XLIII. On the application of force within a limited space, required to produce spherical solitary waves, or trains of periodic waves, of both species, equivoluminal and irrotational, in an elastic solid

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XLIII. *On the Application of Force within a Limited Space, required to produce Spherical Solitary Waves, or Trains of Periodic Waves, of both Species, Equivoluminal* and Irrotational, in an Elastic Solid.* By Lord KELVIN, G.C.V.O., P.R.S.E. †

§ 1. **T**HE complete mathematical theory of the propagation of motion through an infinite elastic solid, including the analysis of the motion into two species, equivoluminal and irrotational, was first given by Stokes in his splendid paper "On the Dynamical Theory of Diffraction" §. The object of the present communication is to investigate fully the forcive which must be applied to the boundary, S, of a hollow of any shape in the solid, in order to originate and to maintain any known motion of the surrounding solid; and to solve the inverse problem of finding the motion when the forcive on, or the motion of, S is given, for the particular case in which S is a spherical surface kept rigid.

§ 2. Let ξ, η, ζ denote the infinitesimal displacement at any point of the solid, of which (x, y, z) is the equilibrium position. The well-known equations of motion || are

$$\left. \begin{aligned} \rho \frac{d^2 \xi}{dt^2} &= (k + \frac{1}{3}n) \frac{d\delta}{dx} + n \nabla^2 \xi, \\ \rho \frac{d^2 \eta}{dt^2} &= (k + \frac{1}{3}n) \frac{d\delta}{dy} + n \nabla^2 \eta, \\ \rho \frac{d^2 \zeta}{dt^2} &= (k + \frac{1}{3}n) \frac{d\delta}{dz} + n \nabla^2 \zeta, \end{aligned} \right\} \quad . \quad . \quad . \quad (1)$$

where δ denotes $\frac{d\xi}{dx} + \frac{d\eta}{dy} + \frac{d\zeta}{dz}$. Using the notation of Thomson and Tait ¶ for strain-components (elongations; and distortions), $e, f, g; a, b, c$; we have

$$\left. \begin{aligned} e &= \frac{d\xi}{dx}; & f &= \frac{d\eta}{dy}; & g &= \frac{d\zeta}{dz} \\ a &= \frac{d\eta}{dz} + \frac{d\zeta}{dy}; & b &= \frac{d\zeta}{dx} + \frac{d\xi}{dz}; & c &= \frac{d\xi}{dy} + \frac{d\eta}{dx} \end{aligned} \right\} \quad . \quad . \quad (2);$$

* By "equivoluminal" I mean *every part of the solid keeping its volume unchanged during the motion.*

† Communicated by the Author: read before the Royal Society of Edinburgh, May 1, 1899.

§ Stokes, 'Mathematical Papers,' vol. ii. p. 243.

|| See my paper "On the Reflexion and Refraction of Solitary Plane Waves, &c.," Proc. R. S. E. Dec. 1898, and Phil. Mag. Feb. 1899.

¶ Thomson and Tait's 'Natural Philosophy,' § 669, or 'Elements,' § 640.

and with the corresponding notation $P, Q, R; S, T, U$; for stress-components (normal and tangential forces on the six sides of an infinitely small rectangular parallelepiped), we have

$$\left. \begin{aligned} P &= (k + \frac{2}{3}n)e + (k - \frac{2}{3}n)(f + g); & Q &= (k + \frac{2}{3}n)f + (k - \frac{2}{3}n)(g + e); \\ R &= (k + \frac{2}{3}n)g + (k - \frac{2}{3}n)(e + f) \\ S &= na; & T &= nb; & U &= nc \end{aligned} \right\} \quad (3).$$

Let now σ be an infinitesimal area at any point of the surface S ; λ, μ, ν the direction-cosines of the normal; and $X\sigma, Y\sigma, Z\sigma$ the components of the force which must be applied from within to produce or maintain the specified motion of the matter outside. We have

$$\left. \begin{aligned} -X &= P\lambda + U\mu + T\nu \\ -Y &= Q\mu + S\nu + U\lambda \\ -Z &= R\nu + T\lambda + S\mu \end{aligned} \right\} \quad . \quad . \quad . \quad (4);$$

whence by (3)

$$\left. \begin{aligned} -X &= (k - \frac{2}{3}n)\lambda\delta + n(2\lambda e + \mu c + \nu b) \\ -Y &= (k - \frac{2}{3}n)\mu\delta + n(2\mu f + \nu a + \lambda c) \\ -Z &= (k - \frac{2}{3}n)\nu\delta + n(2\nu g + \lambda b + \mu a) \end{aligned} \right\} \quad . \quad . \quad (5).$$

These equations give an explicit answer to the question, What is the force? when the strain of the matter in contact with S is given. We shall consider in detail their application to the case in which S is spherical, and the motions and forces are in meridional planes through OX and symmetrical round this line. Without loss of generality we may take

$$z = 0; \text{ giving } \nu = 0, a = 0, b = 0, Z = 0 \quad . \quad . \quad (6)$$

