On the Flow of Electricity in a uniform plane conducting Surface - Part I

This content has been downloaded from IOPscience. Please scroll down to see the full text.

1874 Proc. Phys. Soc. London 1 113

(<http://iopscience.iop.org/1478-7814/1/1/315>)

View [the table of contents for this issue](#), or go to the [journal homepage](#) for more

Download details:

IP Address: 128.192.114.19

This content was downloaded on 01/10/2015 at 11:11

Please note that [terms and conditions apply](#).

PROCEEDINGS
OF
THE PHYSICAL SOCIETY
OF LONDON.

February 27, 1875.

XV. *On the Flow of Electricity in a uniform plane conducting Surface.*—Part I. By G. CAREY FOSTER, F.R.S., and OLIVER J. LODGE.

[With Two Plates.]

1. THE objects of the following paper are:—first, to show that the most important laws relating to the flow of electricity in a plane conducting sheet of uniform conductivity can be established by mathematical considerations of such a simple kind that there is no reason why they should not be introduced into ordinary teaching, as well as into the more complete elementary text-books of electricity; and in the second place, to describe methods that may conveniently be employed for testing experimentally the theoretical conclusions, and to record some results obtained by these methods.

The general subject treated of in this paper has attracted the attention of a considerable number of mathematicians and physicists. The earliest published investigation relating to it is contained in a remarkable memoir by Kirchhoff, which appeared in Poggendorff's *Annalen* in 1845 (vol. lxiv. p. 497). In this paper Kirchhoff established the general mathematical theory of the flow of electricity in an unlimited uniformly conducting-

sheet, and in a limited sheet with a circular boundary, with so much completeness as to leave little for others to do beyond working out the application to special cases of the general principles he laid down, or finding other methods of establishing the conclusions he deduced from them.

We cannot better indicate the general plan of Kirchhoff's investigation than by quoting the following account of it from a paper by Professor W. Robertson Smith*, to which we shall have to make further reference immediately:—"By an application of Ohm's law, he [Kirchhoff] expressed analytically the condition to be satisfied by v [the potential]. When the electricity enters and issues by a number of individual points, he found (apparently by trial) that an integral of the form $\Sigma(\alpha \log r)$, where $r_1, r_2, \&c.$ are the distances of the point (x, y) from the successive points of entrance and issue, satisfied the conditions when the plate is infinite. For a finite plate, it is necessary that the boundary of the plate should be orthogonal to the curves

$$\Sigma(\alpha \log r) = \text{const.} \quad . \quad . \quad (3)$$

He was thus led to form the orthogonal curves whose equation he gives in the form

$$\Sigma(\alpha[r, R]) = \text{const.}, \quad . \quad . \quad (4)$$

where $[r, R]$ is the angle between r and a fixed line R . These equations he applies to the case of a circular plate, completely determining the curves when there is one exit and one entrance point in the circumference, and showing that in any case a proper number of subsidiary points would make the equipotential lines determined by (3) cut the circumference at right angles. Kirchhoff's paper is throughout properly busied with the function v , and the stream-lines are only dealt with incidentally. There is no attempt to give a physical meaning to the equation (4)." To this we have only to add that Kirchhoff proved the accuracy of his theoretical deductions by determining experimentally the form and distribution of the equipotential lines on a circular disk with two electrodes on the edge, as well as (Pogg. *Ann.* vol. lxvii. p. 344, 1846) the strength of the current at various parts of the disk; and that, from the expression for the difference of potential between the electrodes, he deduced by Ohm's law the resistance of a circular disk with two small electrodes

* Proc. Edinb. Roy. Soc. 1869-70, pp. 79-99.

anywhere upon it. In order to test experimentally the value thus obtained, he seems to have devised independently the arrangement now commonly known as Wheatstone's bridge; but, owing to the smallness of the resistance to be measured, he was unable to obtain satisfactory results.

Soon after the publication of Kirchhoff's paper, Smaasen* gave an investigation of the flow of electricity in a plane conducting sheet, in which he takes account, in determining the potential, of the electricity given off to the air, and deduces the resistance of an infinite sheet with two small circular electrodes by a process which, though longer than that employed by Kirchhoff, may be regarded as more direct. It consists in finding the resistance of the space between two lines of flow at an infinitely small distance apart, and then extending by integration the expression thus obtained so as to make it apply to the unlimited sheet. In a subsequent paper† Smaasen determined by an analogous process the resistance of a conducting sphere, or of an unlimited conductor of three dimensions‡. Smaasen's treatment of the subject is, like Kirchhoff's, based chiefly upon the mode of distribution of the potential; the only investigation we are acquainted with which deals specially with lines of flow is contained in the paper by Professor W. Robertson Smith from which we have already quoted. The starting-point adopted by Professor Smith is the same as that from which we have set out in the following communication; and, indeed, we found, after making some progress in our own work, that several of our demonstrations (which we at first thought were new) had been already given by him, while all the chief conclusions were, as we have said above, implicitly contained in Kirchhoff's original memoir. Consequently, although the present paper contains a few minor results which, so far as we know, have not been pointed out ex-

* Pogg. *Ann.* vol. lxi. p. 161.

† Pogg. *Ann.* vol. lxxii. p. 435.

‡ About the same time, the same subject was taken up by Ridolfi of Florence (*Il Cimento*, An. V. 1847, May-June), whose paper, however, we only know from the references of Beetz (in Dove's *Repertorium*, vol. viii. p. 147) and Poggeudorff (Pogg. *Ann.* vol. lxxii. p. 449). No reference to this paper is given either in the Royal Society's Catalogue of Scientific Papers, or in the carefully compiled "Bibliographie" of the mathematical theory of the voltaic pile in Verdet's *Conférences de Physique* (*Œuvres de Verdet*, vol. iv. p. 351).

plicitly before, we do not claim for its contents any essential novelty; and our only reason for venturing to publish it is that the mode of treatment adopted in it has enabled us to establish by strictly elementary mathematical methods, and so to render accessible to a greater number of readers, many of the conclusions which previous writers had arrived at by more abstruse processes.

2. In order to avoid ambiguity, it may be well to begin by giving a few general explanations and definitions of terms that will be frequently used in what follows. It will be assumed that, in all cases considered, the flow of electricity has attained a *stationary* condition—that is to say, that on the whole there is neither gain nor loss of electricity at any part of the sheet, but that at every instant the quantity that enters any part and the quantity that leaves it are equal. Positive electrodes, or such as supply positive electricity to the sheet, will be spoken of as “sources,” while negative electrodes will be termed “sinks.” The rate at which a source or a sink supplies or removes electricity will be spoken of as the “strength” of the source or sink.

Through every point of the sheet a straight line can be drawn such that no electricity crosses it at that point; the direction of a line so drawn through a given point is the “direction of the current” at that point. A “line of flow” or “stream-line” is a line drawn so that no electricity crosses it anywhere; or, in other words, it is a line whose direction at all points coincides with the direction of the current. The quantity of electricity flowing between two consecutive lines of flow is everywhere the same; for since no electricity crosses either of them, whatever quantity starts between them must remain between them throughout their whole course. It may be useful to observe that flow-lines are not loci of constant strength of flow, they are lines of direction simply.

The “strength of the current” at any point is given by the quantity of electricity which crosses, in unit time, a short line normal to the flow-line through the point, divided by the length of the line.

If a line be drawn so as to intersect the lines of flow everywhere at right angles, there will be no flow of electricity along it; but since the sheet on which the line is drawn is a conductor, this implies that all points of the line are at the same potential, or that the line is an “equipotential line.”

3. It is obvious that a line of flow and an equipotential line might be drawn through every point of a conducting sheet traversed by a current, so that, if it were possible to draw them all, the sheet would be completely covered by two sets of lines cutting each other at right angles. The general distribution, however, of the lines of each set can be correctly shown without drawing more than a limited number, if those that are drawn are selected according to a definite and easily recognizable rule. For this purpose the most convenient rule to adopt in relation to the lines of flow, is to place them so that the total flow of electricity between each pair of consecutive lines is the same—and with regard to the equipotential lines, to place them so that the change of potential in passing from any one to the next consecutive line is the same. In the sequel, whenever a *system* of lines of flow, or of equipotential lines, is referred to, it is to be understood that the lines are placed so as to fulfil the above conditions.

One Pole in an Infinite Sheet.

4. In this case it is evident that the lines of flow are straight lines radiating out from the pole, and that the spaces between each pair of consecutive lines will convey equal currents if each line makes the same angle with the next. If a circle of radius r be drawn about the pole as centre, the quantity of electricity which crosses the whole circumference in unit of time, will evidently be the same whatever the value of r , and will be equal to the quantity Q supplied in the same time by the source. Hence, at distance r from the source, the strength of the current will be

$$\frac{Q}{2\pi r},$$

or inversely proportional to the distance r .

It is evident also, either from general considerations of symmetry, or from the condition that equipotential lines and lines of flow must intersect each other at right angles, that the equipotential curves for the case we are considering are circles having the source at their common centre.

