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The Conic Determined by Five Given Points

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*Example.*—Find the present worth of a perpetual annuity of £1, payable at the end of the first year, the yearly payments increasing by £1 each year, and the rate of interest being  $3\frac{1}{2}$  per cent.

Here £1 is the annual interest on £30.

To obtain £1 interest at the end of the first year we should therefore have to invest £30.

To obtain £2 interest at the end of the second year we should have to invest another £30 in one year's time, making £60 in all.

To obtain £3 interest at the end of the third year we should have to invest another £30 in two years' time, and so on.

Hence the given annuity is equivalent to a series of payments of £30 once a year, *i.e.*, to a perpetual annuity of £30, the first payment taking place immediately.

Therefore by (1) its present value =  $£30 \times 30 + £30 = £930$ .

It will now be easy for the teacher or student to apply the same methods to almost any problem on annuities that may be proposed.

G. H. BRYAN.

## THE CONIC DETERMINED BY FIVE GIVEN POINTS.

The majority of text-books on Geometrical Conics do not give any proof of the fundamental property that one and only one conic passes through any five given points, no three of which lie on a straight line. Special attention is drawn to this omission by the Rev. Dr. C. Taylor in an article published in the *Gazette* for May, 1895. The following is a description of three methods which various attempts at obtaining an elementary proof of the theorem have suggested. It is, however, very desirable to have a simpler proof than any of these, if such can be given.

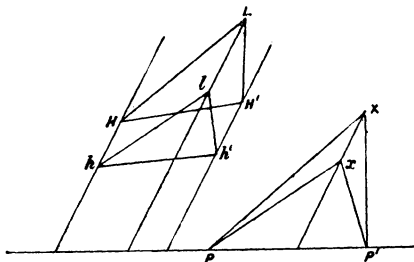
We cannot expect to be able to prove the theorem in question by simply showing that there is a fixed point  $S$  such that the distance of each of the given points from  $S$  bears the same ratio to its distance from a fixed straight line. For in order to show this we must find  $S$ , which is a focus of the conic through the given points. But a construction for one focus will determine all the foci; and such a construction will involve first finding the centre and axes of the conic. Hence, in whatever way we attempt to prove the theorem, we shall, in one form or another, have to make use of the properties of the centre and diameters.

One proof of the theorem, which will only be referred to here, can be deduced from the property that if through any point  $T$  two straight lines be drawn in fixed directions, cutting a given conic in  $P, P'$  and  $Q, Q'$ , then the ratios of the products  $TP \cdot TP', TQ \cdot TQ'$  is constant for all positions of  $T$ . The objection to this method of proof is the necessity of paying attention to the sign of each product.

A second proof, which, however, cannot be rendered general without appealing to the principle of continuity, is by oblique parallel projection; which, when effected in the plane, consists in altering the ordinates of points in a fixed direction to a given base line in any constant ratio. Thus parallel lines project into parallel lines; and it can be proved that any oblique projection of a circle is an ellipse.

Let  $H, H', L, M, N$  be the five given points. We may suppose that  $L, M, N$  are all on the same side of  $HH'$ . For the sake of simplicity the points  $M, N$ , and those points mentioned later whose construction depends on  $M, N$ , are not inserted in the figure. Take any point  $X$  and any base line  $PP'$ ; draw  $XP, XP'$  parallel to  $LH, LH'$ ;  $XQ, XQ'$  parallel to  $MH, MH'$ ; and  $XR, XR'$  parallel to  $NH, NH'$ . The locus of points at which  $PP', QQ'$  subtend equal angles (or supplementary angles if  $PP', QQ'$  are in opposite directions) is, by a known theorem, a circle.

Similarly there is a second circle corresponding to  $PP', RR'$ . If these circles are real, and have a real intersection, the given points will lie on an ellipse, otherwise they will lie on a hyperbola. For suppose the circles cut in a real point  $x$  on the same side of  $PP'$  as  $X$ . Project with



respect to  $PP'$  as base line, so that  $x$  may be the projection of  $X$ ; and let  $H, H', L, M, N$  project into  $h, h', l, m, n$ . Then the angles  $PxP', hlk'$  are equal; for  $XP, XP'$  are parallel to  $LH, LH'$ , and their projections  $xP, xP'$  are parallel to  $lh, lh'$ . And since the angles  $PxP', QxQ', RxR'$  are equal, by construction, the angles  $hlk', hmk', hnk'$  are also equal. Thus  $h, h', l, m, n$  lie on a circle, and therefore  $H, H', L, M, N$  lie on an ellipse.

