from Everett* the values $n = 8.19 \times 10^{11}$, $\rho = 7.85$, in C. G. S. measure, I find that the frequency $\tau^{-1} = 136000$, about. For a steel globe of any other dimensions, this result must be divided by the radius in centimetres. For a globe of the size of the earth [$a = 6.37 \times 10^8$], I find that the period $\tau = 1$ hr. 18 m.†

On some Formulæ arising from the Differentiation of Elliptic Functions with regard to the Modulus. By Rev. M. M. U. WILKINSON.

[Read June 8th, 1882.]

1. If we take the identity,

$$2 \operatorname{cn}(a-\beta) \operatorname{cn}(\beta-\gamma) \operatorname{cn}(\gamma-a) = 2 - \operatorname{sn}^2(a-\beta) - \operatorname{sn}^2(\beta-\gamma) - \operatorname{sn}^2(\gamma-a) \\ + k^2 \operatorname{sn}^2(a-\beta) \operatorname{sn}^2(\beta-\gamma) \operatorname{sn}^2(\gamma-a),$$

and differentiate with respect to k , we have

$$2k \operatorname{sn}^2(a-\beta) \operatorname{sn}^2(\beta-\gamma) \operatorname{sn}^2(\gamma-a) \\ = \{-2 \operatorname{sn}(a-\beta) \operatorname{cn}(\beta-\gamma) \operatorname{cn}(\gamma-a) + 2 \operatorname{sn}(a-\beta) \operatorname{cn}(a-\beta) \\ - 2k^2 \operatorname{sn}(a-\beta) \operatorname{cn}(a-\beta) \operatorname{sn}^2(\beta-\gamma) \operatorname{sn}^2(\gamma-a)\} \frac{d \cdot \operatorname{am}(a-\beta)}{dk} \\ = 2 \{-\operatorname{dn}(a-\beta) - k^2 \operatorname{sn}(\beta-\gamma) \operatorname{sn}(\gamma-a)\} \\ \times \operatorname{sn}(a-\beta) \operatorname{sn}(\beta-\gamma) \operatorname{sn}(\gamma-a) \frac{d \cdot \operatorname{am}(a-\beta)}{dk} \\ = -2 \operatorname{dn}(\beta-\gamma) \operatorname{dn}(\gamma-a) \operatorname{sn}(a-\beta) \operatorname{sn}(\beta-\gamma) \operatorname{sn}(\gamma-a) \frac{d \cdot \operatorname{am}(a-\beta)}{dk};$$

whence

$$\frac{1}{\operatorname{dn}(a-\beta)} \frac{d \cdot \operatorname{am}(a-\beta)}{dk} + \frac{1}{\operatorname{dn}(\beta-\gamma)} \frac{d \cdot \operatorname{am}(\beta-\gamma)}{dk} + \frac{1}{\operatorname{dn}(\gamma-a)} \frac{d \cdot \operatorname{am}(\gamma-a)}{dk} \\ = - \frac{k \operatorname{sn}(a-\beta) \operatorname{sn}(\beta-\gamma) \operatorname{sn}(\gamma-a)}{\operatorname{dn}(a-\beta) \operatorname{dn}(\beta-\gamma) \operatorname{dn}(\gamma-a)}.$$

2. It follows from this, that if

$$x_1 + x_2 + x_3 + x_4 = 0,$$

* *Units and Physical Constants*, p. 53.

† Cf. Thomson, *Phil. Trans.*, 1863, p. 573.