5. If R be the resistance to the radial flow of electricity across the annular portion of the sheet bounded by circles at the potentials V_1 and V_2 , we have, by Ohm's law,

$$V_1 - V_2 = QR ;$$

whence, since Q is constant, equal differences of potential will be found at the boundaries of annular belts of equal resistance. The relation between the radii of successive equipotential circles with a constant difference of potential may be deduced from this condition as follows:—The resistance of any very narrow belt must be proportional to its breadth, or the difference between the radii of its external and internal circumferences; and it must be inversely proportional to the mean circumference, which, seeing that the circumference varies directly as the radius, will be simply the arithmetic mean of the external and internal circumferences. That is, if r_1, r_2, r_3 are radii of successive circles, these circles will bound spaces of equal resistance if

$$\frac{r_2 - r_1}{\pi(r_2 + r_1)} = \frac{r_3 - r_2}{\pi(r_3 + r_2)},$$

or

$$r_1 r_3 = r_2^2.$$

Hence the radii of successive equipotential circles form a geometrical progression whose common ratio may be called μ .

6. To find the resistance of a belt bounded by concentric circles whose radii are r_1 and r_2 respectively, let

$$\frac{r_2}{r_1} = \mu^n,$$

so that there are n equipotential circles contained in the belt, where

$$n = \frac{1}{\log \mu} \log \frac{r_2}{r_1}.$$

Denote the thickness of the conducting sheet by δ and its conductivity by κ ; then, by the last section, the resistance of a narrow ring whose bounding circles have the radii 1 and $1 + \Delta r (= \mu)$ will be

$$R_0 = \frac{1}{\kappa \delta} \cdot \frac{\Delta r}{\frac{1}{2}(2 + \Delta r) 2\pi} = \frac{1}{\pi \kappa \delta} \cdot \frac{\Delta r}{2 + \Delta r}.$$

But the whole belt we are considering is made up of n rings, each of the resistance R_0 . So the resistance of the whole belt is

$$R = nR_0 = \frac{1}{(2 + \Delta r) \pi \kappa \delta} \cdot \frac{\Delta r}{\log(1 + \Delta r)} \log \frac{r_2}{r_1};$$

or, letting Δr decrease without limit,

$$R = \frac{1}{2\pi \kappa \delta} \log \frac{r_2}{r_1}. \quad . \quad . \quad . \quad . \quad . \quad (1)$$

7. Putting this value of R into the expression for the difference of potential of two circles of radii r_1 and r_2 , we have

$$V_1 - V_2 = \frac{Q}{2\pi\kappa\delta} \cdot \log\left(\frac{r_2}{r_1}\right),$$

whence the potential at any point of the sheet is at once given in terms of the strength of the source and the potential at unit distance from the source. Thus let V_1 be the potential at unit distance, then the potential V at any distance r becomes

$$V = V_1 - \frac{Q}{2\pi\kappa\delta} \log r. \quad . \quad . \quad . \quad . \quad (2)$$

8. The same result may be obtained without previously calculating the resistance of a belt of finite breadth as follows. The radii of successive equipotential circles, beginning at the pole itself, are (§ 5)

$$\mu^{-\infty} \dots \mu^{-2}, \mu^{-1}, 1, \mu, \mu^2 \dots \mu^{\infty}.$$

Let the common difference of potential between consecutive circles be Δv ; then, if $r = \mu^n$, the potential at distance r is less than the potential at unit distance by n times Δv , or

$$V_r = V_1 - n\Delta v = V_1 - \frac{\Delta v}{\log \mu} \log r.$$

But $\frac{\Delta v}{\log \mu}$ is constant; and putting $1 + dr$ for μ , where dr is the breadth of the infinitely narrow belt whose inner boundary is the circle of unit radius, we have

$$\frac{\Delta v}{\log \mu} = \frac{dv}{\log(1 + dr)} = \frac{dv}{dr}.$$

Multiplying this by the circumference of the belt (2π) and by the conductivity and thickness of the sheet, we get the strength of the current across the whole belt, or

$$2\pi\kappa\delta \frac{dv}{dr} = Q,$$

$$\frac{\Delta v}{\log \mu} = \frac{Q}{2\pi\kappa\delta},$$

whence

$$V = V_1 - \frac{Q}{2\pi\kappa\delta} \log r,$$

as before.

Two equal Opposite Poles in an Infinite Sheet.

9. If there are two or more electrodes in an infinite uniform conducting sheet, the strength and direction of the current and the potential at any part of the sheet are obtained by the simple superposition of the effects which each electrode would produce at that part if it were the only electrode in the sheet. This might be regarded as probable *à priori*; and it is proved by experiment to be the case. Hence the effect of any number of poles in a sheet may be deduced by properly extending the conclusions already arrived at with regard to a single pole. We will first discuss the case of a single source and a single sink of equal strength.

10. *Lines of Flow.*—Let A (Plate IV. fig. 1) be the source, and B the sink. The flow-lines due to these, taken separately, would be two equiangular pencils of straight lines drawn outwards from A and inwards towards B. These two systems of lines subdivide the whole surface of the conducting sheet into quadrilaterals, such as P Q and Q R. The actual direction of the flow at any point P, due to the combined action of the source and sink, will evidently be intermediate between the directions of the primary streams A P and P B through the same point; similarly the direction of the resultant current at Q will be intermediate between the direction of the primary streams A Q and Q B; in other words, the line of resultant flow through P will enter the quadrilateral P Q at P, and the line of resultant flow through Q will pass at Q from the quadrilateral P Q into the quadrilateral Q R. And it can be shown that the points P, Q, and R are points on the same line of flow—that is, that a continuous curve can be drawn through P, Q, and R such that no electricity flows across it. Thus, let the points of intersection of the primary flow-line A P with the primary flow-lines through B which are nearest to P B on either side of it, be marked P' and P_i respectively; and similarly let Q' and Q_i, R' and R_i be the points where A Q and A R intersect the next flow-lines on either side of Q B and R B respectively. Then, since the total flow of electricity from A between the lines A P and A Q is everywhere the same and is equal to the flow towards B between the lines P B and Q B, which is likewise everywhere the same, the quantity of electricity flowing in a given time across any of the lines P Q', P_i Q, P P_i, Q' Q, which bound the quadrilateral

PQ , is equal. Consequently, considering either of the triangles $PQ'Q$ or PP_iQ , the flow inwards across PQ' or PP_i is equal to the flow outwards across $Q'Q$ or P_iQ ; and therefore there cannot on the whole be any flow across a line drawn inside the quadrilateral from P to Q . Evidently also, by drawing a sufficient number of straight lines through A in directions intermediate between AP and AQ , and an equal number of lines through B in directions lying between PB and QB , keeping the angle between consecutive lines constant in both cases, we can subdivide the line PQ into portions as short as we please, and such that no electricity flows across any of them. Hence P and Q are points on the same line of flow; and it follows, similarly, that the point R is also situated on this line. The same kind of reasoning also proves that a second line of flow passes through the points P' , Q' , and R' , and a third through the points P_p , Q_p , and R_p . Moreover, since the strength of the flow between the lines $P'Q'R'$ and PQR is measured by the quantity of electricity which in unit of time crosses any of the lines $P'P$, PQ' , $Q'Q$, QR' , or $R'R$, and the strength of the flow between the lines PQR and $P_iQ_iR_i$ is measured by the quantity which similarly crosses any of the lines PP_p , P_iQ , QQ_p , Q_iR , or RR_p , and since these quantities are equal to each other, it follows that the lines $P'Q'R'$, PQR , and $P_iQ_iR_i$ are consecutive flow-lines of a system which divides the conducting sheet into portions each of which conveys an equal current.

It is important to observe that the reasoning employed here is general, and not limited to the special case to which it has been applied. The general conclusion to which it leads may be thus stated:—If similar* systems of lines of flow be drawn, corresponding to each of two separate systems of sources and sinks, the lines of flow which would result from the combined action of the sources and sinks of both systems will be obtained by drawing curves through the alternate angles of the quadrilaterals produced by the intersections of the two primary systems of flow-lines, in directions concurrent with both the primary flow-lines that intersect each other at each angle.

11. The method which, as we have seen, allows the flow-lines

* By *similar* systems is here to be understood systems such that the total flow between any pair of consecutive lines of the one set is the same as that between any two consecutive lines of the other set.

for two equal opposite poles to be drawn, also enables us to deduce very easily their general form. Let α be the constant angle between consecutive flow-lines of the pencil diverging from A and of that converging to B. Then, evidently,

$$\angle APB = \angle AQB = \angle ARB = \dots = n\alpha,$$

where n is a constant integer. Also

$$\angle AP'B = \angle A'QB = \dots = (n+1)\alpha,$$

and

$$\angle AP_nB = \angle A_nQB = \dots = (n-1)\alpha.$$

Hence the lines of flow due to one source and one sink of equal strength are arcs of circles passing through the poles, each one differing from the next by a constant change ($=\alpha$) in the angle which the radii vectores from the poles make with each other; or, what comes to the same thing, they are arcs of circles cutting each other at the poles with a constant difference of angle equal to the constant difference of angle ($=\alpha$) between the rectilinear flow-lines which either the source or the sink would produce by itself.

12. The whole number of flow-lines is therefore $\frac{2\pi}{\alpha}$, or the same as the number of lines leaving the source or entering the sink when either of them is by itself in the sheet. This is evident also if we consider that infinitely near to either pole the effect of the other vanishes in comparison, and therefore the lines close to each pole are the same in all respects as they would be if the other pole were absent.

13. The circular form of the flow-lines for the case we are considering can be demonstrated in various other ways. We will give here one additional proof, on account of its great simplicity.

The flow through any point P due to a source at A and a sink at B, being the resultant of the currents through the same point due to A and B taken separately, will be represented in strength and direction by the third side of a triangle whose other two sides represent the currents from A and towards B, respectively, in strength and direction. But the strengths of the two component currents are inversely as the distances AP and BP respectively (§ 4): hence the following construction (fig. 2). From B draw BT parallel to AP, and make its length a third proportional to AP and PB; then PT gives the direction of

the flow at P, and its length is proportional to the strength of the current at P. The similarity of the triangles A P B and P B T gives the angle B P T equal to the angle P A B; and consequently (converse of Euclid, III. 32) the locus of P is a circle through A and B.

