx)$.

Thus $(1+px)^{n-2}(1+qx)^{-(n+2)}$ decreases as x increases from 0; and consequently is less than 1 for positive values of x .

$$\text{That is, } \frac{d^2y}{dx^2} < \frac{n^2-1}{2n^2} x \quad \text{or} \quad \frac{d}{dx} \left(\frac{dy}{dx} - \frac{n^2-1}{4n^2} x^2 \right) < 0, \quad x \text{ being positive}$$

Now $\frac{dy}{dx} - \frac{n^2-1}{4n^2} x^2$ is zero for $x=0$, and so is negative for positive values of x .

$$\text{That is, } \frac{d}{dx} \left(y - \frac{n^2-1}{12n^2} x^3 \right) < 0,$$

and by the same argument,

$$y - \frac{n^2-1}{12n^2} x^3 < 0.$$

Similarly, since $\frac{d^2y}{dx^2}$ is positive for positive values of x , we infer that y is positive; so that

$$0 < y < \frac{1}{12}(1-1/n^2)x^3, \text{ if } x \text{ is positive.}$$

* To reduce this to the form given, write $x = N/a^n$, and multiply both sides by a .

Since $(1+px)/(1+qx)$ is greater than 1, it now follows that *

$$0 < (1+x)^{\frac{1}{n}} - (1+px)/(1+qx) < y/n;$$

so that the error in the approximation (for positive values of x) is less than

$$\frac{1}{12}(n^2-1)x^3/n^3,$$

provided n has any rational value greater than 1.

Mr. Jackson gave this result as an approximation to the error, but without noticing that the estimated error was really in excess of the true error, for positive values of x .

In the special cases $n=2, 3$ (which are those of chief interest, perhaps), it is easy to verify by direct multiplication that

$$(1+x)(1+\frac{1}{4}x)^2 - (1+\frac{3}{4}x)^2 = \frac{1}{16}x^3 < \frac{1}{16}x^3(1+\frac{1}{4}x)^2,$$

$$(1+x)(1+\frac{1}{3}x)^3 - (1+\frac{2}{3}x)^3 = \frac{1}{27}(2x^3+x^4) < \frac{1}{27}x^3(1+\frac{1}{3}x)^3,$$

leading to

$$(1+x)^{\frac{1}{2}} - (1+\frac{3}{4}x)/(1+\frac{1}{4}x) < \frac{1}{32}x^3,$$

$$(1+x)^{\frac{1}{3}} - (1+\frac{2}{3}x)/(1+\frac{1}{3}x) < \frac{1}{81}x^3.$$

These are the same results as were found above by the aid of the calculus; but to extend the algebraic method to any integral value of n seems rather laborious, although some of the transformations are interesting.

It may be worth while to recall the fact that Dedekind makes use of this approximation (for $n=2$) in his theory of irrational numbers.

T. J. F'a. BROMWICH.

185. [A. 1. c.] *Note on a point in the demonstration of the Binomial Theorem.*

When x is numerically less than 1, the series $1+mx+\frac{m(m-1)}{1 \cdot 2}x^2+\dots$ is absolutely convergent for all values of m , and denoting it by $f(m)$ we establish the addition theorem that $f(m) \cdot f(n) = f(m+n)$.

The Binomial Theorem is, that $f(m)$ is, for all values of m , equal to $(1+x)^m$; that is, to the positive quantity which, under the theory of Indices, is represented by the expression $(1+x)^m$.

Usually, the theorem is first shown to be true for positive integral values of m , and then it is considered for positive fractional values, and finally for negative values of m .

In dealing with the case when m is a positive fraction $=\frac{p}{q}$, what is usually done is, after proving that the q th power of $f(\frac{p}{q})$ is equal to $(1+x)^p$, it is quietly assumed that $f(\frac{p}{q}) = (1+x)^{\frac{p}{q}}$. But if q be an even integer, might not $f(\frac{p}{q})$ be rather equal to $-(1+x)^{\frac{p}{q}}$?

This difficulty (so far as I know) is raised and discussed only in *Chrystal*; but the discussion there is by no means elementary. The point, however, admits of an elementary treatment, by taking the cases as follows.

CASE I. If m is a positive integer, we have $f(m) = \{f(1)\}^m = (1+x)^m$, and there is no ambiguity.

CASE II. If m is a negative integer, $=-n$ say, we have

$$f(-n) \cdot f(n) = f(0) = 1;$$

* If a, b, n are all greater than 1, $a^n - b^n > n(a-b)$. This is obvious on division if n is an integer; if n is not an integer we have

$$\frac{d}{dx}(x^n - 1) = nx^{n-1} > n$$

if $x > 1$. Thus $x^n - 1 > n(x-1)$. Write $x=a/b$, and the result follows.