Equations (5) therefore become

$$\left. \begin{aligned} -X &= (k - \frac{2}{3}n)\lambda\delta + n(2\lambda e + \mu c) \\ -Y &= (k - \frac{2}{3}n)\mu\delta + n(2\mu f + \lambda c) \end{aligned} \right\} \quad . \quad . \quad (7).$$

§ 3. In §§ 5-26 of his paper already referred to, Stokes gives a complete solution of the problem of finding the displacement and velocity at any point of an infinite solid, which must follow from any arbitrarily given displacement and velocity at any previous time, if after that, the solid is left to itself with no force applied to any part of it. In a future communication I hope to apply this solution to the diffraction of solitary waves, plane or spherical. Meantime I confine myself to the subject stated in the title of the

present communication, regarding which Stokes gives some important indications in §§ 27-29 of his paper.

§ 4. Poisson in 1819 gave a complete solution of the equation

$$\frac{d^2 w}{dt^2} = v^2 \nabla^2 w \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

in terms of arbitrary functions of x, y, z representing the initial values of w and $\frac{dw}{dt}$; and showed that for every case in which w depends only on distance (r) from a fixed point, it takes the form

$$w = \frac{1}{r} \left\{ F\left(t - \frac{r}{v}\right) + f\left(t + \frac{r}{v}\right) \right\} \quad . \quad . \quad . \quad (9),$$

where F and f denote arbitrary functions. In my Baltimore Lectures of 1884 I pointed out that solutions expressing spherical waves, whether equivoluminal (in which there is essentially different range of displacement in different parts of the spherical surface) or irrotational (for which the displacement may or may not be different in different parts of the spherical surface), can be very conveniently derived from (9) by differentiations with respect to x, y, z . It may indeed be proved, although I do not know that a formal proof has been anywhere published, that an absolutely general solution of (8) is expressed by the formula

$$\Sigma \left(\frac{d}{dx} \right)^h \left(\frac{d}{dy} \right)^i \left(\frac{d}{dz} \right)^j \left\{ \frac{1}{r} \left[F\left(t - \frac{r}{v}\right) + f\left(t + \frac{r}{v}\right) \right] \right\};$$

$$r = \sqrt{(x-x')^2 + (y-y')^2 + (z-z')^2} \quad . \quad (10),$$

where Σ denotes sums for different integral values of h, i, j , and for any different values of x', y', z' .

§ 5. I propose at present to consider only the simplest of all the cases in which motion at every point $(x, 0, 0)$ and $(0, y, z)$ is parallel to $X'X$; and for all values of y and z , ξ is the same for equal positive and negative values of x . For this purpose we of course take $x' = y' = z' = 0$; and we shall find that no values of h, i, j greater than 2 can appear in our expressions for ξ, η, ζ , because we confine ourselves to the simplest case fulfilling the specified conditions. Our special subject, under the title of this paper, excludes waves travelling inwards from distant sources, and therefore annuls $f(t + r/v)$.

§ 6. In §§ 5-8 of his paper, Stokes showed that any motion whatever of a homogeneous elastic solid may, throughout every part of it experiencing no applied force, be analysed

into two constituents, each capable of existing without the other, in one of which the displacement is equivoluminal, and in the other it is irrotational. Hence if we denote by (ξ_1, η_1, ζ_1) the equivoluminal constituent, and by (ξ_2, η_2, ζ_2) the irrotational constituent, the complete solution of (1) may be written as follows:—

$$\xi = \xi_1 + \xi_2; \quad \eta = \eta_1 + \eta_2; \quad \zeta = \zeta_1 + \zeta_2 \quad . \quad . \quad . \quad (11),$$

where ξ_1, η_1, ζ_1 and ξ_2, η_2, ζ_2 fulfil the following conditions, (12) and (13), respectively:—

$$\left. \begin{aligned} \frac{d\xi_1}{dx} + \frac{d\eta_1}{dy} + \frac{d\zeta_1}{dz} &= 0 \\ u^2 \nabla^2 \xi_1 &= \frac{d^2 \xi_1}{dt^2}; \quad u^2 \nabla^2 \eta_1 = \frac{d^2 \eta_1}{dt^2}; \quad u^2 \nabla^2 \zeta_1 = \frac{d^2 \zeta_1}{dt^2}; \quad u^2 = \frac{n}{\rho} \end{aligned} \right\} (12);$$

$$\left. \begin{aligned} \xi_2 &= \frac{dw}{dx}; \quad \eta_2 = \frac{dw}{dy}; \quad \zeta_2 = \frac{dw}{dz} \\ w \text{ being any solution of} \\ v^2 \nabla^2 w &= \frac{d^2 w}{dt^2}; \quad v^2 = \frac{k + \frac{4}{3}n}{\rho} \end{aligned} \right\} . \quad . \quad (13).$$

The first equation of (12) shows that in the (ξ_1, η_1, ζ_1) constituent of the solution there is essentially no dilatation or condensation in any part of the solid; that is to say, the displacement is equivoluminal. The first three equations of (13) prove that in the (ξ_2, η_2, ζ_2) constituent the displacement is essentially irrotational.

