JOURNAL

OF THE

INSTITUTE OF ACTUARIES.

On the Necessity for a Universal Notation. Report presented to the International Congress of Actuaries held at Brussels, September 1895, by A. BÉGAULT, Actuary of the Compagnie Belge d'Assurances générales sur la Vie, Brussels. Translated from the French by G. KING., V.P.I.A.

[Read before the Institute, 30 March 1896.]

THE favour with which the convening of the present Congress has been received by actuaries of all nations, and the very questions submitted to it by the members, fully show that everywhere the same desire has sprung up, namely that actuaries should know each other personally, so that later on there may be more effective inter-communication of ideas by the interchange of the writings which are periodically published. The question of language is a great obstacle, which it is in our interest to reduce to a minimum. A foreign tongue becomes easier to us, when the technical symbols employed, assuming a form which is familiar to us, indicate the drift of the paragraph. On the other hand, when the same symbol represents, according to the author using it, diametrically opposite ideas, the passage is made more difficult to understand, and the formulas, instead of being an index to its meaning, become a source of error.

In all sciences of recent development—electricity for instance—the necessity for a uniform notation has made itself felt. The

Electrical Congress of 1881 fixed electrical units, and succeeding Congresses found that their labours had thereby been greatly lightened.

The science of the actuary is more than any other international in its scope. Everywhere the same questions crop up, whether we are conducting scientifically financial operations in their infinite variety, or whether we are engaged in the serious problem, which is met with in every country, in connection with provision for old age and assurance against sickness. This department of the social question can be dealt with, with safety and equity, only by the aid of actuarial science. It must be said, and repeated with emphasis, that every legislative effort which goes beyond the rigorous laws laid down by mathematics, bears within itself the germ of decay which surely results in destruction. Equity will be absent, and the apparent advantages secured by the earlier beneficiaries will be certainly and dearly paid for by the more deserving, those, namely, who are called upon to make sacrifices, without a guarantee, other than honeyed words, for the carrying out of the promises made to them.

The questions of invalidity and sickness will give rise in different countries to numberless researches. Statistics will be collected, and tables of every kind will be calculated. How then will it be possible to compare the work done in one country with that done in another, if we are not first in accord upon the definition of the subjects with which we are dealing? What is "invalidity"? What is meant by "a day's sickness"?

In the report presented by M. Léon Marie in the name of the Statistical and Financial Commission of Friendly Societies, we see expressed the same idea in connection with a comparison of the sickness tables of different countries. He therein says:

"Sickness of Soldiers: Here we have the only statistics of 'sickness where the meaning of the word 'sickness' is clearly defined. That which in the army, in all countries of the world, is meant by 'a day's sickness' is a day of incapacity for duty caused by a pathological state," and further on:

"The definition of 'sickness' of the soldier (incapacity for duty) being distinctly the same in every country, it results that "the statistics of different armies are closely comparable. (Report "of Dr. Bertillon)."

It is no more our intention, than it is in our power, to force a notation on the actuaries of different countries; but if the

actuaries come to understand that the notation which we prefer is simple, rational, and complete, it is to be hoped that they will employ it in preference to any other. Once the movement started, those authors who wish to be read, will certainly follow it; and once more free discussion will have achieved that which it is impossible to obtain by pressure. We therefore propose to examine

- (1) The notations in use in different countries.
- (2) Whether we can find one among them which satisfies all the conditions of a notation universally acceptable.

THE PRINCIPAL NOTATIONS IN USE IN DIFFERENT COUNTRIES.

To assist in the examination of this question, the reader is begged to refer to the synoptical table in the appendix. It shows:

- (1) The notation employed by English and American actuaries. (The Institute of Actuaries and the Actuarial Society of America.)*
- (2) The notations used by the French actuaries (Dormoy, and the Committee of four Life Assurance Companies).
- (3) The notations employed in Germany (Karup, senior).
- (4) The notations employed in Holland (C. Landré).

The first two groups have many points in common, as also have the second two groups. That evidently follows from the respective characters of the languages.

A suggestive symbol has always been sought, and the initial of the word has naturally occurred to the mind. Hence the differences arise from the differences in the words employed in languages of German origin, and those of Latin origin. The terminology of English actuarial science is almost entirely of Latin origin. Nevertheless, this rule of initials can never

* The commutation system employed in America is that of Farr. That is to say that the American N_x corresponds to the English N_{x-1} . Moreover, the Actuarial Society of America has adopted the following symbols:

$$k_x = \frac{d_x}{l_{x+1}} = \frac{C_x}{D_{x+1}}$$

$$u_x = \frac{l_x}{l_{x+1}} (1 + i) = \frac{D_x}{D_{x+1}}$$

S = sum assured.

K = cost of the assurance = $q(S - V)$.

Π = the single premium (as also does A).

be absolute; because, in view of the great number of terms employed, duplication would be unavoidable. Hence, for the less frequent benefits, letters have been adopted without reference to the benefits which they represent. This also explains the impossibility of selecting one part of a system of notation from one scheme and another part from another. The letters employed in the first would most likely be found in the second, and with other meanings. It is therefore absolutely necessary, unless we should decide on creating an entirely new system, to adopt in its entirety that system which is the most perfect; it being understood that there will be imported into it later on, such additions as may be rendered necessary by the development of actuarial science.

The systems in force are based upon the following ideas :

- (1) The use of a suggestive symbol to indicate the function in question.

England and the United States	l	d	a	A	V
France	l, v	$-, d$	X, a	P, π	V, G
Germany	λ	r	1R	P	$\mathfrak{L}$
Holland	l	d	r	C	V

- (2) The use of suffixes placed below, and to the right of the principal symbol, to indicate the ages of the lives in question.

England and the United States	l_x	a_{xy}	A_{xyz}
France	l_x, v_x	X_{xy}, a_{xy}	P_{xyz}, Π_{xyz}
Germany	λ_a	$R_{a,b}$	$P_{a,b,c}$
Holland	l_a	$r_{a,b}$	$C_{a,b,c}$

- (3) The use of supplementary indices to the right or to the left, to indicate whether the benefit is temporary or deferred.

England and the United States	${}_n a$	${}_n A$	${}_n \overline{a}$	${}_n \overline{A}$
France	${}_n X, a^{(n)}$	${}_n P, -$	$\overline{X}^n, a^n$	$-, -$
Germany	${}_n R$	${}_n P$	${}_n R$	${}_n P$
Holland	${}_n R$	$-$	$-$	$-$

- (4) The indication of fractional parts of a year in the benefit. When the number of these becomes infinite, the benefit is said to be continuous.

England and the United States	$a^{(m)}$	$\bar{a}$
France	$\frac{1}{m}X, -$	$\bar{X}, -$
Germany	$\frac{m}{m}R$	$-$
Holland	$R^{\frac{n}{n}}$	$\bar{R}$

- (5) Lastly in the English system alone, we meet with special signs to indicate the one ($^{\circ}$) that the benefit is complete (complete expectation of life, annuities payable to the moment of death), the other ($\wedge$) that in the case of a survivorship annuity, the epoch of the first payment becoming due depends on the date of death of the life tenant, and not on the date of the initiation of the transaction. These benefits are represented thus.

$\bar{e}, \bar{a}, \bar{d}.$

Thus, each of the different systems solves in a manner, more or less complete, the problem of notation.

The systems in use by German and Dutch actuaries are found, however, in certain respects to be inferior to the others. The German actuaries have varied the principal symbol according to the benefit to be represented. Thus ${}_nP_x$, ${}^n\mathfrak{P}_x$, ${}^n\mathfrak{P}_x$ represent the single premium for a Temporary Assurance, an Endowment, and an Endowment Assurance for n years, respectively. Nothing in the symbols P , $\mathfrak{P}$ or $\mathfrak{P}$ indicates that difference to the reader. We can thus say that the German notation is not suggestive. Compare it with the English Notation for the same benefits.

$$A_{x:\overline{n}|}^1, A_{x:\overline{n}|}^1, A_{x:\overline{n}|}^1.$$

We see here that A represents the single premium for the assurance, and the index indicates to us the nature of the benefit. ${}_{x:\overline{n}|}$ means payable at the termination of the status ${}_{x:\overline{n}|}$, that is to say by the death of x or at the expiration of the fixed term of n years. That is the Endowment Assurance. ${}_{x:\overline{n}|}^1$ means payable at the termination of the status ${}_{x:\overline{n}|}$ provided that the life drops first, that is to say, dies before the expiration of n years. That is a Temporary Assurance. ${}_{x:\overline{n}|}^1$ means payable at the termination of the status, if the n years run out before the death of x , that is to

say x must still be alive after the n years. This is the Endowment. In fact very few of the symbols adopted by the German actuaries agree with those of the other systems.

