112. [A further consideration of the problem 391]

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112. [K. 20. e.] A further consideration of the problem 391. [K. 20. e.] by M. E. N. Barisien may prove to be of interest. A solution was given by Mr. C. E. Youngman in the previous issue (p. 180).

"In any triangle prove that

$$(a+b-2c)^2 \sec^2 \frac{C}{2} + (a-b)^2 \operatorname{cosec}^2 \frac{C}{2} = \text{two similar expressions,}$$

and interpret geometrically."

The first expression

$$\begin{aligned} &= 4R^2 \left\{ (\sin A + \sin B - 2 \sin C)^2 \sec^2 \frac{C}{2} + (\sin A - \sin B)^2 \operatorname{cosec}^2 \frac{C}{2} \right\} \\ &= 4R^2 \left\{ 4 \left(\cos \frac{A-B}{2} - 2 \sin \frac{C}{2} \right)^2 + 4 \sin^2 \frac{A-B}{2} \right\} \\ &= 16R^2 \left\{ 1 - 4 \cos \frac{A-B}{2} \cos \frac{A+B}{2} + 2(1 - \cos C) \right\} \\ &= 16R^2 \{ 3 - 2(\cos A + \cos B + \cos C) \} \\ &= 16R^2 \left(1 - 8 \sin \frac{A}{2} \sin \frac{B}{2} \sin \frac{C}{2} \right) \\ &= 16(R^2 - 2Rr) = 16OI^2, \end{aligned}$$

where O is the circumcentre and I the incentre.

Obviously the other two expressions are equal to this.

As suggested by Mr. Youngman in the last issue, on BC , CA , AB take $BD=c$, $CE=a$, $AF=b$; and again on CB , BA , AC take $CD'=b$, $BF'=a$, $AE'=c$. Then

$\overline{DE'} = \overline{DB} + \overline{BA} + \overline{AE'}$ = sum of three vectors whose directions are along the sides of the triangle taken in order, and whose lengths are each equal to c .

I propose to show that $\overline{DE'} = \frac{c}{R} \cdot \overline{OI}$ and is perpendicular to it.

Let the perpendiculars from O to the sides of the triangle meet the circum-circle in A' , B' , C' respectively.

Then it can be shown that I is the orthocentre of the $\triangle A'B'C'$.

Hence if OK be drawn perpendicular to $A'B'$ and bisecting it, we have

$$\overline{C'I} = 2\overline{OK} = \overline{OA'} + \overline{OB'};$$

$$\therefore \overline{OA'} + \overline{OB'} + \overline{OC'} = \overline{OI}.$$

Hence three vectors along BC , CA , AB , each of length R add up to OI , perpendicular to OI ;

$$\therefore \overline{DE'} = \frac{c}{R} \cdot \overline{OI}, \text{ and is perpendicular to } OI.$$

It immediately follows that

$$\overline{EF'} = \frac{a}{R} \overline{OI}, \quad \overline{FD'} = \frac{b}{R} \overline{OI};$$

$$\therefore \overline{DE'} : \overline{EF'} : \overline{FD'} = c : a : b,$$

and the three are parallel to each other, each being perpendicular to OI .

Finally, since $\overline{DE'} = \frac{c}{R} \overline{OI} = 2OI \sin C$, the circumradius of the $\triangle DE'C$ is OI ; as is also the case with the two other triangles. F. GLANVILLE TAYLOR.