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A.B. Basset M.A.

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XVI. *Note on the Induction of Electric Currents, in an Infinite Plane Current Sheet, which is rotating in a Field of Magnetic Force.* By A. B. BASSET, M.A.*

1. **T**HE determination of the currents induced in a thin plane conducting sheet of infinite extent, which is rotating about an axis perpendicular to itself in a field of magnetic force, may be effected either by employing the equations of electromotive force referred to moving axes, or by dealing with the problem directly by means of the method of images. The former method has been employed by Maxwell† and Prof. C. Niven‡, and is probably the preferable one when the currents cannot be conveniently represented by a system of images; but when the magnetic field is produced by a system of magnets or currents, the problem can be easily solved by finding an analytical expression for the magnetic potential of the moving trail of images, to which the currents are equivalent.

In the present note I have employed the latter method of dealing with the problem, and have thence deduced Niven's results; and I have also worked out the solution in the case of a current sheet rotating about an axis, in the presence of a short magnet which is fixed at right angles to the axis of revolution.

2. To solve the problem by means of images, let us choose the instant at which we desire to observe the currents as the origin of the time, and reckon the time backwards from this epoch.

Adopting Maxwell's notation, let $F(z, \tau)$ be the value of P' at time τ ago (where $-dP'/dz$ is the magnetic potential of the inducing system), and let the system P' be suddenly introduced and kept at rest; a system of currents will be instantaneously generated for which the value of P (where $-dP/dz$ is the magnetic potential of currents) is $f(z, \tau)$, the form of the function f being determined from the fact that it is the image of F with respect to the sheet.

This system of currents, as soon as it has been generated, will begin to decay, the law of decay being such that, at the end of an interval T ,

$$P = f(z + RT, \tau),$$

where $2\pi R$ is the specific resistance of the sheet.

If, therefore, at the end of an interval $d\tau$ the system P' be

* Communicated by the Author

† Electricity and Magnetism, art. 668.

‡ Phil. Trans. 1881.

removed to the position which it occupied at time $\tau - d\tau$ ago, the value of P corresponding to the sudden introduction and removal of the system P' will be

$$f(z + R d\tau, \tau) - f(z, \tau) = R \frac{d}{dz} f(z, \tau) d\tau.$$

At the instant under consideration (that is, after an interval $\tau - d\tau$), the value of this is

$$\begin{aligned} & R \frac{d}{dz} f\{z + R(\tau - d\tau), \tau\} d\tau \\ &= R \frac{d}{dz} f(z + R\tau, \tau) d\tau. \end{aligned}$$

ultimately. Summing up all similar terms between the limits t and 0 of τ , we have finally

$$P = R \frac{d}{dz} \int_0^t f(z + R\tau, \tau) d\tau + f(z, 0), \quad \dots \quad (1)$$

the last term being the potential of the currents generated at the instant at which we are observing them.

If the electromagnetic system is rotating about the axis of z in the same direction as that in which ϕ is measured, and with uniform angular velocity ω , the value of $f(z + R\tau, \tau)$, expressed in cylindrical coordinates, will be

$$f(z + R\tau, \rho, \phi + \omega\tau);$$

and the value of P , after the currents have become steady, will be obtained by integrating between the limits ∞ and 0; whence

$$P = R \frac{d}{dz} \int_0^\infty f(z + R\tau, \rho, \phi + \omega\tau) d\tau + f(z, \rho, \phi), \quad \dots \quad (2)$$

which represents a spiral trail of images.

3. From the foregoing expression for P we can at once obtain Niven's results. For

$$\frac{df}{dz} = R \frac{df}{dz} + \omega \frac{df}{d\phi}; \quad \dots \quad (3)$$

$$\therefore R \frac{dP}{dz} + \omega \frac{dP}{d\phi} = R \frac{d}{dz} \int_0^\infty \frac{df}{d\tau} d\tau + \left(R \frac{df}{dz} + \omega \frac{df}{d\phi} \right)_{\tau=0} = \omega \left(\frac{df}{d\phi} \right)_{\tau=0}. \quad (4)$$

since $f=0$ when $\tau=\infty$.