14. It was shown in § 10 how a system of lines of flow can be traced out by successive points. To be able to draw them continuously with compasses we only require to know the position of the centres; and these are easily found from the following considerations. Since the circles of which the flow-lines are arcs pass through the poles A and B, their centres lie in the straight line at right angles to A B, through O its middle point. If C (fig. 3) be the centre of the circle which gives the flow-line through any point P, the angle at C is equal to the angle at P—the angle characteristic of the given flow-line; and therefore the angle O A C is the complement of the angle at P. Putting $A B = 2a$, we have

$$O C = a \cdot \tan O A C = a \cdot \tan \left(\frac{\pi}{2} - A P B \right).$$

Let the number of lines to be drawn be m , then the constant difference between the angles contained under consecutive lines will be

$$\frac{2\pi}{m} = \alpha,$$

and the several lines will be given by making the angle at the circumference successively equal to

$$0, \alpha, 2\alpha, \dots, \pi - \alpha, \pi, \pi + \alpha, \dots, 2\pi - 2\alpha, 2\pi - \alpha, 2\pi,$$

or, what is the same thing, to

$$0, \alpha, 2\alpha, \dots, \pi - \alpha, -\pi, -(\pi - \alpha), \dots, -2\alpha, -\alpha, 0,$$

where the values $0, \pi, -\pi$, and 2π are represented by the straight line through A and B, and negative angles indicate flow-lines passing on the lower side of A B. But, as is evident from the figure, the same circle gives the two flow-lines whose characteristic angles are $n\alpha$ and $-(\pi - n\alpha)$; hence the number of circles to be drawn or of centres to be found is equal to half the number of flow-lines; and we need only consider those characterized by angles between 0 and π , of which the complements are

$$\frac{\pi}{2}, \frac{\pi}{2} - \alpha, \dots, \alpha, 0, -\alpha, \dots, -\left(\frac{\pi}{2} - \alpha\right), -\frac{\pi}{2}.$$

Consequently the required centres are obtained by drawing lines from A making the above angles with AB, and letting them intersect the perpendicular to AB through O; or, without measuring angles, by laying off from O in both directions along the perpendicular to AB lengths proportional to

$$0, a \tan \alpha, a \tan 2\alpha, \dots a \tan \left(\frac{\pi}{2} - \alpha \right), a \tan \frac{\pi}{2}.$$

Plate IV. shows a system of lines of flow for which $\alpha = 20^\circ$.

15. The *strength of the current* at any point P due to a source at A and an equal sink at B is represented in the construction given in § 13 by the length of the line PT (fig. 2); but by the similarity of the triangles PBT and APB,

$$\frac{PT}{BT} = \frac{AB}{PB};$$

or, putting $AB = 2a$, $BT = \frac{1}{r}$, and $PB = r'$, the strength of the current is

$$s = \frac{Q}{2\pi} \cdot PT = \frac{Q}{2\pi} \cdot \frac{2a}{rr'}. \quad \dots \quad (3)$$

Hence, for given poles at a given distance apart, the strength of the current is the same at all points of the sheet for which the product of the distances from the two poles is constant; or the loci of equal flow are a system of *lemniscates**.

16. *Equipotential Lines*.—The flow-lines for the case of one source and one equal sink having been determined, the form of the equipotential lines is at once given by the consideration that the two sets of lines cut each other orthogonally (§ 2); and it is a well-known geometrical result that the system of lines orthogonal to the system of circles which, as has been seen, represents the flow-lines for this case, is another system of circles having

* The strength of the current at various parts of a circular disk with equal opposite poles on the edge was examined experimentally by Kirchhoff, by suspending a small magnet, with a mirror attached, close above the disk, and was found to agree closely with the results of calculation. Mach has since given a more direct proof that the lines passing through points of equal flow are lemniscates. A disk of silver-leaf was coated with a thin film of wax; and on allowing a strong current to traverse it for a few moments, the wax was melted within a space bounded by a lemniscate and the edge of the disk (Carl's *Repertorium f. experim. Physik*, vol. vi. p. 11, 1870).

their centres on the line through the poles, each circle cutting this line once internally and once externally in points situated harmonically with respect to its extremities. The simplest general expression for such a system of circles is the equation

$$\frac{r}{r'} = c,$$

where c is a quantity which remains constant for each circle but varies from each circle to the next, and r and r' are the distances from any point of the curve to the source and sink respectively.

These general properties of the equipotential lines are easily established by referring to the construction employed in § 13. We there saw that PT (fig. 2) is a tangent to the line of flow through P ; and consequently it is normal to the equipotential line through the same point. If we produce PT to cut AB produced in C , we have the triangles BCP and PCA similar, and

$$CP^2 = CA \cdot CB;$$

whence it appears that if tangents to the lines of flow be drawn from any point C in AB produced, the distance from C to the points where these tangents touch the lines of flow is constant and depends only on the distances of the point C from A and B respectively. Therefore, if a circle be drawn with centre C and radius CP , it cuts all the lines of flow at right angles, and is consequently an equipotential line. If it is only required to find the centre of the equipotential circle through any point P , the simplest method is to make an angle BPC equal to the angle BAP ; then the point where PC and AB intersect is the centre to be found.

The similarity of the triangles BCP and PCA also gives

$$\frac{AP}{PB} = \frac{r}{r'} = \frac{AC}{CP};$$

or the ratio of the *radii vectores* from the two poles is constant for a given circle, as already stated.

17. The above method suffices for drawing the equipotential lines through any number of given points, but not for placing them *systematically* (or so that the difference of potential between consecutive lines may be constant). For this purpose we may have recourse to a process of superposition of the same kind as that employed (§ 10) for placing the lines of flow.

We have seen (§ 5) that the equipotential lines for a single pole are concentric circles, and that the radii of consecutive circles form a geometrical progression. To find the system of equipotential lines for two equal opposite poles, it is only needful to draw for each pole separately a system of equipotential lines with the same difference of potential between any one line and the next as it is intended that there should be between the lines of the combined system, and then to draw lines through the intersections of the two overlapping systems of circles thus obtained, taking care, in going from one intersection to the next, that the changes of potential are in opposite directions for the two primary systems taken separately. Thus, let the lines 1, 2, 3, 4 (fig. 4) represent portions of equipotential lines due to a source at A; and 1', 2', 3', 4' portions of equipotential lines due to a sink at B; and let the potential of the line 1 be V , and let that of the line 1' be V' ; further, let the change of potential in passing from any one line to the next in either system be v , so that the potentials of the lines 2, 3, and 4 are $V-v$, $V-2v$, and $V-3v$ respectively, and the potentials of 2', 3', and 4' are $V'+v$, $V'+2v$, and $V'+3v$ respectively. Then, at the points where 1 and 1', 2 and 2', 3 and 3', 4 and 4' respectively intersect each other, the potentials will be the sums of the potentials of the intersecting lines; and therefore the potential at all these points is the same, namely $V+V'$. Consequently P, Q, R, and S are points on the same equipotential line. Similarly it follows that P_1 , Q_1 , and R_1 are points on the line whose potential is $V+V'+v$; and Q' , R' , and S' points on the line whose potential is $V+V'-v$. We thus get the potential of the resultant equipotential lines differing by the constant amount v , which is the same as the difference of potential of the lines of the two primary systems.

It is evident from this that any two systems of equipotential lines whatever, which have the same constant difference of potential, can be compounded, so as to give a single resultant system, by tracing lines through alternate angles of the quadrilaterals produced by the mutual intersection of the lines of the two systems, and also that the constant difference of potential between the lines of the resultant system will be the same as that between the lines of each of the component systems.

18. Let μ be the common ratio of the radii of the equipoten-

tial circles of the two primary systems considered in § 17, and let the radius of the circle 1 be μ^n and that of the circle 1' be μ^m (see § 8) ; the radii of the successive circles of the one set are then $\mu^{n+1}, \mu^{n+2}, \dots$, and of those of the other $\mu^{m+1}, \mu^{m+2}, \dots$. Hence for the ratio of the radii vectores from the two poles for the points P, Q, R, S, we have

$$\frac{\mu^n}{\mu^m}, \frac{\mu^{n+1}}{\mu^{m+1}}, \frac{\mu^{n+2}}{\mu^{m+2}}, \frac{\mu^{n+3}}{\mu^{m+3}}$$

respectively ; that is, we have for all these points the common ratio

$$\mu^{n-m}.$$

Similarly, we should find for the points P_1, Q_1 , and R_1 the common ratio

$$\mu^{n-m+1},$$

and for the points Q', R' , and S' the common ratio

$$\mu^{n-m-1}.$$

Hence, not only are the equipotential lines for two equal and opposite poles characterized by a constant ratio of the radii vectores, as already proved (§ 16), but this ratio *changes in a constant ratio* on passing from any one line of the system to the next, the ratio of change ($=\mu$) being the same as the ratio of change of radius on passing from one line of the system due to a single pole to the next.