The Dutch system set forth in the work of M. M. G. SNOER, (*Lijfrente en Levensverzekering*) and more recently by M. CORNEILLE L. LANDRÉ (*Wiskundige Hoofdstukken voor levensverzekering*), while it is derived principally from the German notation,* includes a certain number of English symbols such as i, v, l, d, V ; but it does not give symbols for several important groups of benefits, such as deferred benefits, survivorship benefits, and endowment assurances. It does not even mention benefits depending on several lives where the order of survivorship is fixed in advance. None of the systems so far examined include this.

The system of Dormoy bears a certain analogy to the English system, but it is open to the objection, especially from the French point of view, that it does not respect the law of initials. In fact, Q, X, Y , do not suggest to the mind in any way the words Endowment, Annuity, and Variable Annuity. Lastly, the probabilities of life, of death, and of contingent survivorship, whether simple or compound, and Endowment Assurances, are not specially represented.

The system of the Committee of the four French Life Assurance Companies resembles most, without question, the system of the Institute of Actuaries. The actuaries who compiled it took for their model the system adopted in England, but introduced certain modifications to which we shall refer again later on.

We shall examine successively these two systems in so far as they relate to the principal symbol, the various indications of age, of duration, and of the conditions of the benefit. We shall make note of the similarities, and of the differences, and we shall see whether the modifications adopted by the French actuaries have in reality simplified or improved the English system.

(1) The principal symbols (pages 2 and 3 of the appendix) i, d, p, q, a , have the same meaning, while l, μ, A, P, V of the English system are replaced in the French system by v, χ, II, π, G .

(2) Special symbols (pages 4 *et seq* of the appendix). The indication of the group of lives on which the benefit depends is given in the two systems in the same way.

$$p_x, a_{xyz}, \&c.$$

* One portion of the Dutch notation has been borrowed from the work of A. Zillmer, *Mathematische Rechnungen*, 1st Edition.

It is not so when this group becomes a status* according to the phrase in use in England, that is to say when the benefit has reference not merely to the ages of one or several lives, but to other conditions, such as the continued existence or the disappearance of the whole group, or a determined order of death in the lives involved, &c. Thus for example

(a) $p_{\overline{xy}}$ represents in both notations the probability that the survivor, that is, at least one of the lives shall be in existence at the end of . . . years.

$q_{\overline{xy}}$ is used by English actuaries to signify the death of the survivor, and by the French, the probability of the dissolution of the group by the death of one or of both. English actuaries have given to the horizontal bar above the suffix the meaning of last survivor, as have also the French in the symbol $a_{\overline{xy}}$ representing an annuity payable until the last death. Further, English actuaries write

$$1 - p_{xy} = q_{xy}$$

$$1 - p_{\overline{xy}} = q_{\overline{xy}}$$

$$\text{analogous to } 1 - p_x = q_x$$

equations which do not exist in the French system.

(b) Survivorship benefits (annuities for instance) are indicated
in the English system $a_{y|x}$;
in the French system Π_x^y

the one as simple and as clear as the other.

(3) Deferred and temporary benefits.—The limitation of the benefit for n years is indicated

in the English system by ${}_n a$, or $a_{x:\overline{n}|}$,

in the French system by $a^{(n)}$,

and the postponement of the benefit

in the English system by ${}_n | a$,

and in the French system by a^n ,

notations equally satisfactory from the point of view of clearness. French actuaries claim for their notation the advantage of avoiding the index on the left of the principal

* The word "status" has a wider meaning in England than that given to it here. It includes a single life on which a benefit depends. Thus, in a_x , x is a status.—G. K.

symbol. We do not see anything objectionable in this index, nor even that it presents any difficulty to the reader. On the contrary the vertical bar is in conformity with the English principle of the separation of the statuses.

${}_n|a$ means that n years will elapse before entering on the annuity.

$|{}_na$ signifies that 0 years will elapse (the benefit is, therefore, immediate) and that it will run for n years.

Moreover, the index on the left is only avoided for the moment; and whenever it is desired to indicate in the French system that the annuities are to be payable half-yearly, quarterly, or n times a year, it will be necessary to place the new index on the left of, or above, the principal symbol.

Lastly, a^n is an algebraical symbol, which by universal consent means the n^{th} power of a .

(4) *Assurances*.—In this department the French system is entirely incomplete, the different benefits having no individual symbols. It has been thought sufficient to take Π and π for the single and the annual premiums respectively for all assurances payable only on the death of the Life Assured, and $\Pi^{(n)}$ and $\pi^{(n)}$ for all benefits, assurances or annuities, which last for only a limited number of years.

Evidently this branch of the question was left on one side. In preparing "*P'Aide-mémoire*" the Committee were content to supply only a guide for the clerks charged with the duty of calculating rates by given formulas. From this narrow standpoint it was not found inconvenient that $\Pi_x^{(n)}$ should represent the single premium for a temporary assurance, an endowment assurance, or an insurance for a term-certain, a temporary life annuity or a deferred life annuity, or an endowment, a compound assurance, &c. The heading of the page made all mistake impossible; and from this purely utilitarian point of view, the principle is not open to criticism. Let us say, however, that in all the symbols, where the two systems, English and French, are comparable, there are only trivial differences, and that it is impossible to say whether one is or is not more complex than the other. It is, therefore, to be presumed that, when the Institute of French Actuaries resumes the work begun by the Committee of the Four Companies, with a view to extend to assurances the general principles adopted for annuities, it will be difficult, nay even impossible, to simplify in the slightest degree the English system. Let us see, therefore, whether this

system can be applied to the conditions which French-speaking peoples have a right to require.

2. EXAMINATION FROM THE FRENCH STANDPOINT OF THE SYSTEM OF NOTATION OF THE ENGLISH INSTITUTE OF ACTUARIES.

As we showed at the commencement of this enquiry, the greatness of the number of symbols required in the science of life contingencies does not permit of borrowing one part of a scheme of notation from one system, and another part from another. On the other hand, it would not be reasonable to advise actuaries to adopt a system entirely out of harmony with the requirements which would be laid down, were an entirely new system to be devised.

Let us therefore examine the English symbols which the French actuaries have rejected, and let us see if there is any real impossibility in adopting them.

1. The instantaneous death rate $= \mu$; μ is the limit of m , and m at once suggests the word "mortality" (French, mortalité).

2. Single Premium (present value) for an assurance $= A$; in French likewise A is the initial of "assurance" (French, assurance).

3. Annual premium for a benefit $= P$. The letter P at once makes us think of "premium" (French, prime).

4. Value of a Policy (reserve) $= V$. Once again the initial V is that of "value" (French, valeur).

We must call special attention to this expression which explains perfectly the reserve.

In theory, a constant equation must exist between the assets and the liabilities; but the equality which exists at the commencement, between the premiums to be received and the corresponding assurance, disappears. The policy, the value of which is zero before the payment of the first premium, and 1 (if unity is the sum assured) at the moment of death, passes through a series of intermediate values. The expression "value of the policy" is therefore a perfectly happy one. The word "reserve" has already a more commercial meaning, because while it is true that we must have at each valuation

$$\Sigma \text{ Reserves} = \Sigma \text{ value of policies,}$$

we may say that, taken individually, it is not so, as recent policies for example may have negative values.

Number living = l . This from the French standpoint is the only letter for which nothing can be said. The letter v suggests the word "vivant"; and from this point of view it is not easy to justify its rejection. We should not even see much objection to admitting it, if v had not already with English actuaries its own meaning = $\frac{1}{1+i}$. Now the adoption of v for $\frac{1}{1+i}$, which gets rid of negative exponents in formulas, and much simplifies the operation of writing, seems to us very desirable, and the more so because v has a meaning easy to remember, namely the present value (French, valeur) of one franc payable in a year.

If, therefore, we find that in French v and consequently $v^2, v^3, \dots v^n$ are useful symbols to represent the present value of one franc payable in 1, 2, 3, $\dots n$ years, we should abandon the meaning of v assigned to it by French actuaries, and take away from it, its signification as the number living.

Moreover the rule has been but little observed hitherto by French actuaries, for we find $l_x, y, f(x)$ used by different authors.

Surely it is not possible that this single letter should be an insurmountable obstacle to the unification of the systems of notation. We have too much confidence in the practical good sense of our colleagues, to believe that this single objection should outweigh with them the gain that would follow order evolved from the chaos which reigns supreme to-day.

CHANGES INTRODUCED BY THE COMMITTEE OF FRENCH COMPANIES.

1. Adjustments for selection. The mortality tables H^M and $H^{M(5)}$ are the mathematical embodiment of the influence of medical selection during the first five years of whole-life assurance.