we have

$$\begin{aligned}
 & \frac{1}{\operatorname{dn} x_1} \frac{d \cdot \operatorname{am} x_1}{dk} + \frac{1}{\operatorname{dn} x_2} \frac{d \cdot \operatorname{am} x_2}{dk} + \frac{1}{\operatorname{dn} x_3} \frac{d \cdot \operatorname{am} x_3}{dk} + \frac{1}{\operatorname{dn} x_4} \frac{d \cdot \operatorname{am} x_4}{dk} \\
 &= \frac{k \operatorname{sn} x_1 \operatorname{sn} x_2 \operatorname{sn} (x_1 + x_2)}{\operatorname{dn} x_1 \operatorname{dn} x_2 \operatorname{dn} (x_1 + x_2)} + \frac{k \operatorname{sn} x_2 \operatorname{sn} x_3 \operatorname{sn} (x_2 + x_3)}{\operatorname{dn} x_2 \operatorname{dn} x_3 \operatorname{dn} (x_2 + x_3)} \\
 &= \frac{k \operatorname{sn} (x_1 + x_2)}{\operatorname{dn} (x_1 + x_2)} \left\{ \frac{\operatorname{sn} x_1 \operatorname{sn} x_2}{\operatorname{dn} x_1 \operatorname{dn} x_2} - \frac{\operatorname{sn} x_2 \operatorname{sn} x_3}{\operatorname{dn} x_2 \operatorname{dn} x_3} \right\} \\
 &= \frac{k \operatorname{sn} (x_1 + x_2) (\operatorname{cn} x_1 \operatorname{cn} x_2 - \operatorname{cn} x_2 \operatorname{cn} x_3)}{\operatorname{dn} (x_1 + x_2) \operatorname{dn} x_1 \operatorname{dn} x_2 \operatorname{dn} x_3 \operatorname{dn} x_4} \\
 &= \frac{k \operatorname{sn} (x_1 + x_2) \{ \operatorname{sn} x_1 \operatorname{sn} x_2 \operatorname{dn} (x_1 + x_2) - \operatorname{sn} x_2 \operatorname{sn} x_3 \operatorname{dn} (x_2 + x_3) \}}{\operatorname{dn} (x_1 + x_2) \operatorname{dn} x_1 \operatorname{dn} x_2 \operatorname{dn} x_3 \operatorname{dn} x_4} \\
 &= \frac{k \operatorname{sn} (x_1 + x_2) (\operatorname{sn} x_1 \operatorname{sn} x_2 - \operatorname{sn} x_2 \operatorname{sn} x_3)}{\operatorname{dn} x_1 \operatorname{dn} x_2 \operatorname{dn} x_3 \operatorname{dn} x_4} \\
 &= \frac{k \operatorname{sn} (x_1 + x_2) \operatorname{sn} (x_1 + x_2) \operatorname{sn} (x_2 + x_3) (1 - k^2 \operatorname{sn} x_1 \operatorname{sn} x_2 \operatorname{sn} x_3 \operatorname{sn} x_4)}{\operatorname{dn} x_1 \operatorname{dn} x_2 \operatorname{dn} x_3 \operatorname{dn} x_4} \\
 &= - \frac{k \operatorname{sn} (x_1 + x_2) \operatorname{sn} (x_1 + x_2) \operatorname{sn} (x_2 + x_3) (1 - k^2 \operatorname{sn} x_1 \operatorname{sn} x_2 \operatorname{sn} x_3 \operatorname{sn} x_4)}{\operatorname{dn} x_1 \operatorname{dn} x_2 \operatorname{dn} x_3 \operatorname{dn} x_4}.
 \end{aligned}$$

3. In the case of $x_1 + x_2 + x_3 + x_4 + x_5 = 0$,

we shall require the following formulæ, where

$$u + v + r + s = 0,$$

$$k^2 k^2 \operatorname{sn} u \operatorname{sn} v \operatorname{sn} r \operatorname{sn} s + \operatorname{dn} u \operatorname{dn} v \operatorname{dn} r \operatorname{dn} s = k^2 + k^2 \operatorname{cn} u \operatorname{cn} v \operatorname{cn} r \operatorname{cn} s \dots (1);$$

$$\operatorname{cn} u \operatorname{cn} v \operatorname{dn} r \operatorname{dn} s - \operatorname{cn} r \operatorname{cn} s \operatorname{dn} u \operatorname{dn} v = k^2 (\operatorname{sn} u \operatorname{sn} v - \operatorname{sn} r \operatorname{sn} s) \dots (2);$$

$$k^2 \operatorname{sn} (u + v) (\operatorname{sn} u \operatorname{sn} v - \operatorname{sn} r \operatorname{sn} s) = Zu + Zv + Zr + Zs \dots (3);$$