§ 7. We can now see that the most general irrotational solution fulfilling the conditions of § 5 is

$$\xi_2 = \frac{d^2 F_2}{dx^2 r}; \quad \eta_2 = \frac{d^2 F_2}{dx dy r}; \quad \zeta_2 = \frac{d^2 F_2}{dx dz r} \quad . \quad (14),$$

giving

$$\frac{d\xi_2}{dx} + \frac{d\eta_2}{dy} + \frac{d\zeta_2}{dz} = \frac{1}{v^2} \frac{d}{dx} \frac{\ddot{F}_2}{r} \quad . \quad . \quad . \quad (14');$$

and the most general equivoluminal solution fulfilling the same conditions is

$$\xi_1 = \frac{d^2 F_1}{dx^2 r} - \frac{\ddot{F}_1}{u^2 r}; \quad \eta_1 = \frac{d^2 F_1}{dx dy r}; \quad \zeta_1 = \frac{d^2 F_1}{dx dz r} \quad . \quad (15),$$

giving

$$\frac{d\xi_1}{dx} + \frac{d\eta_1}{dy} + \frac{d\zeta_1}{dz} = 0 \quad . \quad . \quad . \quad (15'),$$

where F_1 and F_2 are put for brevity to denote arbitrary functions of $\left(t - \frac{r}{u}\right)$ and $\left(t - \frac{r}{v}\right)$ respectively. Hence the most general solution fulfilling the conditions of § 5 is

$$\xi = \frac{d^2}{dx^2} \frac{\phi}{r} - \frac{\ddot{F}_1}{u^2 r}; \quad \eta = \frac{d^2}{dx dy} \frac{\phi}{r}; \quad \zeta = \frac{d^2}{dx dz} \frac{\phi}{r} \quad (16),$$

where for brevity ϕ denotes a function of r and t , specified as follows:—

$$\phi(r, t) = F_1\left(t - \frac{r}{u}\right) + F_2\left(t - \frac{r}{v}\right) \quad . \quad . \quad (17).$$

Denoting now by accents differential coefficients with respect to r , and retaining the Newtonian notation of dots to signify differential coefficients with reference to t , we have

$$\phi' = -\left(\frac{\dot{F}_1}{u} + \frac{\dot{F}_2}{v}\right); \quad \phi'' = \frac{\ddot{F}_1}{u^2} + \frac{\ddot{F}_2}{v^2}; \quad \phi''' = -\left(\frac{\ddot{F}_1}{u^3} + \frac{\ddot{F}_2}{v^3}\right) \quad . \quad (18).$$

Working out now the differentiations in (16), we find

$$\left. \begin{aligned} \xi &= x^2 \left(\frac{\phi''}{r^3} - \frac{3\phi'}{r^4} + \frac{3\phi}{r^5} \right) + \frac{\phi'}{r^2} - \frac{\phi}{r^3} - \frac{\ddot{F}_1}{u^2 r} \\ \eta &= xy \left(\frac{\phi''}{r^3} - \frac{3\phi'}{r^4} + \frac{3\phi}{r^5} \right) \\ \zeta &= xz \left(\frac{\phi''}{r^3} - \frac{3\phi'}{r^4} + \frac{3\phi}{r^5} \right) \end{aligned} \right\} \quad . \quad (19).$$

§ 8. For the determination of the force-components by (7), we shall want values of δ , e , f , and c . Using therefore (2) and going back to (16) we see that

$$c = 2 \frac{d\eta}{dx} - \frac{1}{u^2} \frac{d}{dy} \frac{\dot{F}_1}{r} \quad . \quad . \quad . \quad (20).$$

Hence, and by (19), we find

$$\begin{aligned} c = y \left\{ 2 \left[x^2 \left(\frac{\phi'''}{r^4} - \frac{6\phi''}{r^5} + \frac{15\phi'}{r^6} - \frac{15\phi}{r^7} \right) + \frac{\phi''}{r^3} - \frac{3\phi'}{r^4} + \frac{3\phi}{r^5} \right] \right. \\ \left. + \frac{1}{r^2} \frac{\ddot{F}_1}{u^3} + \frac{1}{r^3} \frac{\ddot{F}_2}{u^2} \right\} \quad . \quad . \quad (21). \end{aligned}$$