The annuity tables just published by the Institute of Actuaries are the embodiment of the same influence in so far as it concerns annuity nominees (Joint Life Annuity Tables for Lives of both sexes, and also Single Life Annuity Tables, deduced from the mortality experience of Government Life annuitants). In truth the law of mortality varies with the age at entry, but mortality tables constructed on this principle are based upon observations too restricted, and are therefore untrustworthy. Moreover, to employ them practically is quite impossible. It has, however, been attempted to take account of

the influence of selection specially manifested in annuity nominees above 65 years of age. The ages $x, y, z \dots$ to meet the change in the condition of the lives observed have been represented by the Greek letters $\xi, \eta, \zeta \dots$ *

2. Valuation premiums, and commercial premiums. The loading on the net premium for the different life benefits includes two very distinct portions. The first is destined to meet the outgo which will be incurred by the Company as long as the contract lasts, and which recurs each year (cost of conducting the business and provision for bonus). This may be called "cost of working" (*frais de gestion*). The second is to provide for the special expenses incurred in acquiring new business (remuneration to agents) and the cost of collecting the premium throughout the duration of the contract. This has been called "procuration expenses." These two reasonable divisions have produced two distinct premiums, one taking account of "cost of working", and called Valuation premium, the other bringing in also the "procuration expenses", and called Commercial premium. This last is the premium given in the prospectus. The necessity for the Valuation premium has already been felt, because the English Companies in their valuations introduce an adjustment (whole-life loading) for those contracts under which the premiums cease before the sum assured becomes payable.

Following out these ideas we could designate by A' and P' the Single and Annual Valuation Premiums, and by A'' and P'' the Single and Annual Commercial or Office Premiums corresponding to the net premiums A and P .

COMMUTATION SYMBOLS.

$$D_x = v^x l_x = (1+i)^{-x} l_x$$

has everywhere the same signification.

N_x has among English actuaries † that value which divided by D_x gives the life annuity a_x .

In America and France there has been adopted for N_x the value which divided by D_x gives for its quotient $1 + a_x$, that is the annuity-due. This difference is of small importance, and to take

* On the English method of marking Selection, see Translator's Note on p. 16.—G. K.

† In England the notation has also been adopted $N_x = D_x + D_{x+1} + \&c.$ —G. K.

note of it is sufficient. The French and American notations take account principally of the annuity as an auxiliary quantity, to be used as divisor in the calculation of the annual premiums for life benefits. From a didactic point of view, and looking upon annuities as a class of benefits distinct in themselves, like annuities-certain, no valid reason can be urged in favour of the adoption for N_x of the value $D_x + D_{x+1} + \dots + D_w$; annuities, unless there be a special stipulation to the contrary, being always understood as making the first payment fall due at the end of one year. But this is of little consequence. It is sufficient to pay careful attention to their construction, in using commutation columns constructed on one or on the other system.

$$S_x = \sum_x^w N_x.$$

This being a function of N , will have, following what we have just said, different values according to the method of constructing N .

$$C_x = v^{x+1} d_x$$

but the French actuaries have assigned to it the value

$$C_x = (1+i)^{-(x+\frac{1}{2})} d_x.$$

This modification is the mathematical embodiment of the hypothesis that all deaths take place in the middle of the year. We need not remark that this hypothesis is but a modification of the English hypothesis* that all deaths take place at the end of the year. It must be said, however, in defence of the English hypothesis, that it excellently answers the theoretical purposes of actuarial science, and that supplementary calculations enable us by it to approximate as closely as may be desired to the real facts, without even the necessity of assuming that deaths are uniformly distributed between two consecutive ages.

The only objection that can be made to the French value of C_x , from the strictly mathematical point of view, is in its application to a table of mortality other than that from which it first had its rise, a table where the values of l_x , l_{x+1} , l_{x+2} . . . &c. are replaced by $l_{x+\frac{1}{2}}$, $l_{x+\frac{3}{2}}$. . . &c. In fine, if $C_x = (1+i)^{-(x+\frac{1}{2})} d_x$ is a practical notation, it cannot be accepted from the stand-point of rigorous mathematical reasoning.

$$M_x = \sum_x^w C_x$$

* The English hypothesis does not refer to the time of death, but is based upon the original practice of paying claims six months after death, that is, approximately at the end of the year of death.—G. K.

This being a function of C_x will therefore have for value in the English notation

$$(1+i)^{-(x+1)}d_x + (1+i)^{-(x+2)}d_{x+1} + \&c.$$

and in the French notation

$$(1+i)^{\frac{1}{2}}[(1+i)^{-(x+1)}d_x + (1+i)^{-(x+2)}d_{x+1} + \dots \&c.]$$

$$R_x = \Sigma M_x$$

This will be derived naturally from the value of the function represented by M .

CONCLUSIONS.

The notation adopted by the Institute of Actuaries, the Faculty of Actuaries in Scotland, and the Actuarial Society of America, is rational, and for this reason does not require any effort of memory. It is complete, because it includes all cases met with in practice; and the principles upon which it is based, allow of its extension to the most difficult combinations. All other notations have been inspired by it, and none of them have simplified it. Moreover, it has been employed for many years by numberless English authors, whose works form a veritable monument in science.

It is not this last reason which induces us to give preference to the notation of the Institute of Actuaries, but with the advantages which it possesses over all other systems actually in use, and seeing that the desire has been expressed by members of the profession not to introduce an entirely new notation unless such were found to be absolutely necessary, this circumstance is certainly one argument the more in favour of adopting the English system.

We therefore submit to the Congress the following two resolutions:—

- (1) That the notation of the Institute of Actuaries, be adopted by Actuaries of all countries.
- (2) That such modifications in it as may be found necessary in future, be considered by future International Congresses.

APPENDIX.

COMMUTATION SYMBOLS.

FUNCTION ENGLISH NOTATION (Age x)	Institute of Actuaries (England)	Actuarial Society (America)	Committee of Four Companies (France)	Dormoy — (France)	Karup, Senr. — (Germany)	Landré — (Holland)
$\begin{cases} v^x l_x \\ \{ (1+i)^{\omega-x} l_x \end{cases}$	D_x	D_x	D_x	D_a	D_a	λ_a
$\begin{cases} v^{x+1} l_{x+1} + v^{x+2} l_{x+2} + \dots + v^{\omega} l_{\omega} \\ v^x l_x + v^{x+1} l_{x+1} + \dots + v^{\omega} l_{\omega} \\ (1+i)^{\omega-x} l_x + (1+i)^{\omega-(x+1)} l_{x+1} + \dots + l_{\omega} \end{cases}$	N_x^*	N_x	N_x	N_a	$\Sigma(D_a)$	$\Sigma \lambda_a$
$\begin{cases} v^{x+1} l_{x+1} + 2v^{x+2} l_{x+2} + \dots + (w-x)v^{\omega} l_{\omega} \\ v^x l_x + 2v^{x+1} l_{x+1} + \dots + (w-x+1)v^{\omega} l_{\omega} \end{cases}$	S_x	S_x	S_x	S_a	$\Sigma \Sigma(D_a)$	$\Sigma \Sigma \lambda_a$
$\begin{cases} v^{x+1} d_x \\ \{ v^{x+1} d_x \} \end{cases}$	C_x	C_x	C_x	C_a		τ_{a+1}
$\begin{cases} v^{x+1} d_x + v^{x+2} d_{x+1} + \dots + v^{\omega+1} d_{\omega} \\ v^{x+1} d_x + v^{x+2} d_{x+1} + \dots + v^{\omega+1} d_{\omega} \end{cases}$	M_x	M_x	M_x	M_a		$\Sigma \tau_{a+1}$
$\begin{cases} v^{x+1} d_x + 2v^{x+2} d_{x+1} + \dots + (w-x+1)v^{\omega+1} d_{\omega} \\ v^{x+1} d_x + 2v^{x+2} d_{x+1} + \dots + (w-x+1)v^{\omega+1} d_{\omega} \end{cases}$	R_x	R_x	R_x	R_a		$\Sigma \Sigma \tau_{a+1}$

SYSTEMS OF NOTATION.

	Dormoy	Committee of Four French Companies	Institute of Actuaries	Karup, Senr.	C. Landré
GENERAL SYMBOLS.					
I. Interest.					
Annual Interest per Mon- etary Unit	t	i	i	z	i
Discounted Value $\frac{1}{1+i}$ of 1 payable a year hence . . .	$(1+t)^{-1}$	$(1+i)^{-1}$	v	ρ	v
Nominal Rate of Interest convertible m times a year, when the effective rate is i , $m \{ (1+i)^{\frac{1}{m}} - 1 \}$			$j_{(m)}$		
$\text{Log}_e(1+i)$; Force of Discount	ρ	ι	δ		
Value of i discounted, $\frac{i}{1+i}$ $= 1 - v = vi$		$\frac{i}{1+i}$	d		ρ
Present Value of an Annuity of n payments			$a_{\overline{n} }$		
II. Mortality.					
Number Living	$l, f(x), y$	v	l	λ	l, l'
Number Dying		d	d	τ	d, t

	Dormoy	Committee of Four French Companies	Institute of Actuaries	Karup, Senr.	C. Landré
Population			L		
Probability of Life . . .		p	p		W
Probability of Death . .		q	q	w	w
Limiting Age of the Table	ω		ω		
Correction for Selection .		γ	*		
Age corrected for the In- fluence of Selection		ξ, η, ζ	*		
Force of Mortality . . .	τ	z	μ		μ
Central Death Rate . . .			m		
Expectation of Life . . .			e		
Makeham's Formula . .	$\frac{k}{a^x} g^{q^x}$	$ks^x g^{c^x}$	$ks^x g^{c^x}$		$ck^x g^{q^x}$
III. Benefits.					
Endowment	Q^n	P^n	${}_nE$	${}^n\mathfrak{B}$	
Annuity, Life Interest . .	X	a	a	1R	r
Annuity-due			$a, 1+a$	R	R
Single Premium for an Assurance	P	Π	A	P	C
Annual Premium	p	π	P	p	p
Policy Value, Reserve . .	V	G	V	$\mathfrak{V}$	V

* In England Dr. Sprague's way of marking selection has been officially adopted, viz. :—

$l_{[x]}$ = living aged x , just selected.