19. The actual potential at any point of the sheet, in terms of the distances of this point from the two poles, follows directly from equation (2) in § 7. Let r be the distance of the given point from the source, and r' its distance from the sink ; put V for the potential at the point due to the source alone, and V' for that due to the sink alone. Then we have

$$V = V_1 - \frac{Q}{2\pi\kappa\delta} \cdot \log r,$$

and

$$V' = V'_1 - \frac{Q'}{2\pi\kappa\delta} \cdot \log r';$$

and since the source and sink are equal, $V'_1 = -V_1$ and $Q' = -Q$; therefore the resultant potential, or $V + V'$, is

$$U = \frac{Q}{2\pi\kappa\delta} \cdot \log \frac{r'}{r} \cdot \dots \dots \dots (4)$$

This gives the potential $=0$ at all points of the straight line equidistant from the two poles, positive on the side of this line next the source, and negative on the side next the sink. Also it shows that for equal differences of potential we must have equal differences in the value of $\log \frac{r}{r'}$, which agrees with what was proved above (§ 18).

We may write (4) thus (§ 8),

$$U = \frac{Q}{2\pi\kappa\delta} \log \mu^n = n \cdot \Delta v;$$

whence, regarding the source and sink for the present as mere points, the value $n=\infty$ will correspond to the former, and $n=-\infty$ to the latter, while $n=0$ denotes the straight line at an equal distance from both.

20. Let the distances from O (the middle point of A B) of the points D and D' (fig. 2), in which the circle of potential U cuts A B, be called l and l' ; then, a being as usual half the distance between the poles,

$$\frac{a+l}{a-l} = \frac{l'+a}{l'-a} = \mu^n,$$

where μ^n is the ratio of the radii vectores of the given circle. Hence

$$ll' = a^2,$$

and

$$l = a \frac{\mu^n - 1}{\mu^n + 1}, \quad l' = a \frac{\mu^n + 1}{\mu^n - 1}.$$

For the radius of the circle, we have evidently

$$\rho = \frac{1}{2}(l' - l) = \frac{2a\mu}{\mu^{2n} - 1},$$

and for the distance of its centre from O,

$$d = l + \rho = l' - \rho = a \frac{\mu^{2n} + 1}{\mu^{2n} - 1};$$

also

$$\rho^2 = CA \cdot CB = (d + a)h = d^2 - a^2,$$

where $h(=d-a)$ is the distance of the centre of the equipotential circle from the nearest pole.

21. In order to draw a system of equipotential circles with compasses, it is most convenient to have given the distances of

the centres from the nearest pole, and also the distances from the same point of one of the intersections with AB, the line joining the poles—that is, in fig. 2 the distance BC and either BD or BD'. Now

$$BC = h = d - a = \frac{2a}{\mu^{2n} - 1},$$

$$BD = a - l = \frac{2a}{\mu^n + 1},$$

and

$$BD = l' - a = \frac{2a}{\mu^n - 1}.$$

Any convenient value such as 2 or $\frac{3}{2}$ may be given to μ ; or if a special number of lines be required, a value may be found to suit. Thus let L be the greatest value that is to be given to l , and m the number of lines required on each side of O. We shall have

$$\frac{a+L}{a-L} = \mu^m, \text{ or } \log \mu = \frac{1}{m} \log \frac{a+L}{a-L},$$

which determines the value of μ .

The successive values of BD' are then

$$\frac{2a}{\mu-1}, \frac{2a}{\mu^2-1}, \frac{2a}{\mu^3-1}, \dots, \frac{2a}{\mu^m-1}.$$

The successive values of BC are simply alternate values of these, namely

$$\frac{2a}{\mu^2-1}, \frac{2a}{\mu^4-1}, \dots, \frac{2a}{\mu^{2m}-1}.$$

Plate IV. shows the equipotential lines drawn for a value of $\mu = \frac{5}{3}$.

22. *Resistance*.—The resistance, in the direction of the flow, of the part of the sheet extending between two given equipotential circles follows directly from equation (4) (in § 19). Thus for the potentials of the circles characterized by the ratios $r_1 : r'_1$ and $r_2 : r'_2$ respectively, where r_1 and r_2 are distances from the source, and r'_1 and r'_2 the corresponding distances from the sink, we have

$$U_1 = \frac{Q}{2\pi\kappa\delta} \log \frac{r'_1}{r_1} \text{ and } U_2 = \frac{Q}{2\pi\kappa\delta} \log \frac{r'_2}{r_2},$$

whence

$$U_1 - U_2 = \frac{Q}{2\pi\kappa\delta} \log \frac{r'_1 r_2}{r_1 r'_2}.$$

Consequently the resistance of the part of the sheet between these circles is

$$R = \frac{U_1 - U_2}{Q} = \frac{1}{2\pi\kappa\delta} \cdot \log \frac{r_1' r_2}{r_1 r_2'} \quad . \quad . \quad . \quad (5)$$

If ρ_1 is the radius of the circle which has the greater absolute potential, or the one nearest the source, and ρ_2 is the radius of the circle nearest the sink, the similarity of the triangles APC and BPC in fig. 2, Plate IV., gives

$$\frac{\rho_1}{a + d_1} = \frac{r_1}{r_1'} \quad \text{and} \quad \frac{\rho_2}{a + d_2} = \frac{r_2}{r_2'}$$

for circles round the source, and

$$\frac{\rho_1}{a + d_1} = \frac{r_1'}{r_1} \quad \text{and} \quad \frac{\rho_2}{a + d_2} = \frac{r_2'}{r_2}$$

for circles round the sink. Substituting these values, we get

$$R' = \frac{1}{2\pi\kappa\delta} \cdot \log \frac{\rho_2(a + d_1)}{\rho_1(a + d_2)}$$

if both circles surround the source, and

$$R' = \frac{1}{2\pi\kappa\delta} \cdot \log \frac{\rho_1(a + d_2)}{\rho_2(a + d_1)}$$

if they both surround the sink. In the former case we have $\rho_1 < \rho_2$, and in the latter case $\rho_1 > \rho_2$; but using ρ_1 for the radius of the smaller circle and ρ_2 for that of the larger, we may write

$$R' = \frac{1}{2\pi\kappa\delta} \cdot \log \frac{\rho_2(a + d_1)}{\rho_1(a + d_2)} \quad . \quad . \quad . \quad . \quad (6)$$

for the resistance between two equipotential circles surrounding the same pole, whether that pole be a source or a sink.

If the circles surround opposite poles, the resistance becomes

$$R'' = \frac{1}{2\pi\kappa\delta} \cdot \log \frac{(a + d_1)(a + d_2)}{\rho_1\rho_2} \quad . \quad . \quad . \quad (7)$$

In the case of circles round the same pole, if the other pole is infinitely distant, the value of R' becomes

$$\frac{1}{2\pi\kappa\delta} \log \frac{\rho_2}{\rho_1},$$

which is identical with (1), the value found for the resistance of an annular belt of internal radius ρ_1 and external radius ρ_2 ; and in fact the case supposed (an infinite distance between source and sink) is physically identical with the case of a single pole in an infinite sheet.

When the circles surround opposite poles, if the radii are equal, the resistance becomes

$$\frac{1}{\pi\kappa\delta} \log \frac{a+d}{\rho} \quad \text{or} \quad \frac{1}{\pi\kappa\delta} \log \frac{a + \sqrt{a^2 + \rho^2}}{\rho};$$

or when the common radius is small as compared with a ,

$$R = \frac{1}{\pi\kappa\delta} \log \frac{2a}{\rho}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

23. The following modified forms of (6) and (7) obtained by means of the relations (§ 20)

$$\frac{a+d}{\rho} = \frac{\rho}{h}, \quad a^2 = ll', \quad 2\rho = l - l',$$

may also be noted :

$$\left. \begin{aligned} R' &= \frac{1}{2\pi\kappa\delta} \log \frac{\rho_1 h_2}{\rho_2 h_1}, \\ R'' &= \frac{1}{2\pi\kappa\delta} \log \frac{\rho_1 \rho_2}{h_1 h_2}, \end{aligned} \right\} \quad . \quad (9)$$

$$\left. \begin{aligned} R' &= \frac{1}{2\pi\kappa\delta} \log \frac{(\sqrt{l'_1} - \sqrt{l_1})(\sqrt{l'_2} + \sqrt{l_2})}{(\sqrt{l'_1} + \sqrt{l_1})(\sqrt{l'_2} - \sqrt{l_2})}, \\ R'' &= \frac{1}{2\pi\kappa\delta} \log \frac{(\sqrt{l'_1} + \sqrt{l_1})(\sqrt{l'_2} + \sqrt{l_2})}{(\sqrt{l'_1} - \sqrt{l_1})(\sqrt{l'_2} - \sqrt{l_2})}. \end{aligned} \right\} \quad . \quad (10)$$

24. If two equipotential circles alone are given and it is required to find the resistance between them, the position of the poles A and B being unknown, the expressions given above for R' and R'' are inapplicable (unless the circles are so small that their centres sensibly coincide with the poles); for in such cases the values of r , r' , a , d , h , and l are all unknown. The distance, however, between the centres of the circles is directly measurable; and the resistance can be expressed in terms of this distance and of the known radii as follows.