By (16), (14)', and (15)' we find

$$\delta = \frac{1}{v^2} \frac{d}{dx} \frac{\ddot{F}_2}{r} = -x \left(\frac{1}{r^2} \frac{\ddot{F}_2}{v^3} + \frac{1}{r^3} \frac{\ddot{F}_2}{v^2} \right) \quad . \quad . \quad (22);$$

and by (19) directly used in (2) we find

$$\left. \begin{aligned} e &= x \left\{ x^2 \left(\frac{\phi'''}{r^4} - \frac{6\phi''}{r^5} + \frac{15\phi'}{r^6} - \frac{15\phi}{r^7} \right) + 3 \left(\frac{\phi}{r^3} - \frac{3\phi'}{r^4} + \frac{3\phi}{r^5} \right) + \frac{1}{r^2} \ddot{F}_1 + \frac{1}{r^3} \ddot{F}_1 \right\} \\ f &= x \left\{ y^2 \left(\frac{\phi'''}{r^4} - \frac{6\phi''}{r^5} + \frac{15\phi'}{r^6} - \frac{15\phi}{r^7} \right) + \frac{\phi''}{r^3} - \frac{3\phi'}{r^4} + \frac{3\phi}{r^5} \right\} \\ g &= x \left\{ z^2 \left(\frac{\phi'''}{r^4} - \frac{6\phi'}{r^5} + \frac{15\phi'}{r^6} - \frac{15\phi}{r^7} \right) + \frac{\phi''}{r^3} - \frac{3\phi'}{r^4} + \frac{3\phi}{r^5} \right\} \end{aligned} \right\} \quad (23)$$

Remark here how by the summation of these three formulas we find for $e+f+g$ the value given for δ in (22).

§ 9. These formulas (22) and (23), used in (5), give the force-components per unit area at any point of the boundary (S of § 1) of a hollow of any shape in the solid, in order that the motion throughout the solid around it may be that expressed by (19). Supposing the hollow to be spherical, as proposed in § 2, let its radius be q . We must in (23) and (5) put

$$x = q\lambda; \quad y = q\mu; \quad z = q\nu \quad . \quad . \quad . \quad (24);$$

and putting $\nu = 0$, we have, as in (7), the two force-components for any point of the surface in the meridian $z = 0$, expressed as follows:—

$$\left. \begin{aligned} X &= (k - \frac{2}{3}n)C_2\lambda^2 - n\{2\lambda^2A + 2(2\lambda^2 + 1)B + (\lambda^2 + 1)C_1\} \\ Y &= (k - \frac{2}{3}n)C_2\lambda\mu - n\lambda\mu(2A + 4B + C_1) \end{aligned} \right\} \quad (25),$$

where

$$\left. \begin{aligned} A &= \frac{\Phi'''}{q} - \frac{6\Phi''}{q^2} + \frac{15\Phi'}{q^3} - \frac{15\Phi}{q^4} \\ B &= \frac{\Phi''}{q^2} - \frac{3\Phi'}{q^3} + \frac{3\Phi}{q^4} \\ C_1 &= \frac{\ddot{\mathcal{F}}_1}{qu^3} + \frac{\ddot{\mathcal{F}}_1}{q^2u^2} \\ C_2 &= \frac{\ddot{\mathcal{F}}_2}{qv^3} + \frac{\ddot{\mathcal{F}}_2}{q^2v^2} \end{aligned} \right\} \quad . \quad . \quad (26),$$

Φ and $\mathcal{F}$ denoting ϕ and F with q for r .

§ 10. Returning now to (19), consider the character of the motion represented by the formulas. For brevity we shall call XX' simply the *axis*, and the plane of YY', ZZ' the *equatorial plane*. First take $y=0, z=0$, and therefore $x=r$.

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We find by aid of (18)

$$(\text{axial}) \quad \xi = \frac{1}{r} \frac{\ddot{F}_2}{v^2} + \frac{2}{r^2} \left(\frac{\dot{F}_1}{u} + \frac{\dot{F}_2}{v} \right) + \frac{2}{r^3} (F_1 + F_2); \quad \eta = 0; \quad \zeta = 0 \quad (27).$$

Next take $x=0$ and we find

$$(\text{equatorial}) \quad \xi = -\frac{\dot{F}_1}{r} \frac{\dot{F}_1}{u^2} - \frac{1}{r^2} \left(\frac{\dot{F}_1}{u} + \frac{\dot{F}_2}{v} \right) - \frac{1}{r^3} (F_1 + F_2); \quad \eta = 0; \quad \zeta = 0 \quad (28).$$

Hence for very small values of r we have

$$\left. \begin{aligned} (\text{axial}) \quad \xi &\doteq \frac{2}{r^3} (F_1 + F_2) \\ (\text{equatorial}) \quad \xi &\doteq -\frac{1}{r^3} (F_1 + F_2) \end{aligned} \right\} \dots \dots \dots (29);$$

and for very large values of r ,

$$\left. \begin{aligned} (\text{axial}) \quad \xi &\doteq \frac{1}{r} \frac{\ddot{F}_2}{v^2} \\ (\text{equatorial}) \quad \xi &\doteq -\frac{1}{r} \frac{\ddot{F}_1}{u^2} \end{aligned} \right\} \dots \dots \dots (30).$$

Thus we see that for very distant places, the motion in the axis is approximately that due to the irrotational wave alone; and the motion at the equatorial plane is that due to the equivoluminal wave alone: also that with equal values of F_1 and F_2 the equivoluminal and the irrotational constituents contribute to these displacements inversely as the squares of the propagational velocities of the two waves. On the other hand, for places very near the centre, (29) shows that both in the axis and in the equatorial plane the irrotational and the equivoluminal constituents contribute equally to the displacements.