$l_{[x:n+1]}$ = living aged x , selected a year ago.

	Dormoy	Committee of Four French Companies	Institute of Actuaries	Karup, Senr.	C. Landré
SPECIAL SYMBOLS.					
I. Designation of the Lives to which the Principal Symbol refers.					
Number Living	$l_x, f(x)$	v_x	l_x	λ_a	$l_a, \phi, f(x)$
Number of pairs existing, Two Joint Lives	$l_x \times l_y;$ $f_{(x)} \times f_{(y)}$	v_{xy}	l_{xy}	$\lambda_a \times \lambda_b$	
Expectation of Life. One Life			e_x		
Annuity on One Life . .	X_x	a_x	a_x	1R_a	r_a
Joint Life Annuity ceasing on first death	$X_{xyz} \dots$	$a_{xyz} \dots$	$a_{xyz} \dots$	${}^1R_{a, b, c}$	$r_{a, b, c}$
Joint Life Assurance pay- able on first death	P_{xyz}	Π_{xyz}	A_{xyz}	$P_{a, b, c}$	$C_{a, b, c}$
Probability that at least one of three Joint Lives will fail within a year		q_{xyz}	q_{xyz}		
II. Indication of Increase in Age of all the Lives mentioned in the Suffix.					
Number living, aged $x+n$	$f_{(x+n)}$	v_{x+n}	$l_{x+n}; {}^n l_x$	λ_{a+n}	$l_{x+n},$ $f'(x+n)$
Number of pairs of Lives existing, aged $x+n$ and $y+n$	$l_{x+n} \times l_{y+n}$ $f'(x+n)$ $\times f'(y+n)$	$v_{x+n} v_{y+n}$	${}^n l_{xy}$ also $l_{x+n} y+n$	$\lambda_{a+n} \lambda_{b+n}$	
Probability that three Joint Lives, aged $x+n, y+n$, and $z+n$, will all live a year . .		$p_{x+n, y+n, z+n}$	${}^n p_{xyz}$		
Annuity on two Joint Lives, aged $x+n$ and $y+n$.	$X_{x+n: y+n}$	$a_{x+n: y+n}$	$a_{x+n} y+n$ ${}_n a_{xy}$	${}^1R_{a+n, b+n}$	

	Dormoy	Committee of Four French Companies	Institute of Actuaries	Karup, Senr.	C. Landré
<p>III. Indication of the number of years over which the probability or benefit extends, as shown in the principal symbol.</p> <p>Probability that a Life aged x will live n years</p> <p>Probability that both the Lives x and y will live n years</p> <p>Annual Premium limited to n years</p> <p>Value after n years of a Policy on two Joint Lives .</p> <p>Endowment payable if two Joint Lives, x and y, both survive n years</p>		p_x^n p_{xy}^n $\pi^{(n)}$ G_n P_{xy}^n	$n p_x$ $n p_{xy}$ $n P$ $n V_{xy}$ $n E_{xy}$	 $n p$ $n \left[\begin{smallmatrix} P a, b \\ p a, b \end{smallmatrix} \right]$ $n \mathfrak{P} a, b$	
<p>IV. Indication of the Survivors of Lives.</p> <p>Probability that the Survivor of x and y will live n years</p> <p>Probability that the last Survivor of x, y, and z, will die within a year</p> <p>Annuity payable in full during Joint Lives and to Survivor of x and y . . .</p>	 Q_{xy}^n X_{ab}	 $\overline{p}_{xy}^n$ q_{xyz} $a_{\overline{xy}}$	 $n \overline{p}_{xy}$ $q_{\overline{xyz}}$ $a_{\overline{xy}}$	 $R_a \ b \ (*)$ ₁	

* *General Remark.*—In German Notation R represents the annuity-due. There may, therefore, not be complete analogy between the functions in the fourth column and the others for the same line.

Downloaded from <https://www.cambridge.org/core/terms>. <https://doi.org/10.1017/S0020268100005953>

	Dermoy	Committee of Four French Companies	Institute of Actuaries	Karup, Senr.	C. Landré
Annual Premium payable until the second death for an Assurance on the Survivor of two Lives	p_{xy}	$\pi_{\overline{xy}}$	P_{xy}		
Annual Premium payable until the first death for an Assurance on the Survivor of two Lives		π_{xy}			
Probability that at least r Lives out of m Lives will sur- vive n years			$n p_{xyz \dots (m)}^r$		
Annuity on the last r Sur- vivors of m Lives			$a_{xyz \dots (m)}^r$		
Annuity payable while exactly r Lives out of m Lives are in being, and ceasing on the death of one of the r Lives			$a_{xyz \dots (m)}^{[r]}$		
Assurance on the last r Sur- vivors of m Lives			$A_{xyz \dots (m)}^r$		
Annuity payable during the Life of the last Survivor of the three Lives abc , jointly with all the three Lives xyz .			$a_{xyz : \overline{abc}}$		
Annuity payable during the Life of the last Survivor of abc , jointly with the last Sur- vivor of xyz			$a_{\overline{xyz} \overline{abc}}$		
V. Indication of Survivorship.					
Contingent Annuity to x after y	P	$\Pi_{\overline{y}}^x$	$a_{y/x}$	$R_{a, b}$ e	

	Dormoy	Committee of Four French Companies	Institute of Actuaries	Karup, Senr.	C. Landré
Annual Premium for a Con- tingent Annuity		$\pi_{\frac{y}{x}}$	$Pa_{y x}$	$p_{a,b}$ e	
Contingent Annuity on the Status s after the failure of the Status s'			$a_{s s}$		
Contingent Assurance on the Survivor of x and y against z			$\left\{ \begin{array}{l} A_{x \overline{xy}} \\ A_{z \cdot \frac{1}{xy}} \end{array} \right.$		
Contingent Annuity on the last Survivor of abc , after the death of the last Survivor of xyz			$a_{xyz \overline{abc}}$		
VI. Indication of the Order of Death.					
Probability that of xyz , x will die first within n years .			${}_nq_{xyz}^1$		
Probability that x will die before y in the n th year . . .			${}_{n-1} q_{xy}^1$		
Total probability that x will die before y			Q_{xy}^1		
Assurance payable on the death of w if he die fourth, the other deaths following the order shown in the suffix . .			A_{wxyz}^4 321		
Compound Survivorship An- nuity to x , after the Survivor of y and z , z having died first			$\left\{ \begin{array}{l} a_{yz x}^1 \\ a_{yz x}^2 \end{array} \right.$		

	Dermoy	Committee of Four French Companies	Institute of Actuaries	Karup, Senr.	C. Landré
VII. Indication of the Order of Succession.					
Assurance payable on the death of the n th Life to be nominated in succession after the Life in possession			$\frac{A_{\overline{x(y)(s)}}}{A_{\overline{(1)(2)(3) \dots (n)}}}$		
VIII. Indication of a Limit- ation of time.					
Probability that x and y will both die within n years .		$q_{xy}^{(n)}$	$ nq_{xy}$		
Temporary Annuity	${}_nX_x$	$a_x^{(n)}$	$ {}_na_x; a_{x\overline{n}} $	${}_nR_a$	${}_nra$
Temporary Assurance	${}_nP_x$		$ {}_nA_x; A_{x\overline{n}}^1 $	${}_nPa$	${}_nC_a$
Single and Annual Premium for an Endowment	$Q_x^n; p$	$P_x^n; \pi_x^n$	$A_{x\overline{n}}^1; P_{x\overline{n}}^1$	${}_n\mathfrak{P}_a; {}_n\mathfrak{p}_a$	
Single and Annual Premium for an Endowment Assurance			$A_{x\overline{n}}; P_{x\overline{n}}$	${}_n\mathfrak{P}_a; {}_n\mathfrak{p}_a$	
Endowment Assurance on the Survivor of two Lives .		$\Pi_{xy}^{(n)}$	$A_{xy\overline{n}}$		
Annual Premium for an Endowment Assurance for n years, the Premium being payable for n' years. . . .		$\pi_x^{(n, n')}$	${}_nP_{x\overline{n}}$		
Temporary Contingent An- nuity		$\Pi_{\frac{y}{x}}^{(n)}$	$ {}_na_y x$	${}_nR_a \text{ } b$ e	