Calling the distance between the centres D , we have for circles surrounding the same pole,

$$D = d_2 - d_1 = \sqrt{a^2 + \rho_2^2} - \sqrt{a^2 + \rho_1^2},$$

which gives

$$d_1 = \sqrt{a^2 + \rho_1^2} = \frac{\rho_1^2 - \rho_2^2 - D^2}{2D}, \quad d_2 = \sqrt{a^2 + \rho_2^2} = \frac{\rho_1^2 - \rho_2^2 + D^2}{2D},$$

and

$$a = \frac{\sqrt{(\rho_1 + \rho_2 + D)(\rho_1 + \rho_2 - D)(\rho_1 - \rho_2 + D)(\rho_1 - \rho_2 - D)}}{2D}.$$

Hence (6) may be written *

$$R' = \frac{1}{2\pi\kappa\delta} \log \frac{\rho_2 \sqrt{(\rho_1 + \rho_2 + D)(\rho_1 + \rho_2 - D)(\rho_1 - \rho_2 + D)(\rho_1 - \rho_2 - D) + \rho_1^2 - \rho_2^2 - D^2}}{\rho_1 \sqrt{(\rho_1 + \rho_2 + D)(\rho_1 + \rho_2 - D)(\rho_1 - \rho_2 + D)(\rho_1 - \rho_2 - D) + \rho_1^2 - \rho_2^2 + D^2}}. \quad (11)$$

When the equipotential circles surround different poles, we have

$$D = d_1 + d_2 = \sqrt{a^2 + \rho_1^2} + \sqrt{a^2 + \rho_2^2},$$

which gives

$$d_1 = \sqrt{a^2 + \rho_1^2} = \frac{D^2 + (\rho_1^2 - \rho_2^2)}{2D}, \quad d_2 = \sqrt{a^2 + \rho_2^2} = \frac{D^2 - (\rho_1^2 - \rho_2^2)}{2D},$$

and a the same as before. Hence (7) may be written

$$R'' = \frac{1}{2\pi\kappa\delta} \log \frac{\sqrt{(\rho_1 + \rho_2 + D)(\rho_1 + \rho_2 - D)(\rho_1 - \rho_2 + D)(\rho_1 - \rho_2 - D) - \rho_1^2 - \rho_2^2 + D^2}}{2\rho_1\rho_2}. \quad (12)$$

25. If instead of considering the whole extent of the sheet we confine our attention to that part of it which is contained between any two lines of flow (see Plate IV.), its resistance will be given by any of the above formulæ if instead of 2π in the denominator we put the angle, say γ , which the lines of flow forming the boundaries make with each other at either pole; for the spaces between every pair of consecutive flow-lines convey equal currents; and since the difference of potential between their ends is the same for all, they all have the same resistance. Consequently the resistance of a part of the sheet made up of such spaces is inversely proportional to the number of them which compose it. Now, using as previously α to stand for the angle which two consecutive lines of flow make with each other at the poles, the total number of spaces into which the whole sheet is divided is $\frac{2\pi}{\alpha}$, and the number of spaces lying between lines of

flow whose angle at the poles is γ , is $\frac{\gamma}{\alpha}$; hence, if R_γ is the resist

* This expression was given in a slightly different form by Gauguin on the authority of Blavier (*Ann. de Chim. et de Phys.* Ser. 3. vol. lxvi. p. 203. 1862); and a demonstration has since been published by the latter (*Journal de Physique*, vol. iii. p. 115, April 1874).

ance of the part of the sheet bounded by these lines, and R the resistance of the whole of the sheet, we have

$$\frac{R_\gamma}{R} = \frac{2\pi}{\gamma}.$$

In the case of two flow-lines which form arcs of the same circle drawn through the two poles, we have $\gamma = \pi$, and therefore

$$R_\pi = 2R.$$

Hence, if a small equipotential circle is drawn round each pole, the resistance of the part of the sheet lying between these circles and contained within any circle drawn through the poles is the same, being equal to twice the resistance of the unlimited sheet between the same two equipotential circles, and also equal to the resistance of as much of the unlimited sheet as would be left after cutting out the portion bounded by any circle through the poles. Consequently if ρ is the common radius of the small equipotential circles, we have, for the resistance of a disk on the edges of which the poles are situated, and also for that of the space outside it, according to §§ 22 & 24,

$$R = \frac{2}{\pi\kappa\delta} \cdot \log \frac{2a}{\rho} \quad \text{or} \quad R = \frac{2}{\pi\kappa\delta} \cdot \log \frac{D}{\rho}. \quad (13)$$

26. It may be observed that the method by which the resistance was obtained above (§ 22) is essentially the same as that adopted by Kirchhoff. The method by which we first obtained the expression, however, was founded on the consideration of the equality of the resistance of the curvilinear rectangles into which the sheet is divided by the intersecting systems of lines of flow and equipotential lines (*vide* Plate IV.). This process, which may be stated as follows, we afterwards found to be somewhat similar in principle to that employed by Sinaasen.

Let two circular electrodes be placed upon the sheet so as to coincide with two of the natural equipotential circles, which for simplicity we will assume to have the same radius ($=\rho$). The spaces between consecutive lines of flow are of equal resistance, and may for the present purpose be regarded as so many independent conductors connecting the two poles and combined in multiple arc. Consequently the resistance of the sheet which is made up of these spaces, is equal to the resistance of one of them divided by their whole number. These spaces in their turn are each composed of $2n$ equi-resisting rectangles arranged in

series ; so the resistance of each space is $2n$ times the resistance of one of the rectangles, $2n$ being (as is evident from Plate IV.) the whole number of equipotential lines in the sheet which are not obliterated by the electrodes.

Now consider the resistance of one of the four rectangles at the middle of the figure. Its length $OD = l = a \frac{\mu-1}{\mu+1}$ (§ 20) (for, $n=0$ giving the straight equipotential line through O (§ 19), the next one will be given by $n=1$). Its breadth b , which in fig. 3 is denoted by OQ , will be

$$a \cot \frac{\pi-\alpha}{2} = a \tan \frac{\alpha}{2}.$$

Calling δ the thickness and κ the conductivity of the sheet as before, the resistance of the rectangle will be approximately

$$R_0 = \frac{l}{b\kappa\delta} = \frac{1}{\kappa\delta \tan \frac{1}{2}\alpha} \cdot \frac{\mu-1}{\mu+1}.$$

The number of equipotential lines between the electrodes is $2n$ if the electrodes are of equal size and coincide with the n th equipotential circle on each side of the middle line. The radius of this circle is given in § 20 as

$$\rho = \frac{2a\mu^n}{\mu^{2n}-1};$$

solving which we get

$$\mu^n = \frac{a \pm \sqrt{a^2 + \rho^2}}{\rho}.$$

The positive sign only is admissible, since μ is essentially positive ; whence the number of lines

$$2n = \frac{2}{\log \mu} \log \frac{a + \sqrt{a^2 + \rho^2}}{\rho}.$$

So, then, the resistance of a strip between two consecutive flow-lines is

$$R_1 = 2nR_0 = \frac{2}{(\mu+1)\kappa\delta \tan \frac{1}{2}\alpha} \cdot \frac{\mu-1}{\log \mu} \log \frac{a + \sqrt{a^2 + \rho^2}}{\rho}.$$

The number of flow-lines in the sheet is (§ 12)

$$\frac{2\pi}{\alpha};$$

therefore the resistance of the whole sheet is approximately

$$R = \frac{\alpha}{2\pi} R_1 = \frac{1}{(\mu+1)\pi\kappa\delta} \cdot \frac{\alpha}{\tan \frac{1}{2}\alpha} \cdot \frac{\mu-1}{\log \mu} \log \frac{a + \sqrt{a^2 + \rho^2}}{\rho}.$$

Now, make the rectangles infinitely small (and therefore rectangular), by letting α approach 0 and μ approach 1; then

$$R = \frac{1}{\pi\kappa\delta} \log \frac{a + \sqrt{a^2 + \rho^2}}{\rho},$$

the same expression as that obtained at the end of § 22 for the same case.

The resistance of the segment contained between two flow-lines intersecting each other with the angle γ is, under the same circumstances (§ 25),

$$R_\gamma = \frac{2}{\gamma\kappa\delta} \log \frac{a + \sqrt{a^2 + \rho^2}}{\rho} \quad \dots \quad (14)$$

Any number of Poles in an Infinite Sheet.

27. *Lines of Flow.*—The equation to the flow-lines for two equal and opposite poles, A and B, in an unlimited sheet, was obtained in § 11 in the following form,

$$\angle APB = n\alpha,$$

where n is a parameter whose value changes by unity on passing from one line to the next, and α is the angle between adjacent rays of the equiangular pencil of flow-lines produced by either pole separately. If the straight line AB be taken as the axis of x , the above equation may be written

$$\angle APB = \angle PBX - \angle PAX = n\alpha.$$

Moreover it was shown, in § 10, that the system of flow-lines resulting from the composition of any two similar systems is obtained by drawing lines through the alternate angles of the quadrilaterals produced by the mutual intersection of the lines of the component systems, in directions concurrent with both the flow-lines which intersect each other at each angle. Hence, if the poles A and B (Pl. IV. fig. 1) were of the same sign (both sources or both sinks), the lines of flow of the resultant system, instead of passing through the points P, Q, R and P₁, Q₁, R₁, &c., as there represented, would pass through the supplementary angles of the quadrilaterals, namely from Q' to P₁, from R' to Q₁, &c. It is evident that, as we pass from point to point of a line so

drawn, the angles which the *radii vectores* from the poles make with the axis of x change equally, but in opposite directions; hence this equation may be written

$$\angle PBX + \angle PAX = n\alpha.$$

28. If, instead of the line through the poles, any other straight line in the plane of flow be taken as the direction of reference for measuring the angles made by the *radii vectores*, the form in which the equations to the lines of flow have just been given will still be applicable. Thus, if the line X (Pl. V. fig. 1), making any angle δ with A B, be taken as the direction of reference, and if a and b be the points where it is met by P A and P B respectively, it is clear that the angles P a X and P b X are each of them less than the previous angles P A X and P B X by the angle δ , but that for each flow-line they give a constant sum or difference according as the poles are of the same or of opposite kinds. Hence, denoting the angles which the *radii vectores* make with any fixed line by θ_1 and θ_2 respectively, and agreeing to distinguish sources and sinks by a difference of intrinsic sign of the angles corresponding to them, we may write the equation to the flow-lines due to two equal poles, whether similar or opposite, in the general form

$$\theta_1 + \theta_2 = n\alpha.$$

29. In like manner, the lines of flow due to any number of equal poles are represented by the equation $\Sigma\theta = n\alpha$, the angles which the *radii vectores* from the several poles make with any fixed line being respectively denoted by $\theta_1, \theta_2, \dots$. In order to prove this, it is sufficient to show that, if the expression holds good for any given number of poles, it will still be true for one pole more; for since the formula has been proved for two poles, this would enable us to advance, by successive additions of one pole at a time, to any number.