$$k^2 \operatorname{sn} u \operatorname{sn} v \operatorname{sn} r \operatorname{cn} s \operatorname{dn} s - \operatorname{dn} u \operatorname{dn} v \operatorname{dn} r \operatorname{cn} s \operatorname{sn} s + \operatorname{cn} u \operatorname{cn} v \operatorname{cn} r \operatorname{sn} s \operatorname{dn} s = k^2 \operatorname{sn} (r + s) (\operatorname{sn} u \operatorname{sn} v - \operatorname{sn} r \operatorname{sn} s) \dots (4);$$

$$k^2 \operatorname{sn} u \operatorname{sn} v \operatorname{sn} r \operatorname{dn} s + \operatorname{dn} u \operatorname{dn} v \operatorname{dn} r \operatorname{sn} s = \operatorname{sn} s \operatorname{dn} s + k^2 \operatorname{cn} s \operatorname{sn} (r + s) (\operatorname{sn} u \operatorname{sn} v - \operatorname{sn} r \operatorname{sn} s) \dots (5).$$

The first two are given by Prof. Cayley and Prof. H. J. S. Smith, in *Proceedings* for January and February, 1879 (see p. 97). (3) is obvious.

To establish (4), we have, considering u, v as constant in (1), differentiating, and dividing by k^2 ,

$$\begin{aligned}
 & k^2 \operatorname{sn} u \operatorname{sn} v \operatorname{sn} r \operatorname{cn} s \operatorname{dn} s \\
 & \quad - \operatorname{dn} u \operatorname{dn} v \operatorname{dn} r \operatorname{cn} s \operatorname{sn} s \\
 & \quad + \operatorname{cn} u \operatorname{cn} v \operatorname{cn} r \operatorname{sn} s \operatorname{dn} s = A \\
 &= k^2 \operatorname{sn} u \operatorname{sn} v \operatorname{sn} s \operatorname{cn} r \operatorname{dn} r \\
 & \quad - \operatorname{dn} u \operatorname{dn} v \operatorname{dn} s \operatorname{cn} r \operatorname{sn} r \\
 & \quad + \operatorname{cn} u \operatorname{cn} v \operatorname{cn} s \operatorname{sn} r \operatorname{dn} r;
 \end{aligned}$$

$$\begin{aligned}
2A &= k^2 \operatorname{sn} u \operatorname{sn} v (\operatorname{sn} r \operatorname{cn} s \operatorname{dn} s + \operatorname{sn} s \operatorname{cn} r \operatorname{dn} r) \\
&\quad - \operatorname{dn} u \operatorname{dn} v (\operatorname{dn} r \operatorname{cn} s \operatorname{sn} s + \operatorname{dn} s \operatorname{cn} r \operatorname{sn} r) \\
&\quad + \operatorname{cn} u \operatorname{cn} v (\operatorname{cn} r \operatorname{sn} s \operatorname{dn} s + \operatorname{cn} s \operatorname{sn} r \operatorname{dn} r) \\
&= \operatorname{sn} (r+s) (1 - k^2 \operatorname{sn}^2 r \operatorname{sn}^2 s) \{ k^2 \operatorname{sn} u \operatorname{sn} v - \operatorname{dn} u \operatorname{dn} v \operatorname{cn} (r-s) \\
&\quad + \operatorname{cn} u \operatorname{cn} v \operatorname{dn} (r-s) \} \\
&= k^2 \operatorname{sn} u \operatorname{sn} v \operatorname{sn} (r+s) (1 - k^2 \operatorname{sn}^2 r \operatorname{sn}^2 s) \\
&\quad + \operatorname{sn} (r+s) \{ -\operatorname{dn} u \operatorname{dn} v \operatorname{cn} r \operatorname{cn} s + \operatorname{cn} u \operatorname{cn} v \operatorname{dn} r \operatorname{dn} s \\
&\quad - \operatorname{sn} r \operatorname{sn} s (\operatorname{dn} u \operatorname{dn} v \operatorname{dn} r \operatorname{dn} s - k^2 \operatorname{cn} u \operatorname{cn} v \operatorname{cn} r \operatorname{cn} s) \} \\
&= k^2 \operatorname{sn} u \operatorname{sn} v \operatorname{sn} (r+s) (1 - k^2 \operatorname{sn}^2 r \operatorname{sn}^2 s) \\
&\quad + \operatorname{sn} (r+s) \{ k^2 (\operatorname{sn} u \operatorname{sn} v - \operatorname{sn} r \operatorname{sn} s) - k^2 \operatorname{sn} r \operatorname{sn} s + k^2 k^2 \operatorname{sn} u \operatorname{sn} v \operatorname{sn}^2 r \operatorname{sn}^2 s \} \\
&= 2k^2 \operatorname{sn} (r+s) (\operatorname{sn} u \operatorname{sn} v - \operatorname{sn} r \operatorname{sn} s).
\end{aligned}$$