The last result is perfectly general; but at the surface,

$$-\frac{dP}{dz} = 2\pi\Phi, \quad f = -P',$$

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whence equation (4) becomes

$$\omega \frac{d}{d\phi} (P + P') = \sigma \Phi. \quad . \quad . \quad . \quad . \quad . \quad (5)$$

The value of the electric potential ψ may be obtained as follows. From Maxwell, art 600,

$$\psi = \psi' - F'\dot{x} - G'\dot{y} - H'\dot{z}; \quad . \quad . \quad . \quad . \quad (6)$$

where ψ' is the quantity which appears in art. 658, and which is therefore zero; ψ is the quantity which appears in art. 668*; and F' , G' , H' are the components parallel to moving axes x , y , z of the total vector potential due to the influencing system and the currents. In the present case,

$$\begin{aligned} \dot{x} &= -\omega y, \quad \dot{y} = \omega x, \quad \dot{z} = 0; \\ \therefore \psi &= \omega(F'y - G'x) \\ &= \omega \left(y \frac{d}{dy} + x \frac{d}{dx} \right) (P + P') \\ &= \omega \rho \frac{d}{d\rho} (P + P') \quad . \quad . \quad . \quad . \quad . \quad (7) \end{aligned}$$

at the surface.

If we change the signs of P and P' , it will be seen that (5) and (7) agree with Prof. C. Niven's results†.

4. Another result, which is more convenient for practical purposes, may be obtained as follows.

Taking account of (3), equation (2) may be written

$$P = \int_0^\infty \left(\frac{df}{d\tau} - \omega \frac{df}{d\phi} \right) d\tau + f(z, \rho, \phi);$$

and since $f(z + R\tau, \rho, \phi + \omega\tau)$ vanishes when $\tau = \infty$,

$$P = -\omega \int_0^\infty \frac{df}{d\phi} d\tau,$$

Differentiating with respect to z , we obtain

$$\Omega = -\omega \int_0^\infty \frac{d\Omega'}{d\phi} d\tau; \quad . \quad . \quad . \quad . \quad . \quad (8)$$

where Ω' is the magnetic potential at the instant under consideration of the currents which were instantaneously generated at time τ ago.

5. Let us now suppose that we have a long thin bar-magnet

* It should be noticed that the two ψ 's in Maxwell's investigation do not represent the same quantity.

† Phil. Trans. 1881, p. 342.

whose axis is parallel to the axis of revolution, and whose positive pole is situated at a point whose coordinates are $z=h$, $\rho=c$, $\phi=0$: and let us find the magnetic potential of the currents, when the magnet is rotating about the axis of z , in the same direction as that in which ϕ is measured.

In this case,

$$\Omega' = \frac{m}{\{(z + R\tau + h)^2 + \rho^2 + c^2 - 2\rho c \cos(\phi + \omega\tau)\}^{\frac{1}{2}}},$$

where m is the strength of the positive pole; whence by (8),

$$\Omega = -m\omega \frac{d}{d\phi} \int_0^\infty \frac{d\tau}{\{(z + R\tau + h)^2 + \rho^2 + c^2 - 2\rho c \cos(\phi + \omega\tau)\}^{\frac{1}{2}}}.$$

Let

$$Q = \{\rho^2 + c^2 - 2\rho c \cos(\phi + \omega\tau)\}^{\frac{1}{2}};$$

Then, since

$$\int_0^\infty \epsilon^{-\lambda v} J_0(\lambda Q) d\lambda = \frac{1}{(v^2 + Q^2)^{\frac{1}{2}}},$$

and

$$J_0(\lambda Q) = J_0(\lambda\rho)J_0(\lambda c) + 2\Sigma_1^\infty J_n(\lambda\rho)J_n(\lambda c) \cos n(\phi + \omega\tau),$$

the value of Ω becomes

$$\begin{aligned} \Omega &= -2m\omega \frac{d}{d\phi} \int_0^\infty d\lambda \int_0^\infty \epsilon^{-\lambda(z+R\tau+h)} \Sigma_1^\infty J_n(\lambda\rho)J_n(\lambda c) \cos n(\phi + \omega\tau) d\tau \\ &= 2m\omega \Sigma_1^\infty n \int_0^\infty \frac{\epsilon^{-\lambda(z+h)} (R\lambda \sin n\phi + n\omega \cos n\phi) J_n(\lambda\rho)J_n(\lambda c)}{R^2\lambda^2 + n^2\omega^2} d\lambda. \quad (9) \end{aligned}$$