Let, then, the curves P'' Q' R, P' Q R₁, and P Q₁ R₂ (Pl. V. fig. 2) represent parts of three successive lines of flow due to any number k of equal poles, and let them be characterized respectively by the following values of $\Sigma\theta$, namely $(n+1)\alpha$, $n\alpha$, and $(n-1)\alpha$. If there is another pole of the same strength at L, which we will suppose in the first instance to be a *source*, the pencil of flow-lines diverging from it with the common difference of angle α will intersect the flow-lines from the k sources, producing a number of quadrilaterals through whose opposite angles the

lines of flow will pass which result from the action of the pole at L combined with that of all the rest (§ 10). If the direction of the flow along the lines of the component systems be as represented by the arrow-heads* in the figure, one line of the resultant system will pass through the points P' and Q', another through P, Q, and R, and a third through Q₁ and R₁. Let the lines joining the source L with the points P, Q, and R cut the fixed line X in p , q , and r respectively; then, for the flow-line through Q due to the k poles, we have by hypothesis

$$\theta_1 + \theta_2 + \dots + \theta_k = n\alpha;$$

and for the line through the same point due to the source L we have

$$\angle QqX = m\alpha,$$

and therefore, for both together,

$$\Sigma\theta + QqX = (n + m)\alpha.$$

For the point P, we have similarly

$$\Sigma\theta + PpX = (n - 1)\alpha + (m + 1)\alpha;$$

and for the point R we have

$$\Sigma\theta + RrX = (n + 1)\alpha + (m - 1)\alpha.$$

Consequently for each of these points, which we already know are situated on the same flow-line of the resultant system, we have

$$\theta_1 + \theta_2 + \dots + \theta_k + \theta_l = (n + m)\alpha.$$

In the same way, we should get for the points P' and Q' situated on the line of flow of the resultant system which lies next to the line through P, Q, and R on the left, the respective values which follow:

$$\Sigma\theta + P'pX = n\alpha + (m + 1)\alpha,$$

$$\Sigma\theta + Q'qX = (n + 1)\alpha + m\alpha,$$

* For the purpose of the argument in the text, we may assume arbitrarily either the *direction of flow* along the assumed system of lines of flow, or the *direction of increase* in the value of n (in $n\alpha$), but not both; for a definite assumption regarding one of these conditions determines the other also. If the system of flow-lines for any given set of poles is built up by successive repetitions for one pole at a time of the construction given in § 10, it will be found that the conventional distinction between positive and negative angles leads to the following rule as to the connexion between the two above-named conditions:—Going along a given flow-line *with the flow*, any line on the *left hand* corresponds to a *higher* value of n than the given line, while a line on the *right hand* corresponds to a *lower* value of n .

and therefore for each of them

$$\theta_1 + \dots + \theta_k + \theta_l = (n + m + 1)\alpha.$$

The same reasoning gives for the points Q_1 and R_1 on the line of flow next on the right to that through P , Q , and R ,

$$\theta_1 + \dots + \theta_k + \theta_l = (n + m - 1)\alpha.$$

30. Next suppose the additional pole at L to be a *sink*; then, since the resultant flow must be everywhere concurrent with both the components, the new flow-lines will pass through the other pair of angles of each quadrilateral; for instance, three consecutive lines will pass through the points P' and Q_1 , through P'' , Q , and R_2 , and through Q' and R_1 ; and by applying the same considerations as above, it will be seen that these lines are characterized by the following values of the expression

$$\theta_1 + \dots + \theta_k + \theta_l,$$

namely :—for the line through P' and Q_1 ,

$$(n - \overline{m + 1})\alpha;$$

for that through P'' , Q , and R_2 ,

$$(n - m)\alpha;$$

and for that through Q' and R_1 ,

$$(n - \overline{m - 1})\alpha.$$

Hence it follows that sources and sinks affect the value of the sum of angles denoted by $\Sigma\theta$ in opposite ways, and therefore that, as assumed provisionally above (§ 28), sources and sinks must be distinguished by a difference in the intrinsic sign of the angles which the *radii vectores* drawn from them make with the fixed line. When this is done, the equation

$$\Sigma\theta = n\alpha$$

is true for the lines of flow produced by any number of equal poles, whether of the same sign or not.

Further, if we have poles of unequal strengths supplying (or removing) quantities of electricity in the unit of time denoted, say, by $Q_1, Q_2, \dots$, these may be regarded as each of them produced by the coalescence of a corresponding number of equal poles supplying in unit of time a quantity of electricity q which is taken small enough to be a common measure of all the quantities $Q_1, Q_2, \dots$. In applying the general formula to such a case, the value of θ corresponding to each pole would have to be taken a number of times equal to the number of constituent

poles of strength q required to make up the actual pole. Hence

$$\Sigma(Q\theta) = Q_1\theta_1 + Q_2\theta_2 + \dots = n\alpha \quad . \quad . \quad . \quad (15)$$

is a perfectly general expression for the lines of flow produced by any number of poles of any strength.

31. *Equipotential Lines*.—It was shown in § 17 that the system of equipotential lines resulting from the composition of any two systems, for which the constant difference of potential on passing from one line to another is the same, is given by drawing lines through the alternate angles of the quadrilaterals formed by the mutual intersection of the lines of the component systems, if the angles are taken in such order that on going from any one to the next we pass, in one of the component systems, to a line of higher, and in the other to one of lower potential. We know also (see § 5) that the equipotential lines due to a single pole are concentric circles whose radii vary according to the terms of a geometrical progression, the potential increasing in the case of a source as the radii decrease, and, in the case of a sink, as the radii increase. Hence it follows very simply, by the application of reasoning exactly analogous to that employed in § 29, that if P be a point on an equipotential line due to any number of equal sources at the points $A, B, C, \dots$, and sinks at the points $A', B', \dots$,

$$\frac{AP \cdot BP \cdot CP \dots}{A'P \cdot B'P \dots} = \mu^n = \text{constant},$$

where μ is the constant ratio of the radii of consecutive equipotential circles due to a single pole, while n is a number characteristic of the particular line on which the point P is situated and increasing by unity if this point passes from any given line to the line of next lower potential. For shortness we may put r_1 for AP , r_2 for $BP, \dots$, and write the above equation thus,

$$\frac{r_1 \cdot r_2 \cdot r_3 \dots}{r'_1 \cdot r'_2 \dots} = \mu^n,$$

or

$$\Sigma \log r = n \log \mu.$$

32. No part of the above reasoning will be disturbed if we suppose that each of the equal poles hitherto discussed emits (or absorbs) in unit of time a quantity of electricity equal to unity; and as this supposition will simplify the consideration of poles

of unequal strength, we will adopt it. Then, when the poles are unequal, let them severally emit in unit of time quantities of electricity represented respectively by $Q_1, Q_2, Q_3, \dots$ (the absorption of electricity by sinks being reckoned as negative emission), the effect will be the same as if $Q_1, Q_2, \dots$ poles of unit strength coincided at the points occupied by the actual poles. Hence, in such a case, the *radii vectores* from the several poles must be taken $Q_1, Q_2, \dots$ times in the above formula, and it thus becomes

$$\left. \begin{aligned} & r_1^{Q_1} \cdot r_2^{Q_2} \cdot r_3^{Q_3} \dots = \mu^n \\ \text{or} \quad & \Sigma(Q \log r) = n \log \mu, \end{aligned} \right\} \dots \dots \dots (16)$$

which is a quite general expression for the equipotential lines due to any distribution of poles of any strength.

33. To find the actual potential at any point due to a given set of poles, we must recur to equation (2), § 7. Let $V_1, V_2, \dots$ be the potentials which the various poles would produce at the given point if each acted separately, and let $\phi_1, \phi_2, \dots$ be the potentials produced by each separately at unit distance from itself; then, taking the other symbols in the senses already defined, we have

$$\begin{aligned} V_1 &= \phi_1 - \frac{Q_1}{2\pi\kappa\delta} \cdot \log r_1, \\ V_2 &= \phi_2 - \frac{Q_2}{2\pi\kappa\delta} \cdot \log r_2; \\ &\vdots \qquad \qquad \qquad \vdots \end{aligned}$$

and for the resultant potential at the given point,

$$U = \Sigma V = \Sigma \phi - \frac{1}{2\pi\kappa\delta} \cdot \Sigma(Q \log r) \dots \dots \dots (17)$$

Since, for a given set of poles, $\Sigma \phi$ is constant, this gives for a locus of constant potential

$$\Sigma(Q \log r) = \text{const.} = 2\pi\kappa\delta(\Sigma \phi - U),$$

which is in accordance with (16); for we may write

$$\Sigma \phi - U = n \Delta v,$$

and by § 8, when $Q=1$, we have $\log \mu = 2\pi\kappa\delta \cdot \Delta v$.