To establish (5), we get, from equations (4) and (1),

$$\begin{aligned}
&k^2 \operatorname{dn} u \operatorname{dn} v \operatorname{dn} r \operatorname{sn}^2 s = \operatorname{dn} u \operatorname{dn} v \operatorname{dn} r \operatorname{sn} s (\operatorname{dn}^2 s - \operatorname{cn}^2 s) \\
&= k^2 \operatorname{sn} s \operatorname{dn} s + k^2 \operatorname{cn} u \operatorname{cn} v \operatorname{cn} r \operatorname{cn} s \operatorname{sn} s \operatorname{dn} s - k^2 k^2 \operatorname{sn} u \operatorname{sn} v \operatorname{sn} r \operatorname{sn}^2 s \operatorname{dn} s \\
&\quad - k^2 \operatorname{sn} u \operatorname{sn} v \operatorname{sn} r \operatorname{cn}^2 s \operatorname{dn} s - \operatorname{cn} u \operatorname{cn} v \operatorname{cn} r \operatorname{cn} s \operatorname{sn} s \operatorname{dn} s \\
&\quad + k^2 \operatorname{cn} s \operatorname{sn} (r+s) (\operatorname{sn} u \operatorname{sn} v - \operatorname{sn} r \operatorname{sn} s);
\end{aligned}$$

$$\begin{aligned}
\text{or, } \operatorname{dn} u \operatorname{dn} v \operatorname{dn} r \operatorname{sn}^2 s &= \operatorname{sn} s \operatorname{dn} s - \operatorname{cn} u \operatorname{cn} v \operatorname{cn} r \operatorname{cn} s \operatorname{sn} s \operatorname{dn} s \\
&\quad - \operatorname{sn} u \operatorname{sn} v \operatorname{sn} r \operatorname{dn} s (k^2 \operatorname{sn}^2 s + \operatorname{cn}^2 s) + \operatorname{cn} s \operatorname{sn} (r+s) (\operatorname{sn} u \operatorname{sn} v - \operatorname{sn} r \operatorname{sn} s);
\end{aligned}$$

$$\begin{aligned}
&\text{and} \quad \operatorname{dn} u \operatorname{dn} v \operatorname{dn} r \operatorname{sn} s \operatorname{cn}^2 s \\
&= k^2 \operatorname{sn} u \operatorname{sn} v \operatorname{sn} r \operatorname{cn}^2 s \operatorname{dn} s + \operatorname{cn} u \operatorname{cn} v \operatorname{cn} r \operatorname{cn} s \operatorname{sn} s \operatorname{dn} s \\
&\quad - k^2 \operatorname{cn} s \operatorname{sn} (r+s) (\operatorname{sn} u \operatorname{sn} v - \operatorname{sn} r \operatorname{sn} s);
\end{aligned}$$

$$\begin{aligned}
\text{so that } \operatorname{dn} u \operatorname{dn} v \operatorname{dn} r \operatorname{sn}^2 s &= \operatorname{sn} s \operatorname{dn} s - k^2 \operatorname{sn} u \operatorname{sn} v \operatorname{sn} r \operatorname{dn} s \\
&\quad + k^2 \operatorname{cn} s \operatorname{sn} (r+s) (\operatorname{sn} u \operatorname{sn} v - \operatorname{sn} r \operatorname{sn} s).
\end{aligned}$$

We have now to find simple expressions for F' , when

$$x_1 + x_2 + x_3 + x_4 + x_5 = 0, \quad \text{or} \quad \Sigma x = 0,$$

$$\text{and} \quad kF = \Pi \operatorname{dn} x \cdot \Sigma \frac{1}{\operatorname{dn} x} \frac{d \cdot \operatorname{am} x}{dk}.$$