	Dermoy	Committee of Four French Companies	Institute of Actuaries	Karup, Senr.	C. Landré
IX. Indication of a deferred term.					
Annuity deferred n years .	X_x^n	a_x^n	$n a_x$	${}_nR_a$	
Assurance deferred n years			$n A_x$	${}_nPa$	
Annual Premium for a Deferred Annuity on the Survivor of two lives . . .			$P_n a_{xy}$		
Intercepted Annuity . .	${}_mX_x^n$	$a_x^{n(m)}$	${}_n m a_x$	${}_m{}_nR_a$	
Deferred Contingent Assur- ance			${}_n A_{xy}^1$		
X. Indication of Sub-division of the year.					
Annuity payable m times a year	$\frac{1}{m}X_x$		$a_x^{(m)}$	${}_m{}_nR_a$	$\frac{n}{r^n a}$
Deferred Annuity payable m times a year on the Survivor of two lives	$\frac{1}{m}X_{xy}^n$		${}_n a_{xy}^{(m)}$		
Annual Premium for ditto			$P_n a_{xy}^{(m)}$		
Assurance payable $\frac{1}{2m}$ of a year after death			$A_x^{(m)}$		

	Dormoy	Committee of Four French Companies	Institute of Actuaries	Karup, Senr.	C. Landré
XI. Indication of Continuity.					
Continuous Annuity . . .	$\overline{X}_x$		$\bar{a}_x$		$\overline{R}_a$
Assurance payable at the moment of death: Continuous Assurance			$\overline{A}_x$		
Endowment Assurance pay- able at the moment of death .			$\overline{A}_{xn'}$		
XII. Indication that the Benefit is to be complete.					
Complete Expectation of Life			$\hat{e}_x$	${}_mL_a$	e_a
Deferred Complete Expect- ation of Life			${}_n\hat{e}_x$		
Complete, or Apportionable Annuity			$\bar{a}_x$		
XIII. Indication that Pre- miums are to be payable by Instalments.					
Annual Premium payable by m instalments			$P^{(m)}$	${}_p^m$	
Policy Value when the Premium is payable by instal- ments			$V^{(m)}$		

	Dormoy	Committee of Four French Companies	Institute of Actuaries	Karup, Senr.	C. Landré
XIV. Symbols for Annual Premiums for Fractional or Continuous Benefits.					
Annual Premium for an Assurance payable $\frac{1}{2m}$ of a year after death			$^{(m)}P$		
Policy Value of ditto . .			$^{(m)}V$		
Annual Premium for a Continuous Assurance			$^{(\infty)}P$		
Policy Value of ditto . .			$^{(\infty)}V$		
XV. Symbols for Variable Benefits.					
Annuity commencing at 1 and increasing by 1 each year	Y		(Ia)		
Varying Annuity	Y		(va)		
Assurance commencing at 1 and increasing by 1 each year.			(IA)		
Varying Assurance			(vA)		
Temporary Increasing Annuity	Y		$(Ia)_{\overline{xn} }$		
Temporary Varying Annuity	Y		$(va)_{\overline{xn} }$		

	Dormoy	Committee of Four French Companies	Institute of Actuaries	Karup, Senr.	C. Landré
Temporary Increasing As- surance			$(IA)_{x\overline{n} }$		
Temporary Varying As- surance			$(\nabla A)_{x\overline{n} }$		
Whole-Life Annuity In- creasing for n years . . .	Y		$(I_{\overline{n} }a)_x$		
Whole-Life Annuity Vary- ing during n years	Y		$(\nabla_{\overline{n} }a)_x$		
Whole-Life Assurances In- creasing for n years			$(I_{\overline{n} }A)_x$		
Whole-Life Assurance Vary- ing for n years			$(\nabla_{\overline{n} }A)_x$		

The following is an abstract of the discussion which took place at the Congress on M. Bégault's Report:—

M. BÉGAULT, in submitting the Report said that it was not without misgiving that he had first taken up the subject. The further he had advanced in his studies, and the more he had become acquainted with the conscientious work done in different countries, the more hesitation he had felt in asking actuaries to surrender the notations familiar to them, in order to adopt that of the Institute of Actuaries. What had encouraged him to follow up the task which he had undertaken was, that in several countries actuaries were not in accord in employing a national notation. M. Landré had written that his notation had been derived partly from that of Zillmer, and partly from the English system. M. Karup, who had kindly revised the tables in the appendix to the Report, had said that German actuaries used several systems, and that, so far, they had not come to an agreement. This information, which bore

witness to the service to be rendered to actuarial science, decided him to complete his task.

The Report submitted could be thus summarized:—

1. The German notation of M. Karup, Senr., is neither complete nor suggestive. It requires constant efforts of memory, the same letter, P, representing entirely different functions according to the kind of type used, whether Roman or Gothic. It is therefore not representative, and moreover it is not complete. It presents the same defects as that of Francis Baily.

2. The Dutch notation, which has borrowed many of its symbols from the English system, is incomplete.

3. The notation of Dormoy, incomplete, as also is that of Maas which preceded it, includes symbols which suggest the French names of the functions even less than does the English system itself.

4. Lastly, the notation adopted by the Committee of Four French Life Assurance Companies is based on the principles of the English system.

It is to be remarked that the notation of the Institute of Actuaries could be adopted in its entirety by French-speaking peoples from the point of view of the convenience which results from representing technical terms by their initial letters. It is therefore unnecessary to create an entirely new system, a fact which meets the wish expressed elsewhere by a member of the Institute of French Actuaries.

Moreover, it is undeniable that the slight modifications introduced by the Committee of French Companies into the notation relating to annuities, have not simplified it; and it is to be presumed that analogous alterations in the notation relating to assurances, would not lead to any simplification of the English system.

There is, nevertheless, one improvement introduced by the Committee of Four French Companies, namely, the methods proposed for marking "selection", and for distinguishing valuation premiums and office premiums. It is this which had induced him to submit the second portion of the resolution to be discussed, namely:—That such modifications as may be found necessary hereafter, be considered by future International Congresses.

M. QUIQUET said that, in his capacity of delegate of the Institute of French Actuaries, he had, in accordance with resolutions passed by that Institute, to record his vote for a universal notation, but personally he was not in favour of such a course. Under these circumstances he felt his position to be somewhat delicate. Nevertheless, he wished to explain the reasons which had led him to his own convictions, and, without committing the Institute of French Actuaries, he would oppose the proposals of M. Bégault.

Certainly the idea of a universal notation is attractive, but is it possible? Would it not introduce serious inconveniences?

In the first place, M. Bégault has relied, in order to induce us to adopt the English system, on the example of the Electrical Congress. There is, however, an essential difference between the

resolutions of that Congress and the proposals on which we are asked to vote. We are asked to represent certain functions by appropriate symbols: the electricians had to determine the values of units of measurement. They confined themselves, as regards notation, to a symbol for each of the three fundamental units which have given their name to the system, C.G.S.

The most serious inconvenience which it appeared to him as likely to result from the adoption of a single official system, is the retarding influence it would exercise on initiative. Who has said that no one has imagined a system more simple than that now offered to us? The official adoption of a system would give it a sanction which would make more difficult, and would retard, the adoption of any other.

Who says, on the other hand, that the commutation system is the best? and who will venture to affirm that it will be used for any length of time in the future? Its official adoption will evidently delay, precisely because of the official character attached to it, the substitution of a simplification or an innovation.

Thus, the resolution submitted would have no other effect than to interfere with initiative. It would neither inspire nor stimulate anything. It would tend to enslave science, whose freedom it would be much better to proclaim.

He would go further. A single system would not have the effect of suppressing other systems. We must read the works already published. It will be necessary, therefore, as in the past, to know the systems of notation employed hitherto. That is why the Institute of French Actuaries had proposed the preparation of a dictionary of notations, which should facilitate the study of existing works, and which would be, in his opinion, more useful than a single system. On this account he congratulated M. Bégault on having, in the most perfect manner, arranged the tables in the appendix to his Report. They will be of the greatest assistance to us in our studies.

He did not certainly under-estimate the importance of the labours of his English brethren, and he wished to acknowledge them in passing. They constitute a veritable monument, and that is one of the best arguments that could be urged in favour of adopting their notation. But the publications of the Institute of French Actuaries already form, with the nine volumes which previously appeared during the time of the former *Cercle des Actuaire Français*, a remarkable series, not written in English notation. The French notation includes certain peculiarities which do not suggest the English system. The difference of symbols, and the question of initials, would not be, as M. Bégault seems to think, an obstacle to its adoption: but the Committee of Four French Companies has invented a special notation for pure premiums, valuation premiums, and office premiums, distinctions which are not to be found in the English system. It follows that this system should at any rate be modified before being agreed to. The system adopted would therefore no longer be the English system, but a new system to be added to the already long list of those with which we must be acquainted.

He would add that the English system appeared to him to be redundant and complicated, and to present, from the typographical point of view, very serious difficulties.