34. As already observed (§ 1), both the general equations

$$\Sigma(Q\theta) = \text{const.} \quad \text{and} \quad \Sigma(Q \log r) = \text{const.}$$

were given thirty years ago by Kirchhoff, who obtained the

former as a purely mathematical consequence of the latter. In fact, as the systems of lines represented by them are orthogonal to each other, it follows that their first derived functions differ only in one having $\frac{dy}{dx}$ where the other has $-\frac{dx}{dy}$; hence differentiation and subsequent integration, after making this substitution, convert one expression into the other*. This relation between the two equations is of importance, since the form of the equipotential lines can be readily determined experimentally, whereas no practicable method exists for ascertaining experimentally the course of the lines of flow.

35. Before going further we may point out some general properties of lines of flow and equipotential lines. Physically considered, a line drawn from any one pole to any other, so that no electricity crosses it at any point, may be regarded as an independent line of flow, as was done in treating the case of two equal and opposite poles. The lines, however, given by the equation $\Sigma(Q\theta) = \text{const.}$ are continuous curves each of which passes through every pole; and, in a mathematical sense, the whole of the curve given by any one value of the constant must be regarded as a single line of flow. When flow-lines are spoken of in the sequel, it is to be understood that they are *complete* in the above sense. If all the poles are of equal strength, every flow-line passes once through each of them; hence, if they are unequal, every line passes through each of them respectively as

* Let the coordinates of the several poles be (a_1, b_1) , (a_2, b_2) , &c. The distances of a point from each will be $r_1 = \sqrt{(x-a_1)^2 + (y-b_1)^2}$, $r_2 = \&c.$, and the angles which the joining lines make with the axis of x will be given by $\tan \theta_1 = \frac{y-b_1}{x-a_1}$, $\tan \theta_2 = \&c.$ Then

$$\begin{aligned} \frac{d}{dx} \Sigma(Q \log r) &= \Sigma \cdot Q \frac{d}{dx} \log \sqrt{(x-a)^2 + (y-b)^2} \\ &= \Sigma \cdot Q \left\{ \frac{x-a}{(x-a)^2 + (y-b)^2} + \frac{y-b}{(x-a)^2 + (y-b)^2} \cdot \frac{dy}{dx} \right\}. \end{aligned}$$

Now make the substitution, equivalent to writing $\phi + \frac{\pi}{2}$ for ϕ , and integrate,

$$\begin{aligned} \int \Sigma \cdot Q \left\{ \frac{y-b}{(x-a)^2 + (y-b)^2} - \frac{x-a}{(x-a)^2 + (y-b)^2} \cdot \frac{dy}{dx} \right\} dx \\ = \Sigma \cdot Q \tan^{-1} \frac{y-b}{x-a} = \Sigma(Q\theta). \end{aligned}$$

often as it contains the greatest common measure of the strengths of the whole set. Moreover, if on any flow-line there is a point of no flow, the line will intersect itself at this point one or more times. Separate flow-lines cannot intersect each other anywhere except at a pole.

Equipotential lines form closed curves which always surround one or more poles. At points where the strength of the current is nothing, the same equipotential line cuts itself once or oftener; in such cases there is at least one pole within each loop formed by the line. Separate equipotential lines never intersect each other.

The method of superposition, by which any system of flow-lines or equipotential lines can be drawn, is easily carried out by drawing, in rather strong lines on white paper, one of any two systems that are to be compounded to a resultant system, and drawing the other component system on tracing-paper. Placing the two drawings one over the other in any required manner, and laying a second piece of tracing-paper on the top, the curves which pass through the intersections of the two component systems can be drawn at once. The resultant system may then, if required, be recompounded with one of the components by the same process.

36. The general course of the lines of flow and equipotential lines for a few of the simpler cases is shown in Plate V.

Two equal similar poles (fig. 3).—The equation to the lines of flow for this case is

$$\theta_1 + \theta_2 = n\alpha,$$

which is equivalent to

$$x^2 - y^2 - 2xy \cot n\alpha = a^2,$$

if the straight line through the poles is taken as the axis of x , and its middle point as the origin. They are a system of rectangular hyperbolæ cutting each other at the poles with a constant difference of angle ($=\alpha$). The value $n\alpha = \pi$ gives the two axes, the origin where they intersect being a point of no flow. The equipotential lines for the same case are a system of lemniscates, $r_1 r_2 = \mu^n$, and are identical with the loci of equal flow for two equal opposite poles (§ 15). The strength of the current at any point at a distance r from the origin is $s = \frac{Q}{\pi} \cdot \frac{r}{r_1 r_2}$; and consequently the loci of constant flow are the same as the equipotential lines

for three equal poles—two of the same sign, and one of the opposite sign halfway between them, as shown in fig. 4. The strength of the current on the self-cutting curve is $\frac{Q}{2\pi a}$. For all other values of s the curve consists of two separate branches; when s is $> \frac{Q}{2\pi a}$, one branch encloses each pole; and when s is $< \frac{Q}{2\pi a}$, one branch surrounds both poles and the other surrounds neither, being within the central loop of the self-cutting curve. Each of the loci cuts any straight line through the origin harmonically with respect to its intersections with the circle $r=a$.

Three equal poles.—A special case coming under this head, which admits of some important applications, is that in which there are three equal poles in a straight line, one of the outside ones being of opposite sign to the other two—for instance, two sources A and A' , and one sink C . In this case the flow-line given by $\theta_1 + \theta_2 - \theta_3 = \pi$ consists of two branches, one of which is the straight line passing through the poles; while the other is a circle whose centre is the sink C (fig. 5), and whose radius CP is a mean proportional between the distances from the sink to the two sources respectively (or $CP = \sqrt{CA \cdot CA'}$). If the two sources coincide, the systems of lines of flow and equipotential lines become what are shown in fig. 6, and the circular branch of the flow-line $n\alpha = \pi$ passes through the pole of double strength. In accordance with what was said above (§ 34), every flow-line of the system passes twice through the double source and once through the sink.

Four equal poles.—Four equal poles of the same sign situated at the corners of a square give the system of flow-lines represented in fig. 7. The two diagonals of the square and the two straight lines through the middle point parallel to the sides of the square are given equally by the values $n\alpha = \pi$ and $n\alpha = 0$. For $n\alpha = \frac{1}{2}\pi$ we get a curve of four branches, one of which is situated symmetrically in each quadrant.

A special case of four equal poles, which is important in consequence of its being susceptible of experimental verification, is presented by a combination of two sources and two sinks arranged in a manner that may be regarded as a duplication, with inversion of signs, of the system of three poles shown in fig. 5. Let

the pole at C be a sink, those at A and A' being sources. Let two additional sinks be placed at B and B' (fig. 8), points on another straight line through C, and let a source equal to them be put at C; then the three poles of this new system would give by themselves, as one of their flow-lines, the straight line CB and the circle with centre C and radius $= \sqrt{CB \cdot CB'}$. If, however, the three new poles be made equal to the three poles of the first set, the sink originally at C and the equal source now put there will exactly compensate each other, and the combined system will be reduced to two sources at A and A' and two sinks equal to them at B and B'. If, further, the points B and B' be taken so that $CB \cdot CB' = CA \cdot CA' = CP^2$, the circle of radius CP

Fig. 8.

will be a common line of flow in each of the systems due to C, A, and A' and to C, B, and B', and will therefore remain a line of flow in the resultant system (see fig. 9) due to A, A', B, and B'. Moreover, since these four points are placed so that one circle can be drawn through them all, it is easy to see that this circle will be another flow-line of the resultant system; for the four poles may be grouped in two ways into two pairs of equal opposite poles (either A, B and A', B', or A, B' and A', B), each of which would separately give this circle as a line of flow. If the straight lines AB and A'B' are drawn and produced to their intersection at D (fig. 8), a circle drawn about this point as centre with radius $= \sqrt{DB \cdot DA} = \sqrt{DB' \cdot DA'}$ will be an equipotential line common to the systems due respectively to the source and sink at A and B, and to those at A' and B'; consequently this circle is also an equipotential line of the resultant system due to all four poles. Another equipotential line of the resultant system would

be the circle (in this case imaginary) drawn with the point of intersection of AB' and $A'B$ as centre so as to cut each of these straight lines in points harmonically situated with respect to their extremities.

Again, if four equal poles be taken situated as in the last example, but so that poles of the same sign are diagonally opposite each other—in other words, if the signs of either A and B or of A' and B' be interchanged, the circle passing through the four poles will still be a flow-line of the resultant system, but C , as well as D , will now be the centre of an equipotential circle, while the imaginary circle will be a line of flow.