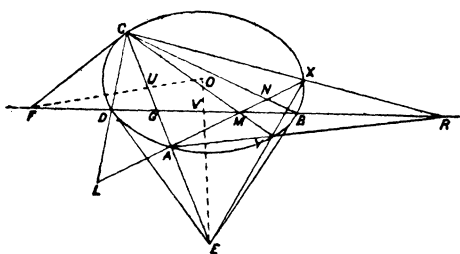
A third proof, which is free from the objections already noticed, but which assumes the properties of poles and polars, and is therefore not so elementary as might be desired, depends on the following theorem:—

*If  $A, B, C, D$  be any four given points, and any line through  $A$  cut  $CD, DB, BC$  in  $L, M, N$ , then the locus of the point  $X$  on this line such that the range  $LMNX$  is harmonic is a conic through  $A, B, C, D$ .*

Let  $AC, BD$  cut in  $G$ ; let  $E, F$  be the harmonic conjugates of  $G$  with respect to  $AC, BD$ , and  $U, V$  the middle points of  $AC, BD$ ; let  $EV, FU$  cut in  $O$ . Then one and only one conic can be described with centre  $O$ , touching  $ED$  at  $D$ , and passing through  $B$ ; for  $OD$  is then a semi-diameter,  $DE$  the direction of

the conjugate diameter, and  $B$  a point on the curve. This conic will pass through  $A$  and  $C$ ; for the pole of  $BD$  is the point  $E$  where the diameter bisecting  $BD$  meets the tangent at  $D$ ; and the polar of  $F$  is  $EG$ , since it passes through  $E$  the pole of  $BD$ , and also through  $G$ . Hence the chord of the conic on the line  $EG$  is bisected by  $FO$  at  $U$ , and is divided harmonically by  $E$  and  $G$ ; and this chord can be no other than  $AC$ .

Take any point  $X$  on the conic; let  $EX$  cut the conic again in  $Y$ ; let  $AY$ ,  $CX$  meet in  $R$ , and  $AX$ ,  $CY$  in  $M$ . Then since



$A, C, X, Y$  are four points on the conic, the triangle  $EMR$  is a self-conjugate triangle. Hence  $M$  and  $R$  lie on the polar of  $E$ , that is on  $BD$ ; and the range  $DMBR$  is harmonic. Therefore the harmonic pencil  $C(DMBR)$  cuts  $AX$  harmonically; hence the locus of the point

$X$  such that the range  $LMNX$  is harmonic is the conic we have been considering.

Let  $B, C, D, X, X'$  be any five given points. Then there is only one line through  $X$  such that the range  $LMNX$  is harmonic,  $XM$  being found by joining  $X$  to the point  $M$  in which the harmonic conjugate of  $CX$  with respect to  $CB, CD$  cuts  $BD$ . Let  $XM$  meet the similar line  $X'M'$  through  $X'$  in  $A$ . Then  $B, C, D, X, X'$  lie on a conic, viz., the conic corresponding to  $A$  and the triangle  $BCD$  as in the theorem.

EDITOR.

## EXAMINATION QUESTIONS AND PROBLEMS.

Questions 62-69 are taken from Woolwich and Sandhurst Entrance papers set in November last; 70-81 from recent Entrance Scholarship papers at Oxford and Cambridge; and 82, 83 from the London Matriculation paper of January.

62. Simplify

$$\frac{1}{1-x^4} \left\{ 1 + \frac{x}{1-x} + \frac{x^2}{1-x^2} + \frac{x^3}{(1-x)(1-x^2)} + \frac{x^3}{(1-x^2)(1-x^3)} \right\}.$$

63. Show that a rectangle of given perimeter has the greatest area when its sides are equal.

64. From a given point without a circle draw, when possible, a straight line which shall have its middle point and its other end upon the circumference.