Now, we have

$$\begin{aligned}
&\frac{1}{\operatorname{dn} x_1} \frac{d \cdot \operatorname{am} x_1}{dk} + \frac{1}{\operatorname{dn} x_2} \frac{d \cdot \operatorname{am} x_2}{dk} + \frac{1}{\operatorname{dn} x_3} \frac{d \cdot \operatorname{am} x_3}{dk} + \frac{1}{\operatorname{dn} (x_4 + x_5)} \frac{d \cdot \operatorname{am} (x_4 + x_5)}{dk} \\
&= \frac{k \operatorname{sn} (x_1 + x_2) \{ \operatorname{sn} x_1 \operatorname{sn} x_2 - \operatorname{sn} x_3 \operatorname{sn} (x_4 + x_5) \}}{\operatorname{dn} x_1 \operatorname{dn} x_2 \operatorname{dn} x_3 \operatorname{dn} (x_4 + x_5)}; \\
&\frac{1}{\operatorname{dn} x_4} \frac{d \cdot \operatorname{am} x_4}{dk} + \frac{1}{\operatorname{dn} x_5} \frac{d \cdot \operatorname{am} x_5}{dk} - \frac{1}{\operatorname{dn} (x_4 + x_5)} \frac{d \cdot \operatorname{am} (x_4 + x_5)}{dk} \\
&= \frac{k \operatorname{sn} x_4 \operatorname{sn} x_5 \operatorname{sn} (x_4 + x_5)}{\operatorname{dn} x_4 \operatorname{dn} x_5 \operatorname{dn} (x_4 + x_5)};
\end{aligned}$$

whence

$$\begin{aligned} \operatorname{dn}(x_4+x_5) F &= \operatorname{dn} x_4 \operatorname{dn} x_5 \operatorname{sn}(x_1+x_2) \{ \operatorname{sn} x_1 \operatorname{sn} x_2 - \operatorname{sn} x_5 \operatorname{sn}(x_4+x_5) \} \\ &\quad + \operatorname{sn} x_4 \operatorname{sn} x_5 \operatorname{dn} x_1 \operatorname{dn} x_2 \operatorname{dn} x_5 \operatorname{sn}(x_4+x_5) \\ &= \operatorname{dn}(x_4+x_5) \operatorname{sn}(x_1+x_2) \{ \operatorname{sn} x_1 \operatorname{sn} x_2 - \operatorname{sn} x_5 \operatorname{sn}(x_4+x_5) \} \\ &\quad + k^2 \operatorname{sn} x_4 \operatorname{sn} x_5 \operatorname{cn}(x_4+x_5) \operatorname{sn}(x_1+x_2) \{ \operatorname{sn} x_1 \operatorname{sn} x_2 - \operatorname{sn} x_5 \operatorname{sn}(x_4+x_5) \} \\ &\quad + \operatorname{sn} x_4 \operatorname{sn} x_5 \operatorname{dn} x_1 \operatorname{dn} x_2 \operatorname{dn} x_5 \operatorname{sn}(x_4+x_5) \\ &= \operatorname{dn}(x_4+x_5) \operatorname{sn}(x_1+x_2) \{ \operatorname{sn} x_1 \operatorname{sn} x_2 - \operatorname{sn} x_5 \operatorname{sn}(x_4+x_5) \} \\ &\quad + \operatorname{sn} x_4 \operatorname{sn} x_5 \{ \operatorname{sn}(x_4+x_5) \operatorname{dn}(x_4+x_5) - k^2 \operatorname{sn} x_1 \operatorname{sn} x_2 \operatorname{sn} x_5 \operatorname{dn}(x_4+x_5) \}, \end{aligned}$$

by virtue of equation (5). So that

$$\begin{aligned} F &= \operatorname{sn}(x_1+x_2) \operatorname{sn} x_1 \operatorname{sn} x_2 - \operatorname{sn} x_5 \operatorname{sn}(x_1+x_2) \operatorname{sn}(x_4+x_5) \\ &\quad + \operatorname{sn} x_4 \operatorname{sn} x_5 \operatorname{sn}(x_4+x_5) - k^2 \Pi \operatorname{sn} x \\ &= \frac{1}{k^2} \Sigma Z x - k^2 \Pi \operatorname{sn} x. \end{aligned}$$