He would therefore ask the Congress to accept the following resolutions:—

1. That there is no need to elaborate a new system of notation.
2. That, without laying down any system, the Congress resolves that authors should, at the head of their works, specify the system of notation used.
3. That the tables prepared by M. Bégault be completed and published.

Mr. MCCLINTOCK said that he shared the opinion of M. Quiquet, that a universal notation is certainly not absolutely necessary. Nevertheless, it appeared to him, as well as to the other American actuaries present at the Congress, that the report of M. Bégault treats the question as well as it could be treated under existing circumstances. Also they were disposed to adopt his conclusions, although they hoped that the modifications in details which the Americans have made in the English system, would be adopted. There are, in fact, two points to be considered. First, is a universal notation desirable? M. Bégault had sufficiently established the necessity of an understanding on this point. Next, should this universal notation be one of the systems already existing? and, if so, which of the systems is most to be preferred? The great seniority of the Institute of Actuaries of London, and the preponderating influence which it has acquired, as well as the important work it has done, seemed to the American actuaries sufficient reasons for adopting its system as the foundation of a universal notation, as proposed in the report of M. Bégault, which justly had been so favourably received. He hoped that the French members of the Congress would accept this proposition.

Mr. PIERSON said the Actuarial Society of America had commissioned him to hand in a memorandum on the question under discussion, in the tenor of the remarks of Mr. McClintock. He had the honour of placing this memorandum on the table.*

M. SAMWER said that M. Karup had expressed the hope that the English system of notation would be adopted by the Congress.

M. BÉGAULT said that M. Quiquet had pronounced in favour of liberty in notation, fearing that initiative would be impeded by the adoption of a uniform system. That fear seemed to him to be baseless. The notation of the Institute of Actuaries of London had been known for a very long time, and he was not aware that so far private initiative had succeeded in introducing any simplifications.

Moreover, since we must learn all systems, in order to read books published in the past, we shall all know the English notation, and it is not asking too much to require all actuaries to use it in the future. M. Quiquet seemed also to think that on account of their official character, obsolete symbols would take too long to disappear; but if, for instance, commutation columns should fall into desuetude, then, *ipso facto*, should they no longer be employed, the symbols attached to them would also disappear. M. Quiquet thought the English

* A copy of this memorandum follows the discussion.

system redundant and complicated. Nevertheless, it was based on a few very simple principles of very easy application. That was his main reason for supporting the recommendations of his report, viz. :—

1. The adoption of the English Notation, and
2. The consideration by future Congresses of modifications to be made in it.

M. QUIQUET said that if modifications were made in the English system, then it would become a new system, and, if the question be remitted to a Commission, to another Congress, when should we realize the benefits of the reform? It would, at the best, be a new system to be studied. The official sanction accorded to a system, he repeated, because the arguments of M. Bégault had not convinced him, would delay progress, because innovations would be with difficulty introduced if they should have for effect to revolutionize part of a universal system.

If he said that the English system is redundant and complicated, he did not mean that as a criticism of the system itself, but he merely wished to point out the difficulties it presents in giving instruction to students. It is plainly impossible, when expounding a theory, to pause every moment, in order to explain the meaning of a complicated symbol employed; but that explanation would be unavoidable in teaching one's students or clerks. The English system was not well fitted for teaching purposes, and that was what he meant when he said it was redundant and complicated.

M. LÉON MARIE said that the question before the Congress was in two parts: 1st, is it desirable to adopt a universal notation? and 2nd, which system should be adopted? He regretted to differ from his colleague M. Quiquet, because he thought it was very desirable that actuaries should speak the same mathematical language. Nevertheless, could we decide on an existing system? He thought not, because, seeing that even the English system, proposed by M. Bégault for adoption, requires modifications, these modifications would require to be examined; and it was not at a meeting of a Congress that they could be discussed. Thus, it would be necessary to come to an understanding as to the meaning of N_x . The letter v , representing the value of 1 due a year hence, did not seem to him to be such an important simplification that it should of necessity be adopted. No mathematician would be troubled, if, instead of bringing into his calculations the symbol v , he were to write $(1 + i)^{-1}$, the negative exponent having nothing to alarm. Then again, the letters A and P each represents a premium; the one the single premium, the other the annual. To employ two similar letters, Π and π for instance, would be more logical. Other modifications for consideration could also be mentioned. Could that be done here? Would it not be better to adjourn the discussion? That is why he would propose:—

1. That the Congress resolve in favour of a universal notation; and
2. That the permanent Committee elaborate a scheme for a universal notation, the question to be decided on at the next Congress.

Mr. KING said that the English actuaries present at the Congress had read with so much interest the Report of M. Bégault that they proposed to ask leave for the Institute of Actuaries to publish a translation of it in their Journal.

The English actuaries had no desire to force their notation on their professional brethren of other nationalities. Nevertheless, that notation was based on very simple principles, and was not nearly so complicated as might be thought at first sight. It had stood, before being finally adopted, the proof of time. It had been employed for many years, and had undergone many modifications before the Institute officially adopted it.

It had been urged against the English notation that it includes special symbols for sundry benefits of various kinds unknown elsewhere. That arose from the peculiarities of the British legal system, whence these peculiar benefits sprang, for which it was necessary to provide suitable symbols; but it would always be easy for other actuaries to omit from the notation those symbols not required by them; and, as it might seem to them fit, but always adhering to the principles on which the system of notation is based, to introduce modifications suggested to them by special conditions unknown in England.

As to the difficulty of introducing into an official system changes as required, that argument did not appear to him to be of much weight. It is not the science which must be limited by the system of notation, but the system itself which must be pliable as the necessity arises. English actuaries have always been guided by this principle. In England from 150 to 200 students annually undergo the course of training of the Institute, and the difficulty in teaching spoken of by M. Quiquet has never been experienced.

Lastly, the English actuaries trusted that the Congress would bear in mind the fact that English books and periodicals for many years past have all been written in the same notation, while the already long catalogue of works by French actuaries are written in various notations. Scarcely any inconvenience could therefore be produced by adopting the English system.

Seeming, however, that it was not the desire of English actuaries to force their system on others, he would suggest slightly to amend the resolutions of M. Bégault as follows, namely:

1. It is desirable that the notation of the Institute of Actuaries should be employed by actuaries of all countries.
2. That such modifications as may hereafter be found necessary be discussed at future Congresses.

The PRESIDENT (M. Mahillon) said that four propositions were before the Congress, namely those of Messrs. Bégault, Quiquet, Marie, and King. He thought the Congress should first of all decide on the question whether a universal system of notation should be adopted; and he asked M. Quiquet whether he saw any objection to this question being in the first place determined.

M. QUIQUET replied in the negative. Although he was personally averse to a universal notation, yet, as delegate of the Institute of French Actuaries, he would vote for it.

On the resolution in favour of adopting a system of universal notation being put to the Congress, it was carried unanimously.

The PRESIDENT said it remained to settle the second point, which system should be adopted. There were three proposals before the Congress, those of Messrs. Bégault, Léon Marie, and King.

M. GERKRATH thought that the English system should be adopted.

M. LÉON MARIE said that the resolution he had proposed would defer for the longest time the full settlement of the question. Nevertheless, it could be amended. As a transitional measure the Congress might recommend the English system, leaving it to a Commission to submit a system of notation to the next Congress.

Mr. MACAULAY supported the adoption of the English system as proposed by M. Bégault.

The PRESIDENT read the text of the amended proposal of M. Léon Marie, namely :

The Congress decides that the question shall be discussed at the next Congress. In the meantime recommends the use of the notation of the Institute of Actuaries of London.

M. GOEMAERE thought that this proposal would be the most reasonable solution of the question. It would satisfy those who preferred the English system, and time would be given before the next Congress to elaborate a system acceptable to all actuaries.

M. RAFFMANN supported the English system of notation.

The PRESIDENT thought the following proposals would be unanimously agreed to, namely :

1. The notation of the Institute of Actuaries shall be employed in preference by actuaries of all countries.
2. Such modifications of it as may be found necessary shall be considered at future International Congresses.

Thus formulated, the proposals would not insist on the use of the English system, but only recommend it ; and they were of a kind, he thought, to give satisfaction to all.

These resolutions were put to the vote, and carried unanimously.

*Report of MR. ISRAEL C. PIERSON, Ph. D. Cor. memb. I.A.F.,
Actuary of the "Washington Life Insurance Company",
Secretary of the Actuarial Society of America, New York.*

The Council of the Actuarial Society of America requested me to prepare a brief article on the subject of "The Arrangement of the Commutation Columns." This topic comes naturally under the head of one of the questions in the programme which the Committee of Organization of the International Congress of Actuaries presented for the consideration of the members of the Congress: "La nécessité d'une notation universelle."