§ 11. Equations (25) and (26) give us full specification of the force which must be applied to the boundary of our spherical hollow to cause the motion to be precisely through all time that specified by (19), with F_1 and F_2 any arbitrary functions. Thus we may suppose $F_1(t-q/u)$ and $F_2(t-q/v)$ to be each zero for all negative values of t , and to be zero again for all values of t exceeding a certain limit τ . At any distance r from the centre, the disturbance will last during the time

$$\left. \begin{aligned} \text{from} \quad & t = \frac{r-q}{v} \quad \text{to} \quad t = \frac{r-q}{v} + \tau, \\ \text{and from} \quad & t = \frac{r-q}{u} \quad \text{to} \quad t = \frac{r-q}{u} + \tau. \end{aligned} \right\} \dots \dots (31)$$

Supposing $v > u$, we see that these two durations overlap by an interval equal to

$$\left. \begin{aligned} & \frac{r-q}{v} + \tau - \frac{r-q}{u} \\ \text{if } & r-q < \tau / \left(\frac{1}{u} - \frac{1}{v} \right) \end{aligned} \right\} \dots \dots \dots (32).$$

On the other hand, at every point of space outside the radius $q + \tau/(1/u - 1/v)$ the wave of the greater propagational velocity passes away outwards before the wave of the smaller velocity reaches it, and the transit-time of each wave across it is τ . The solid is rigorously undisplaced and at rest throughout all the spaces outside the more rapid wave, between the two waves, and inside the less rapid of the two.

§ 12. The expressions (25) and (26) for the components of the surface-force on the boundary of the hollow required to produce the supposed motion, involve $\ddot{\mathcal{F}}_1$ and $\ddot{\mathcal{F}}_2$. Hence we should have infinite values for $t=0$ or $t=\tau$, unless $\ddot{\mathcal{F}}_1$ and $\ddot{\mathcal{F}}_2$ vanish for $t=0$ and $t=\tau$, when $r=q$. Subject to this condition the simplest possible expression for each arbitrary function to represent the two solitary waves of § 11, is of the form

$$\mathcal{F} = (1 - \chi^2)^3, \quad \text{where } \chi = \frac{2t}{\tau} - 1 \quad \dots \dots (33).$$

Hence, by successive differentiations, with reference to t ,

$$\left. \begin{aligned} \dot{\mathcal{F}} &= -\frac{12}{\tau} \chi (1 - \chi^2)^2 \\ \ddot{\mathcal{F}} &= \frac{24}{\tau^2} (-1 + 6\chi^2 - 5\chi^4) \\ \ddot{\mathcal{F}} &= \frac{48}{\tau^3} (12\chi - 20\chi^3) \end{aligned} \right\} \dots \dots \dots (34).$$

The annexed diagram of four curves represents these four functions (33) and (34).

§ 13. Take now definitively

$$\left. \begin{aligned} F_1 \left(t - \frac{r}{u} \right) &= c_1 q^3 (1 - \chi_1^2)^3 \\ F_2 \left(t - \frac{r}{v} \right) &= c_2 q^3 (1 - \chi_2^2)^3 \end{aligned} \right\} \dots \dots (35),$$

Scale of $\dot{\mathcal{F}}$ = $\frac{1}{5}$ scale of $\mathcal{F}$
" $\ddot{\mathcal{F}}$ = $\frac{1}{25}$ " $\mathcal{F}$
" $\ddot{\mathcal{F}}$ = $\frac{1}{125}$ " $\mathcal{F}$

where

$$\left. \begin{aligned} \chi_1 &= \left(t - \frac{r-q}{u} - \frac{1}{2}\tau \right) \div \frac{1}{2}\tau \\ \chi_2 &= \left(t - \frac{r-q}{v} - \frac{1}{2}\tau \right) \div \frac{1}{2}\tau \end{aligned} \right\} \quad . \quad . \quad . \quad (36).$$

Consider now separately the equivoluminal and the irrotational motions. Using (19), (18), (35), (34), and taking the equivoluminal constituents, we have as follows:—

$$\left\{ \begin{array}{l} \text{Equatorial, } x=0; \quad \eta_1=0 \\ \xi_1 = -c_1 \left(\frac{q^3}{r^3} \mathcal{X}_1 + \frac{12q^3}{r^2 u \tau} \mathcal{X}_1' + \frac{24q^3}{r u^2 \tau^2} \mathcal{X}_1'' \right) \end{array} \right\} \quad . \quad (37)$$