Another form for F is worthy of notice. We have

$$\begin{aligned} &\operatorname{sn}(x_1+x_2) [\operatorname{cn}(x_1+x_2) - \operatorname{cn} x_1 \operatorname{cn} x_2] \operatorname{dn} x_1 \operatorname{dn} x_2 \operatorname{dn} x_5 \\ &\quad - \operatorname{sn}(x_1+x_2) [\operatorname{dn}(x_1+x_2) - \operatorname{dn} x_1 \operatorname{dn} x_2] \operatorname{cn} x_5 \operatorname{cn} x_4 \operatorname{cn} x_5 \\ &= \operatorname{sn}(x_1+x_2) \operatorname{sn} x_1 \operatorname{sn} x_2 [k^2 \operatorname{cn}(x_1+x_2) \operatorname{cn} x_5 \operatorname{cn} x_4 \operatorname{cn} x_5 \\ &\quad - \operatorname{dn}(x_1+x_2) \operatorname{dn} x_5 \operatorname{dn} x_4 \operatorname{dn} x_5] \\ &= k^2 \operatorname{sn}(x_1+x_2) \operatorname{sn} x_1 \operatorname{sn} x_2 [k^2 \operatorname{sn}(x_1+x_2) \operatorname{sn} x_5 \operatorname{sn} x_4 \operatorname{sn} x_5 - 1]; \end{aligned}$$

and

$$\begin{aligned} &k^2 \operatorname{sn} x_5 \operatorname{sn} x_4 \operatorname{sn} x_5 \operatorname{cn}(x_1+x_2) \operatorname{dn} x_1 \operatorname{dn} x_2 \\ &= k^2 \operatorname{sn} x_5 \operatorname{sn} x_4 \operatorname{sn} x_5 \operatorname{cn}(x_1+x_2) \operatorname{dn}(x_1+x_2) \\ &\quad + k^2 k^2 \operatorname{sn} x_1 \operatorname{sn} x_2 \operatorname{sn} x_5 \operatorname{sn} x_4 \operatorname{sn} x_5 \operatorname{cn}^2(x_1+x_2); \end{aligned}$$

so that

$$\begin{aligned} &\operatorname{sn}(x_1+x_2) \operatorname{cn} x_1 \operatorname{cn} x_2 \operatorname{dn} x_5 \operatorname{dn} x_4 \operatorname{dn} x_5 \\ &\quad - \operatorname{sn}(x_1+x_2) \operatorname{dn} x_1 \operatorname{dn} x_2 \operatorname{cn} x_5 \operatorname{cn} x_4 \operatorname{cn} x_5 \\ &\quad - k^2 \operatorname{cn}(x_1+x_2) \operatorname{dn} x_1 \operatorname{dn} x_2 \operatorname{sn} x_5 \operatorname{sn} x_4 \operatorname{sn} x_5 \\ &= \operatorname{sn}(x_1+x_2) \{ \operatorname{cn}(x_1+x_2) \operatorname{dn} x_5 \operatorname{dn} x_4 \operatorname{dn} x_5 - \operatorname{dn}(x_1+x_2) \operatorname{cn} x_5 \operatorname{cn} x_4 \operatorname{cn} x_5 \} \\ &\quad + k^2 \operatorname{sn}(x_1+x_2) \operatorname{sn} x_1 \operatorname{sn} x_2 - k^2 \operatorname{sn} x_5 \operatorname{sn} x_4 \operatorname{sn} x_5 \operatorname{cn}(x_1+x_2) \operatorname{dn}(x_1+x_2) \\ &\quad - k^2 k^2 \Pi \operatorname{sn} x = k^2 \operatorname{sn}(x_1+x_2) \operatorname{sn} x_1 \operatorname{sn} x_2 \\ &\quad + k^2 \operatorname{sn} x_4 \operatorname{sn} x_5 \operatorname{sn}(x_4+x_5) + k^2 \operatorname{sn}(x_1+x_2+x_3) \operatorname{sn}(x_1+x_2) \operatorname{sn} x_4 \\ &\quad - k^2 k^2 \Pi \operatorname{sn} x = k^2 F. \end{aligned}$$