37. *Resistance*.—Equation (17) gives, for the difference of potential of any two points whose respective distances from the several poles are $r_1, r_2, r_3, \dots$, and $r'_1, r'_2, \dots$ the value

$$\begin{aligned} U - U' &= \frac{1}{2\pi\kappa\delta} [\Sigma(Q \log r') - \Sigma(Q \log r)] \\ &= \frac{1}{2\pi\kappa\delta} \Sigma \left(Q \log \frac{r'}{r} \right), \end{aligned}$$

since, for each pole, there are the corresponding terms $Q_n \log r_n$ and $Q_n \log r'_n$. If there are altogether K poles, k being of one sign and k' of the opposite sign, where k is not less than k' , and if, further, they are all of the same strength, the quantity of electricity crossing each complete equipotential line in unit of time is kQ ; consequently the resistance of the portion of the sheet lying between the equipotential lines which pass through the given points is

$$R = \frac{U - U'}{kQ} = \frac{1}{2\pi\kappa\delta \cdot k} \cdot \Sigma \left(\log \frac{r'}{r} \right). \quad . \quad . \quad (18)$$

This formula is in principle quite general; but the practical application of it in actual cases requires that we should know the position of the poles from which the distances $r_1, r_2, \dots$ are to be measured; and these cannot be ascertained (or at least not by elementary methods), except for comparatively few and simple cases; for although it is comparatively easy to determine the equipotential lines for a given set of poles, the inverse problem, of finding the distribution of poles required to produce equipotential lines coinciding with two given curves on the conducting sheet, presents in general very great mathematical difficulties, and has hitherto received only partial solutions.

38. We will give here, in the first place, the approximate application of the general formula to the case of two sources and two equal sinks at the angles of a quadrilateral inscribed in a circle and so placed that unlike poles are diagonally opposite each other. This is the arrangement shown in figs. 8 and 9, where A and A' may be taken as sources and B and B' as sinks. It is evident that the equipotential lines very near the poles will each consist of two branches, one of them surrounding one source (or sink), and the other surrounding the other source (or sink), and also that they will be very approximately circles having the poles at their centres. Hence, if the sources and sinks are formed by four circular electrodes, whose common radius ρ is a small fraction of the distance between any two of them, and if they are so placed that the distances of their centres A, A', B, and B' from a fixed point C fulfil the condition $CA \cdot CA' = CB \cdot CB'$ (which is equivalent to saying that their centres are on the circumference of one and the same circle), we may without serious error regard the circles of contact between the electrodes and the conducting sheet as forming together a pair of equipotential lines due to such a distribution of poles as that referred to at the beginning of this paragraph. Then, taking points on the circumference of the circles round A and B as the points to which the values of r and r' in equation (18) respectively refer, we have approximately

$$\Sigma \left(\log \frac{r'}{r} \right) = \log \frac{(AB)^2 \cdot AB' \cdot A'B}{\rho^2 \cdot AA' \cdot BB'};$$

and since for the case supposed $k = k' = 2$, the resistance of the part of the sheet extending between the pair of circles round A and A' and the pair round B and B' is represented with similar approximation by

$$R = \frac{1}{4\pi\kappa\delta} \cdot \log \frac{(AB)^2 \cdot AB' \cdot A'B}{\rho^2 \cdot AA' \cdot BB'} \dots \dots \dots (19)$$

If a circle be drawn with the centre C and radius

$$CP = \sqrt{CA \cdot CA'} = \sqrt{CB \cdot CB'},$$

this circle will coincide, as already pointed out (§ 35), with a flow-line of the system due to the combined action of the four poles. Consequently no electricity passes into or out of this circle; and therefore the whole of the electricity supplied by the source A flows to the sink B *inside* the circle (fig. 9), while the whole of what is supplied by the source A' flows to B' *outside*

the circle. It follows, since the sources A and A' were assumed of the same strength, that the resistance offered by the part of the sheet lying within the circle of radius CP to the flow of electricity between the electrodes A and B is the same as the resistance of the part of the sheet lying outside this circle to the flow of electricity between the electrodes A' and B' . Hence also the resistance of a disk bounded by the circle in question and containing the two electrodes A and B is equal to *twice* the resistance of the entire sheet to all four poles, and is therefore represented by

$$R' = \frac{1}{2\pi\kappa\delta} \cdot \log \frac{(AB)^2 \cdot AB' \cdot A'B}{\rho^2 \cdot AA' \cdot BB'}, \quad \dots \quad (20)$$

which is the formula referred to in § 1 (page 115) as having been given by Kirchhoff for the resistance of a circular disk with two small circular electrodes anywhere upon it.

39. As already stated, this formula is only approximate, and in certain special cases it entirely fails. For instance, if one (or both) of the poles passes to the edge of the disk, then, in order that the circumference may still continue to be a line of flow, the second pole of the same sign must coincide with it; consequently in such a case AA' or BB' , or both, will vanish, and the expression for the resistance fails by becoming infinite. The reason evidently is that equation (20) was got by assuming ρ to be very small in comparison with any of the distances between the poles; and this can no longer be true when any two of them coincide. By slightly modifying the notation, we can obtain an expression which does not fail in the same way. Thus, let the points to which the values of $r_1, r_2, \dots$ refer be respectively a point F where the straight line AB cuts the equipotential circle round A , and a point G where the same line cuts the equipotential circle round B . Then we have $AF = BG = \rho$, and the expression for the resistance of the sheet to the current from all four poles becomes

$$R = \frac{1}{4\pi\kappa\delta} \cdot \log \left(\frac{BF \cdot B'F}{AF \cdot A'F} \cdot \frac{AG \cdot A'G}{BG \cdot B'G} \right), \quad \dots \quad (19a)$$

while the resistance of the disk bounded by the circular flow-line may be written

$$R' = \frac{1}{2\pi\kappa\delta} \cdot \log \frac{AG \cdot BF \cdot A'G \cdot B'F}{\rho^2 \cdot A'F \cdot B'G}. \quad \dots \quad (20a)$$

In this form the expression for R' admits of extension to some

special cases. For example, let the pole A pass to the edge of the disk, while B goes to the centre; then A' will coincide with A, and B' will go to an infinite distance. We have then sensibly $AG=BF=A'G$ = the radius of the disk ($=P$), also $A'F=\rho$, and the resistance to the flow between the pole at the edge and the pole at the centre of the disk becomes

$$\frac{3}{2\pi\kappa\delta} \cdot \log \frac{P}{\rho} \dots \dots \dots (21)$$

If both poles go to the edge, A' coincides with A, and B' with B, the system being reduced to two equal opposite poles at a distance $AG=BF=2a$ from each other. In accordance with this, the resistance of the whole sheet becomes

$$\frac{1}{\pi\kappa\delta} \cdot \log \frac{2a}{\rho},$$

which is the value already given for the same case in equation (8).

40. If, all else remaining as before, the signs of the poles A' and B' are interchanged (that is, if A' becomes a sink and B' a source), the circle drawn with the centre C and radius $\sqrt{CA \cdot CA'}$ becomes an equipotential line instead of a line of flow, the circle with centre D and radius $\sqrt{DB \cdot DB'}$ remains an equipotential line, and the only circular lines of flow are the circle through the four poles and the imaginary circle (§ 36) whose centre is the intersection of AB' and A'B. The resistance of the sheet to the flow of electricity between this new arrangement of poles is given by interchanging in (19a) the signs of the *radii vectores* drawn from A' and B', whereby we get

$$R = \frac{1}{4\pi\kappa\delta} \cdot \log \frac{AG \cdot BF \cdot A'F \cdot B'G}{\rho^2 \cdot A'G \cdot B'F} \dots \dots (19b)$$

The equipotential circles with the centres C and D being common to the system due to the source at A and sink at A', and also to that due to the sink at B and source at B', and having equal but opposite potentials when taken as belonging to either of these systems separately, will, in the system due to the four poles, form the two branches of the line of zero potential. The other equipotential lines of the system due to the four poles consist also each of them of two branches, both of which never lie within the same one of the two circles in question. From this it follows that each circle divides the sheet into two equi-

resisting portions; and consequently the resistance of each of them is

$$R'' = \frac{1}{2\pi\kappa\delta} \cdot \log \frac{AG \cdot BF \cdot A'F \cdot B'G}{\rho^2 \cdot A'G \cdot B'F} * . . . (22)$$

41. Since half the lines of flow due to each pole lie within the circle drawn through them all, the resistance of the portion of the sheet bounded by this circle is (§ 25) twice the resistance of the unlimited sheet; consequently it is equal to R' or to R'' according to the arrangement of the poles.

42. It was pointed out in § 38 that the resistance of the part of the sheet outside the circular flow-line with centre C, to the flow from the source A to the sink B, is the same as the resistance of the part inside this circle to the flow from A to B. Accordingly the value of equation (20) remains unaltered when A' and B' are put for A and B, and *vice versa*, and at the same time ρ^2 (or $\rho_A \rho_B$) is replaced by $\rho_A \rho_{B'} = \rho_A \rho_B \left(\frac{A'B'}{AB} \right)^2$. The circles round the two sources then coincide approximately with the two branches of a single equipotential line; and the same is true for those surrounding the sinks.

Similar remarks are applicable to equation (22).

Experimental verifications of some of the conclusions here arrived at will be given in Part II. of this communication.

XVI. *The Electrolysis of certain Metallic Chlorides.* By J. H. GLADSTONE, Ph.D., F.R.S., Fullerian Professor of Chemistry in the Royal Institution, and ALFRED TRIBE, F.C.S., Lecturer on Chemistry in Dulwich College.

WE have previously shown that nitrate of copper brought into tension by silver and copper in conjunction is decomposed by free oxygen in solution†. Thinking that chlorine

* It may be noted that by adding together the values of R' and R'' we get $\frac{1}{2\pi\kappa\delta} \cdot \log \left(\frac{AG \cdot BF}{\rho^2} \right)^2$, which, written in the simpler form

$$\frac{2}{\pi\kappa\delta} \cdot \log \frac{AB}{\rho},$$

is the resistance of a circular disk on whose edge the poles A and B are placed (see equation 13).

† Proc. Roy. Soc. vol. xx. p. 290.