$$\left\{ \begin{array}{l} \text{Cone of latitude } 45^\circ, \quad x^2 = xy = \frac{1}{2}r^2 \\ \xi_1 = c_1 \left(\frac{1}{2} \frac{q^3}{r^3} \mathcal{X}_1 + \frac{6q^3}{r^2 u \tau} \mathcal{X}_1' - \frac{12q^3}{r u^2 \tau^2} \mathcal{X}_1'' \right) \\ \eta_1 = \frac{1}{2} c_1 \left(\frac{3q^3}{r^3} \mathcal{X}_1 + \frac{36q^3}{r^2 u \tau} \mathcal{X}_1' + \frac{24q^3}{r u^2 \tau^2} \mathcal{X}_1'' \right) \end{array} \right\} \quad . \quad (38)$$

$$\left\{ \begin{array}{l} \text{Axial, } x^2 = r^2, \quad \eta_1 = 0 \\ \xi_1 = c_1 \left(\frac{2q^3}{r^3} \mathcal{X}_1 + \frac{24q^3}{r^2 u \tau} \mathcal{X}_1' \right) \end{array} \right\} \quad . \quad . \quad (39);$$

where

$$\mathcal{X}_1 = (1 - \chi_1^2)^3; \quad \mathcal{X}_1' = (1 - \chi_1^2)^2 \chi_1; \quad \mathcal{X}_1'' = -1 + 6\chi_1^2 - 5\chi_1^4. \quad (40).$$

§ 14. Similarly for the irrotational constituents;

$$\left\{ \begin{array}{l} \text{Equatorial, } x=0; \quad \eta_2=0 \\ \xi_2 = -c_2 \left(\frac{q^3}{r^3} \mathcal{X}_2 + \frac{12q^3}{r^2 v \tau} \mathcal{X}_2' \right) \end{array} \right\} \quad . \quad . \quad (41);$$

$$\left\{ \begin{array}{l} \text{Cone of latitude } 45^\circ, \quad x^2 = xy = \frac{1}{2}r^2 \\ \xi_2 = c_2 \left(\frac{1}{2} \frac{q^3}{r^3} \mathcal{X}_2 + \frac{6q^3}{r^2 v \tau} \mathcal{X}_2' + \frac{12q^3}{r v^2 \tau^2} \mathcal{X}_2'' \right) \\ \eta_2 = \frac{1}{2} c_2 \left(\frac{3q^3}{r^3} \mathcal{X}_2 + \frac{36q^3}{r^2 v \tau} \mathcal{X}_2' + \frac{24q^3}{r v^2 \tau^2} \mathcal{X}_2'' \right) \end{array} \right\} \quad . \quad (42);$$

$$\left\{ \begin{array}{l} \text{Axial, } x^2 = r^2; \quad \eta_2 = 0 \\ \xi_2 = c_2 \left(\frac{2q^3}{r^3} \mathcal{X}_2 + \frac{24q^3}{r^2 v \tau} \mathcal{X}_2' + \frac{24q^3}{r v^2 \tau^2} \mathcal{X}_2'' \right) \end{array} \right\} \quad . \quad (43);$$

where $\mathcal{X}_2$ &c. are given by (40) with χ_2 for χ_1 ; χ_1 and χ_2 being given by (36).

§ 15. The character of the motion throughout the solid, which is fully specified by (19), will be perfectly understood after a careful study of the details for the equatorial, conal, and axial places, shown clearly by (37) . . . (43) for each constituent, the equivoluminal and the irrotational, separately. The curve $\mathcal{H}$ in the diagram of § 12 shows the history of the motion that must be given to any point of the surface S , for either constituent alone, and therefore for the two together, in any case in which q is exceedingly small in comparison with the smaller of the two quantities $u\tau$, $v\tau$, which for brevity we shall call the wave-lengths. The curve $\mathcal{H}$ shows the history of the motion produced by either wave when it is passing any point at a distance from the centre very great in comparison with its own wave-length. But the three algebraic functions $\mathcal{H}$, $\mathcal{H}'$, $\mathcal{H}''$ all enter into the expression of the motion due to either wave when the faster has advanced so far that its rear is clear of the front of the slower, but *not* so far as to make its wave-length (which is the constant thickness of the spherical shell containing it) great in comparison with its inner radius. Look at the diagram, and notice that in the origin at S , a mere motion of each point in one direction and back, represented by $\mathcal{H}$, causes in very distant places, a motion ($\mathcal{H}''$) to a certain displacement d , back through the zero to a displacement $1.36 \times d$ in the opposite direction, thence back through zero to d in the first direction and thence back to rest at zero. Remark that the direction of d is radial in the irrotational wave and perpendicular to the radius in the equivoluminal wave. Remark also that the d for every radial line varies inversely as distance from the centre.