Doubtless all will agree that it is extremely desirable to have a universal system of notation, to be employed in the Actuarial profession,

and to be used by the several Institutes and Societies in their text-books, journals and publications. That it is needed does not require any argument, although there may be a difference of opinion or choice as to the significance of certain symbols and the manner of employing them. Under this division of the subjects for discussion it seems very important to consider the proper position of the numbers in the **N** and **S** columns of the Commutation Columns. There are two systems in general use:

1. That of Dr. William Farr, in which

$$\begin{aligned}D_x &= v^x l_x. \\ N_x &= D_x + D_{x+1} + D_{x+2} + \text{etc.} \\ S_x &= N_x + N_{x+1} + N_{x+2} + \text{etc.} \\ C_x &= v^{x+1} d_x. \\ M_x &= C_x + C_{x+1} + C_{x+2} + \text{etc.} \\ R_x &= M_x + M_{x+1} + M_{x+2} + \text{etc.}\end{aligned}$$

sometimes called the *initial* form.

2. That of Mr. Griffith Davies:

$$\begin{aligned}D_x &= v^x l_x. \\ N_x &= D_{x+1} + D_{x+2} + D_{x+3} + \text{etc.} \\ S_x &= N_x + N_{x+1} + N_{x+2} + \text{etc.} \\ C_x &= v^{x+1} d_x. \\ M_x &= C_x + C_{x+1} + C_{x+2} + \text{etc.} \\ R_x &= M_x + M_{x+1} + M_{x+2} + \text{etc.}\end{aligned}$$

sometimes called the *terminal* form.

The difference between the two systems is apparent at once on inspection.

The actuarial world is greatly indebted to Frederick Hendriks, Esq., for his "Memoir of the Early History of Auxiliary Tables for the Computation of Life Contingencies", the first article of the first volume of the Journal of the Institute of Actuaries. He has given a record of the invention of the Commutation method made by two men independently, in separate countries, but nearly simultaneously; the first, Johann Nicolaus Tetens, a Dano-German "highly educated professor of mathematics"; the second, George Barrett, an English "self-educated country farmer." The work of Tetens was published in 1785. (*) Barrett's first conception of his system occurred in 1786 as an independent discovery, for it appears that, although Barrett did not make public his method until 1811, he had been working at it for twenty-five years. Tetens based his first "auxiliary columns" upon the number living at each age, and made

$$C_x = v^x l_x, \text{ and } E_x = C_x + C_{x+1} + C_{x+2} + \text{etc.}$$

Barrett based his two columns also on the number living at each age, and made $A_x = l_x(1+i)^{w-x}$, and $B_x = A_x + A_{x+1} + A_{x+2} + \text{etc.}$

(*) Correspondence between Mr. George Barrett and Mr. Francis Bailly. Professor De Morgan, *J.I.A.*, iv, 185.

In the two cases the methods are different, but the ideas and mathematical principles are the same.

Mr. Griffith Davies extended and increased the utility of the Commutation Columns, but he changed the method of the formation of the second column. Exactly why he made the change does not clearly appear. It has been suggested that he did it to make the values of "*a*" symmetrical.

Dr. Wm. Farr in his English Life Tables, and Mr. David Chisholm in his works, restored the arrangement of the Commutation Columns to the Initial form as devised by the two independent inventors:

$$\begin{array}{ccccccc} \text{TETEN'S SYSTEM} & & \text{BARRETT'S SYSTEM} & & \text{DAVIES' SYSTEM} & & \text{FARR'S SYSTEM} \\ \frac{E_x}{C_x} & = & \frac{B_x}{A_x} & = & \frac{N_{x-1}}{D_x} & = & \frac{N_x}{D_x} = 1 + a_x. \end{array}$$

In 1890, it was resolved by the Actuarial Society of America to adopt the symbols of the Text-book of the Institute of Actuaries, with the exception that the letters denoting Commutation Columns, according to the system of Dr. Farr, customarily followed by American Actuaries, be printed in full-face type, and to make the additions noted below.

$$k_x = \frac{d_x}{l_{x+1}} = \frac{C_x}{D_{x+1}}; u_x = \frac{l_x}{l_{x+1}} (1+i) = \frac{D_x}{D_{x+1}}; S = \text{Sum insured.}$$

K = Cost of insurance = $q(S - V)$; Π = Single Premium (alternative for $\mathcal{A}$).

The Gothic letters, **D**, **N**, etc., were selected to make it evident at sight what they stand for respectively, as distinguished from the Roman letters, *D*, *N*, etc., or the Italic letters *D*, *N*, etc., which are used in the system of Mr. Griffith Davies.

$$D_x = v^x l_x; N_x = D_x + D_{x+1} + \text{etc.}; S_x = N_x + N_{x+1} + \text{etc.}$$

$$C_x = v^{x+1} d_x; M_x = C_x + C_{x+1} + \text{etc.}; R_x = M_x + M_{x+1} + \text{etc.}$$

As has been remarked by the author of the Institute of Actuaries' Text-book, Part II, in the practical application of the science of life contingencies premiums (which are annuities due, " $1+a$ ") are of much more frequent occurrence than annuities " a ."

Statistics extracted from latest published reports, so far as they are available, exhibit the comparative relations between the financial transactions in life insurance and in annuities during the last year:

COMPANIES	RECEIPTS FOR PREMIUMS ON LIFE INSURANCE	PAYMENTS FOR ANNUITIES
*United States	\$180,334,000	\$2,335,000
Canada	5,850,000	39,000
Great Britain	83,115,000	6,403,000
Continent of Europe, excepting France	73,181,000	4,321,000

(*) 31 December 1894. Companies of the United States, Assets \$1,060,000,000. Amount of insurance in force, \$4,660,000,000.

In the United States the life insurance companies are obliged to return to the Insurance Commissioners of the several States, annually, on 31 December, a detailed report of receipts, disbursements, assets and liabilities. Some of the commissioners require the data of all the insurance and annuity contracts outstanding, and make a valuation of these liabilities. The Insurance Commissioner of the State of Massachusetts, to whose department the largest number of such reports is made, says that to make valuations of the policies of the companies, it is necessary to use about seven hundred different formulæ for the varying forms of insurance, and about twenty-five different formulæ for the several annuity forms.

In a single case, the New York Life Insurance Company, which has outstanding a larger number of annuity contracts than all the other American companies together, the number of varying formulæ employed for computing premiums and reserves on insurances is 260, while of annuity formulæ there are twelve different varieties.

The statistics and statements which have just been given exhibit these ratios :

LIFE INSURANCE RELATIVE TO ANNUITY TRANSACTIONS :

COMPANIES

United States	77 to 1
Canada	150 to 1
Great Britain	13 to 1
Continent of Europe, excepting France	17 to 1

LIFE INSURANCE RELATIVE TO ANNUITY FORMULÆ USED.

State of Massachusetts	28 to 1
New York Life Insurance Company	22 to 1

The Initial or Farr System has been used in America in all the Life Insurance publications containing Commutation Columns and formulæ, and in all the manuscript work in the offices, although the Gothic letters have not been employed until within a few years. The principal works are mentioned in Note A.

Mr. George King, in the Institute of Actuaries' Text-book, says : (*) "Since, in the practical application of the science of life contingencies, premiums are of much more frequent occurrence than annuities, Davies' alteration in the summation of the columns has the effect of destroying the symmetry of the great majority of the formulas, and is therefore of doubtful benefit." With this opinion many other eminent English Actuaries agree. See Note B.

It is evident :

1. That the prevailing occasions on which the Commutation Tables are employed are those for computations to ascertain the premiums required for life contingencies, annuities due. These computations are made simply and symmetrically if the second column is constructed according to the initial form.

2. That the general opinion is that the summation of the second column of the Commutation Tables, as made originally by Tetens and Barrett and restored by Farr, is the natural one and is mathematically correct.

(*) Institute of Actuaries' Text-book, p. 109.

But the Davies, or Terminal form, is used generally by Actuaries outside of the United States, and is the rule for the Journal of the Institute of Actuaries.

Mr. King says that as Davies' "method of summation has been almost universally adopted, we shall accept it" for the Text-book.

The Actuarial Society of America has adopted the notation of the Institute of Actuaries, having added somewhat to it. They are prepared to agree to an improvement or extension of this system in any way that may be thought best, but at the present time recommend the acceptance of this system with the addition of symbols for the Initial form of the Commutation Columns.

It is proposed to distinguish the Commutation Columns of the Farr or Initial system by the Gothic letters, **D**, **N**, etc., and those of the Davies, or Terminal system, by the Roman or Italic letters, *D*, *N*, etc. Then $\mathbf{D}_x = D_x$; $\mathbf{N}_x = N_{x-1}$ etc.

It is advisable to secure a system which will not inconveniently disturb existing tables, formulæ and custom.

Dr. Thomas B. Sprague, in connection with his select tables, proposes to adopt the Initial form, and suggests an open letter, *e.g.*, **N**, to indicate the numbers in that column. His suggestion meets with general approval, but since the Gothic letter **N**, etc., is more easily written than the open letter **N**, etc., it seems best to use Gothic letter.

This important question, "the necessity of a universal notation", has been brought forward very opportunely by the Committee of Organization. It is to be hoped that a system will be approved by this Congress, or in the near future, which general opinion, custom and experience show to be the best to be used in practice, and to be employed in the future in publications in the field of the science of life insurance, whatever the language may be.