The forces which act on the pole are given by the following equations:—

$$\left. \begin{aligned} -\frac{1}{c} \frac{d\Omega}{d\phi} &= -\frac{2mR}{c} \Sigma n^2 \int_0^\infty \frac{\lambda \epsilon^{-2\lambda h}}{R^2\lambda^2 + n^2\omega^2} J_n^2(\lambda c) d\lambda, \\ -\frac{d\Omega}{dz} &= 2m\omega^2 \Sigma n^2 \int_0^\infty \frac{\lambda \epsilon^{-2\lambda h}}{R^2\lambda^2 + n^2\omega^2} J_n^2(\lambda c) d\lambda, \\ -\frac{d\Omega}{d\rho} &= -2m\omega^2 \Sigma n^2 \int_0^\infty \frac{\lambda \epsilon^{-2\lambda h}}{R^2\lambda^2 + n^2\omega^2} J'_n(\lambda c) J_n(\lambda c) d\lambda. \end{aligned} \right\} (10)$$

Now we have supposed the magnetic pole to be rotating in the same direction as that in which ϕ is measured, therefore the first equation indicates a force tending to oppose the motion of the magnet; hence the magnet exerts a force on the sheet tending to pull it round in the direction of rotation. If therefore the magnet is fixed and the sheet be made to rotate, the magnet will experience a dragging force parallel to the direction of motion of the sheet.

The second equation represents a repulsive force acting from the disk ; and the third equation represents a force acting towards the axis of the disk.

These results are equivalent to those obtained by Maxwell and observed by Arago (see art. 669).

If the magnet were not sufficiently long to enable us to leave the action of the negative pole out of account, the magnetic potential of the currents excited by the latter will be found by changing m into $-m$, h into $h+a$, where a is the length of the magnet ; and the combined effect of the two poles will be obtained by adding the results.

It is, however, simpler to place a *short* magnet, of length $2c$, at right angles to the axis of revolution, and consider the combined effect due to both poles. The magnetic potential of the currents due to the action of the negative pole will be obtained in this case by changing m into $-m$, and ϕ into $\pi + \phi$; whence the expressions for the magnetic forces acting on the positive pole will be found from equations (10) by changing m into $2m$, and n into $2n+1$, and effecting the summation with respect to n from ∞ to 0.

The calculation can then be effected as follows :—Expand the Bessel's functions in powers of λc ; the expressions for the forces will then assume the form of series every term of which is of the form

$$\int_0^\infty \frac{\lambda^p e^{-2\lambda h} d\lambda}{R^2 \lambda^2 + (2n+1)^2 \omega^2},$$

where p and n are positive integers. Now

$$\int_0^\infty \frac{e^{-2\lambda h} d\lambda}{R^2 \lambda^2 + (2n+1)^2 \omega^2} = \frac{1}{(2n+1) R \omega} \left\{ \cos q \left(\frac{\pi}{2} - \text{Si } q \right) + \text{sin } q \text{Ci } q \right\}.$$

where $q = 2(2n+1)h\omega/R$. Hence, by differentiating with respect to h , the first-mentioned integral can be expressed in terms of the sine and cosine integrals, the values of which have been calculated by Mr. Glaisher, whence numerical values of the magnetic forces might be found.

If we neglect powers of c higher than the fifth, the magnetic forces in the ϕ , z , and ρ directions will be found to be respectively equal to

$$\begin{aligned} & mR \int_0^\infty \frac{\lambda^3 e^{-2\lambda h}}{R^2 \lambda^2 + \omega^2} \left(c - \frac{\lambda^2 c^3}{4} + \frac{5\lambda^4 c^5}{192} \right) d\lambda + \frac{mRc^5}{64} \int_0^\infty \frac{\lambda^7 e^{-2\lambda h}}{R^2 \lambda^2 + 9\omega^2}; \\ & m\omega^2 \int_0^\infty \frac{\lambda^3 e^{-2\lambda h}}{R^2 \lambda^2 + \omega^2} \left(c^2 - \frac{\lambda^2 c^4}{4} \right) d\lambda; \\ & -m\omega^2 \int_0^\infty \frac{\lambda^2 e^{-2\lambda h}}{R^2 \lambda^2 + \omega^2} \left(c - \frac{\lambda^2 c^3}{2} + \frac{5\lambda^4 c^5}{64} \right) d\lambda - \frac{3m\omega^2 c^5}{64} \int_0^\infty \frac{\lambda^6 e^{-2\lambda h}}{R^2 \lambda^2 + 9\omega^2}. \end{aligned}$$