§ 16. Draw any line OPK in any fixed direction through O , the centre of the spherical surface S at which the forcive originating the whole motion is applied. In the particular case of §§ 12 . . . 15, and in any case in which $F_1\left(t - \frac{q}{u}\right)$ and $F_2\left(t - \frac{q}{v}\right)$ are each assumed to be, from $t=0$ to $t=\tau$, of the form $t^3\left(\frac{1}{2}\tau - t\right)^3 A_i t^i$, where i denotes an integer, the time-history of the motion of P is $B_0 + B_1 t + \dots + B_{6+i} t^{6+i}$, and its space-history (t constant and r variable) is $C_{-3} r^{-3} + C_{-2} r^{-2} + \dots + C_{5+i} r^{5+i}$; the complete formula in terms of t and r being given explicitly by (19). The elementary algebraic character of the formula: and the *exact nullity* of the displacement for every point of the solid for which $r > q + vt$; and between

$r=q+v(t-\tau)$ and $r=q+ut$, when $v(t-\tau)>ut$; and between $r=q$ and $r=q+u(t-\tau)$, when $t>\tau$: these interesting characteristics of the solution of a somewhat intricate dynamical problem, are secured by the particular character of the originating forcive at S, which we find according to §§ 8, 9 to be that which will produce them. But all these characteristics are lost except the first (nullity of motion through all space outside the spherical surface $r=q+vt$), if we apply an arbitrary forcive to S*, or such a forcive as to produce any arbitrary deformation or motion of S. Let for example S be an ideal rigid spherical lining of our cavity; and let any infinitesimal arbitrary motion be given to it. We need not at present consider infinitesimal rotation of S: the spherical waves which this would produce, particularly simple in their character, were investigated in my Baltimore Lectures, and described in recent communications to the British Association and Philosophical Magazine†. Neither need we consider curvilinear motion of the centre of S, because, the motion being infinitesimal, independent superposition of x -, y -, z -motions produces any curvilinear motion whatever.

§ 17. Take then definitively $\mathcal{E}(t)$, or simply $\mathcal{E}$, an arbitrary function of the time, to denote excursion in the direction OX, of the centre of S from its equilibrium-position. Let $\partial^{-1}\mathcal{E}$, $\partial^{-2}\mathcal{E}$ denote $\int_0^t dt \mathcal{E}$ and $\int_0^t dt \int_0^t dt \mathcal{E}$. Our problem is, supposing the solid to be everywhere at rest and unstrained when $t=0$, to find (ξ, η, ζ) for every point of the solid ($r>q$) at all subsequent time (t positive); with

$$\text{at } r=q, \quad \xi=\mathcal{E}(t), \quad \eta=0, \quad \zeta=0 \quad . \quad . \quad . \quad (44).$$

These, used in (19), give

$$0 = \frac{\ddot{\mathcal{F}}_1(t)}{u^2} + \frac{\ddot{\mathcal{F}}_2(t)}{v^2} + 3 \left\{ \frac{1}{q} \left[\frac{\dot{\mathcal{F}}_1(t)}{u} + \frac{\dot{\mathcal{F}}_2(t)}{v} \right] + \frac{1}{q^2} [\mathcal{F}_1(t) + \mathcal{F}_2(t)] \right\} \quad (45).$$

and

$$\mathcal{E}(t) = -\frac{1}{q^2} \left[\frac{\dot{\mathcal{F}}_1(t)}{u} + \frac{\dot{\mathcal{F}}_2(t)}{v} \right] - \frac{1}{q^3} [\mathcal{F}_1(t) + \mathcal{F}_2(t)] - \frac{1}{q} \frac{\ddot{\mathcal{F}}_1}{u^2} \quad . \quad . \quad (46),$$

* If the space inside S is filled with solid of the same quality as outside, the solution remains algebraic, if the forcive formula is algebraic, though discontinuous. The displacement of S ends, not at time $t=\tau$ when the forcive is stopped, but at time $t=\tau+2q/u$ when the last of the inward travelling wave produced by it has travelled in to the centre, and out again to $r=q$.

† B. A. Report, 1898, p. 783; Phil. Mag. Nov. 1898, p. 494.

Adding $3q \times (46)$ to (45), we find

$$3q\mathcal{E}(t) = -2 \frac{\ddot{\mathcal{X}}_1(t)}{u^2} + \frac{\ddot{\mathcal{X}}_2(t)}{v^2} \quad . \quad . \quad . \quad (47);$$

whence

$$\frac{\ddot{\mathcal{X}}_2(t)}{v^2} = 2 \frac{\ddot{\mathcal{X}}_1(t)}{u^2} + 3q\partial^{-2}\mathcal{E}(t) \quad . \quad . \quad . \quad (48);$$

and by this eliminating $\mathcal{X}_2$ from (46),

$$\left[\partial^2 + \frac{1}{q}(u+2v)\partial + \frac{1}{q^2}(u^2+2v^2) \right] \mathcal{X}_1(t) = \mathcal{G}(t) \quad . \quad (49),$$

where ∂ denotes d/dt , and

$$\mathcal{G}(t) = -qu^2 \left(1 + \frac{3v}{q}\partial^{-1} + \frac{3v^2}{q^2}\partial^{-2} \right) \mathcal{E}(t) \quad . \quad (50).$$