NOTE A.

Massachusetts Reports on Life Insurance, 1859-1865. Mr. Elizur Wright, 1865.

New York Insurance Department Reports, 1864-1868.

Valuation Tables, published by the New York Insurance Department.

Agents' Monetary Life and Valuation Tables, by David Parks Fackler, A.M., Ex-President of the Actuarial Society of America, 70 pages. Insurance Monitor, N. Y., 1870.

Principles and Practice of Life Insurance, by Mr. Nathan Willey. First edition published in 1872. The sixth edition, 413 pages, prepared in 1892, by two members of the Actuarial Society of America. New text by Mr. Henry Worthington Smith, A.M., LL.B. All tables revised, with the co-operation of Mr. John Tatlock, Jr., A.M., F.R.A.S., Assistant Actuary of the Mutual Life Insurance Company. The Spectator Company, N.Y., publishers.

Life Tables, with appropriate formulas for the solution of questions pertaining to Life Contingencies, by Lewis C. Lawton and Benjamin Griffen, of the Mutual Life Insurance Company of New York. Second edition, 481 pages. Published by the Company in 1873.

System and Tables of Life Insurance: a treatise developed from the experience and records of thirty American Life Offices, under the direction of a committee of Actuaries, among whom were Messrs. Sheppard Homans, George W. Phillips, D. Parks Fackler, Emory McClintock, Wm. D. Whiting, Charlton T. Lewis and James Weir Mason, members of the Actuarial Society of America. By Levi W. Meech, Actuary in charge, 551 pages. Published in 1881.

Experience of the Washington Life Insurance Company. Published in 1889. Papers and Transactions of the Actuarial Society of America.

NOTE B.

Mr. Griffith Davies, to whom science owes improved adaptions of Barrett's method, committed one mistake, which has led to confusion; he raised the number in columns N and S one place, and thus adopted an anomalous arrangement of the columns of the same table. Entertaining the highest respect for Griffith Davies, and desirous to retain his forms, I had to choose between the alternative of perpetuating his fortuitous derangement of the natural order of the columns, to the confusion of future students, or of embarrassing, by change of symbols, those who had already grown accustomed to the derangement. But I decided in favor of the natural arrangement of the columns, which has been ever since adhered to, and found to work satisfactorily, and I retained the letters N and S.—See pp. cxix. and cxx., English Life Tables, Dr. Wm. Farr.

It is difficult to conceive the amount of confusion which is avoided by taking $N_x = D_x + D_{x+1} \dots$, precisely as $M_x = C_x + C_{x+1} \dots$; and also by taking $a_x = \frac{N_x}{D_x}$, which Professor De Morgan calls "an annuity-due."—English Life Table, p. 157.

Mr. Davies, in his D and N columns, preferred to shift the latter one line downwards, so that his N_x always corresponds with D_{x-1} or N_{x+1} with D_x . No reason can be assigned for this arbitrary change, unless it be, that he wished the expression for finding an annuity to be always $\frac{N_x}{D_x}$, and not, as it properly

ought to be $\frac{N_{x+1}}{D_x}$.—H. Fillipowski, Editor of "The Doctrine of Life Annuities and Assurances, etc., by Francis Baily."

Although the N's are here stated, according to Jones' arrangement of the column, I greatly prefer to use those of Mr. Chisholm. Joshua Milne, *J.I.A.*, viii, p. 118.

Which of these two systems is the better? By mathematical analogy, no doubt the Initial system would be preferred Again, for the computer's convenience, I should also be inclined to prefer the Initial system, on the principle that in life assurance, which much exceeds annuity business in amount, the annuity-due a premium is of more importance than the simple annuity.—Prof. De Morgan, *J.I.A.*, x, pp. 304, 305.

DISCUSSION.

THE PRESIDENT (Mr. A. J. Finlaison, C.B.) said the Institute was under an obligation to Mr. King for the vigorous and scholarly translation of M. Bégault's paper. The paper was a talented and powerful address, advocating the system of notation promulgated by the Institute of Actuaries. He ventured to think that it would prove of permanent use as a dictionary of the notation for works already in print, and would act as a valuable guide to the student in the notation of the Institute itself. The paper was considered an important one at the Congress, and had tended to add to the estimation in which the Institute of Actuaries was held on the Continent.

Mr. H. W. MANLY said M. Bégault had been the principal translator of the Text-Book of the Institute into French. He, therefore, had become familiar with the Institute notation, and had learnt to appreciate its adaptability to every status which might require to be symbolized. He believed it was generally known that it had been the intention of the French Actuaries to propose that

the whole subject be deferred until a dictionary of notations used in the various countries had been compiled, and it very much surprised them that M. Bégault had taken the subject out of their hands by writing his paper and compiling the dictionary itself. The proposition in question, therefore, necessarily fell to the ground. Notwithstanding the resolution finally come to at Brussels, it would be seen that there had been a general feeling at the Congress that it would be desirable to adopt a universal notation, and that the English would be the best to adopt. The Institute of Actuaries was very much indebted to M. Bégault, not only for introducing our Text-Book on the Continent, but for the production of the paper now under discussion.

MR. J. CHISHOLM said he rose to refer to a small point dealing with one symbol only, which, however, revived memories of battles fought long ago, but still undecided. The Commutation Tables adopted by the Institute differed from those in use in America, and he wished to emphasize, now that they were on the point of adopting a universal notation, that they should do all in their power to prevent the further use of a double system, such as was in vogue at the present time, and he hoped that no further works would be published on the Commutation system other than in the notation adopted by the Institute. He did not think there was a pin's point to choose between them, but it was desirable that under all circumstances the same symbol should have the same meaning, and seeing that Griffith Davies's system had been adopted by the Institute he thought that all should use it.

THE PRESIDENT, having proposed a vote of thanks to Mr. King, which was unanimously accorded,

MR. G. KING, in reply, acknowledged the vote of thanks, and said, with reference to Mr. Chisholm's remarks, with all deference, he thought they were a little beside the point. They did not relate to notation, but to the system of columns to be adopted, and he did not see why both forms of columns should not be used concurrently. It would be right to retain the ordinary N to represent Davies's form of column known as the terminal form, and a different symbol should be adopted for the initial form of N which had been used by Mr. Chisholm, sen., and the late Dr. Farr, and which was in use in America. As a matter of fact, the form of N printed in open type, had already been used in Great Britain to represent the initial form and had been adopted by the Institute. He thought the open N much more suitable than the thick N proposed by the American Actuaries, because it was difficult, in writing, to distinguish the thick N from the ordinary letter. He was pleased to see how loyally their French colleagues adopted the notation of the Institute of Actuaries after the resolution of the Congress had been carried. For a system of notation to be good, it must have at least five characteristics. It must be easy to remember; it must be easy to teach; it must be easy to write; it must be easy to print, and it must readily admit of additions. He thought all those five characteristics belonged to the British system. He was not surprised that their French friends found a little difficulty at first in understanding the Institute notation, for he must confess in the volume, "Institute of Actuaries' Life Tables", where the notation was first

promulgated, no attempt was made to arrange it in any order of principle. He had ventured to alter that in the Text-Book. He had not given there a long array of symbols, but the principles on which those symbols were ranged, with illustrations; and he believed such a method was clearer than merely giving a list of symbols; and now that their French friends had the Text-Book translated into their own language, he had no doubt they would see the simplicity of the British system. It was founded on a few broad principles easy to remember, and therefore it was easy to teach. He thought M. Quiquet was entirely mistaken in thinking there was any difficulty in teaching our system, and he spoke from a large experience. The question of writing was an important one, and it was very desirable that the symbols should be easy of formation. He had tried various systems, and he had found the British by far the most simple. As to the printing, that was also comparatively easy. They had had no difficulty in the printing of the Text-Books and Journals, and there was also no difficulty in printing the Text-Book in Brussels. That this was not the case as regards some of the other systems, had been demonstrated in rather an amusing manner, Messrs. Layton not having some of the type required for the tables appended to the paper, M. Bégault was sending some over. With regard to the last of the five characteristics he had mentioned, namely, the ease of adding to the notation, the notation in the Text-Book had developed to a certain extent from that of the Institute of Actuaries' Life Tables. For instance, symbols for increasing benefits had been introduced, the principle for the symbols remaining identical. A very important addition, also, had been made by Dr. Sprague, and adopted by the Institute, for indicating "selection." One or two incidental points had been mentioned at the discussion of M. Bégault's paper at the Congress. For instance, as to the use of v for the present value of 1 discounted. He would be sorry to drop that symbol, as it was very convenient, and he ventured to add that there was a theoretical difference between v and $(1+i)^{-1}$, although they had of necessity the same numerical value. If he might take an illustration from music, it was something like the difference between A sharp and B flat, and every musician knew that, although these were represented by the same note on the piano, they had, nevertheless a very different meaning in musical composition.