§ 18. I hope later to work out this problem for the case of motion commencing from rest at $t=0$, and $\mathcal{E}(t)$ an arbitrary function; but confining ourselves meantime to the case of $\mathcal{S}$ having been, and being, kept perpetually vibrating to and fro in simple harmonic motion, assume

$$\mathcal{E}(t) = h \sin \omega t \quad . \quad . \quad . \quad (51).$$

With this, (50) gives

$$\mathcal{G}(t) = hqu^2 \left[\left(\frac{3v^2}{q^2\omega^2} - 1 \right) \sin \omega t + \frac{3v}{q\omega} \cos \omega t \right] \quad . \quad (52).$$

To solve (49) in the manner most convenient for this form of $\mathcal{G}(t)$, we now have

$$\begin{aligned} \mathcal{X}_1(t) &= \frac{1}{\partial^2 + \frac{1}{q}(u+2v)\partial + \frac{1}{q^2}(u^2+2v^2)} \mathcal{G}(t) \\ &= \frac{\partial^2 + \frac{1}{q^2}(u^2+2v^2) - \frac{1}{q}(u+2v)\partial}{\left[\partial^2 + \frac{1}{q^2}(u^2+2v^2) \right]^2 - \left[\frac{1}{q}(u+2v)\partial \right]^2} \mathcal{G}(t) \\ &= hqu^2 \times \\ &\quad \frac{\left\{ \frac{3v^2}{q^4\omega^2}(u^2+2v^2) + \frac{1}{q^2}(v^2+3vu-u^2) + \omega^2 \right\} \sin \omega t + \frac{u-v}{q\omega} \left(\frac{3uv}{q^2} + \omega^2 \right) \cos \omega t}{\left[\frac{1}{q^2}(u^2+2v^2) - \omega^2 \right]^2 + \frac{\omega^2}{q^2}(u+2v)^2} \quad . \quad . \quad (53). \end{aligned}$$

With $\mathcal{H}_1$ thus determined, (48) gives $\mathcal{H}_2$ as follows,

$$\frac{\mathcal{H}_2(t)}{v^2} = \frac{2\mathcal{H}_1(t)}{u^2} - \frac{3hq}{\omega^2} \sin \omega t \quad . \quad . \quad . \quad (54).$$

For ξ, η, ζ by (19) we now have

$$\left. \begin{aligned} \xi &= B(r, t)x^2 - \frac{1}{r^3} \left[\frac{\ddot{\mathcal{H}}_1(t_1)}{u} + \frac{\ddot{\mathcal{H}}_2(t_2)}{v} \right] - \frac{1}{r^3} [\mathcal{H}_1(t_1) + \mathcal{H}_2(t_2)] - \frac{1}{r} \frac{\ddot{\mathcal{H}}_1(t_1)}{u^2} \\ \eta &= B(r, t)xy \\ \zeta &= B(r, t)xz \end{aligned} \right\} (55),$$

where (compare § 9 above)

$$\left. \begin{aligned} B(r, t) &= \frac{1}{r^3} \left[\frac{\ddot{\mathcal{H}}_1(t_1)}{u^2} + \frac{\ddot{\mathcal{H}}_2(t_2)}{v^2} \right] + \frac{3}{r^4} \left[\frac{\ddot{\mathcal{H}}_1(t_1)}{u} + \frac{\ddot{\mathcal{H}}_2(t_2)}{v} \right] \\ &\quad + \frac{3}{r^5} [\mathcal{H}_1(t_1) + \mathcal{H}_2(t_2)] \end{aligned} \right\} \quad . \quad (56).$$

$$\left. \begin{aligned} t_1 &= t - \frac{r-q}{u}; & t_2 &= t - \frac{r-q}{v} \end{aligned} \right\}$$

§ 19. The wave-lengths of the equivoluminal and rotational waves are respectively $\frac{2\pi u}{\omega}$ and $\frac{2\pi v}{\omega}$. For values of r very great in comparison with the greater of these, the second members of (55) become reduced approximately to the terms involving $\ddot{\mathcal{H}}_1$ and $\ddot{\mathcal{H}}_2$. These terms represent respectively a train of equivoluminal waves, or waves of transverse vibration, and a train of irrotational waves, or waves of longitudinal vibration; and the amplitude of each wave as it travels outwards varies inversely as r .

§ 20. For the case of an incompressible solid we have $v = \infty$, which by (53) gives

$$\frac{\ddot{\mathcal{H}}_1(t)}{u^2} = -\frac{3}{2}hq \sin \omega t \quad . \quad . \quad . \quad (57);$$

and by (55) we have, for r very great,

$$\left. \begin{aligned} \xi &\doteq -\frac{3}{2}hq \sin \omega t \left(\frac{x^2}{r^3} - \frac{1}{r} \right) \\ \eta &\doteq -\frac{3}{2}hq \sin \omega t \frac{xy}{r^3} \\ \zeta &\doteq -\frac{3}{2}hq \sin \omega t \frac{xz}{r^3} \end{aligned} \right\} \quad . \quad . \quad . \quad (58),$$

which fully specify, for great distances from the origin, the wave-motion produced by a rigid globe of radius q , kept moving to and fro according to the formula $h \sin \omega t$.

[To be continued.]