Note on Clebsch's Second Method for the Integration of a Pfaffian Equation. By J. BRILL, M.A. Received November 6th, 1899. Communicated November 9th, 1899.

1. In a paper recently published by the Society,* I have shown that there exists a complete set of multilinear differential covariants of a Pfaffian expression, each derivable from the preceding, and not simply a bilinear one, as was formerly supposed. In that paper I gave the form of these covariants in terms of the quantities that occur in the Pfaffian expression and their derivatives. I propose now to discuss the form which they take when expressed in terms of the quantities included in a normal form of the given expression, and to apply my results to complete what is known as "Clebsch's Second Method for the Integration of a Pfaffian Equation."† I shall, of course, assume the results of my former paper, and shall not think it necessary to give any new proofs of known facts in the theory of Pfaffian expressions.

2. In my former paper I indicated a new method for deriving each of the various sets of derived functions, connected with a Pfaffian expression, from the preceding; and stated that the generality of the method could be established by mathematical induction. Since the paper was printed I have hit upon a convenient method for the completion of the induction, and I propose to give a sketch of it here, as the establishment of the identity of the coefficients of the various covariants with the derived functions of the various orders depends upon the generality of the method.

It will only be necessary to deal with the case of the derivation of the derived functions of the odd orders by means of differential operations, as the connexion of those of any even order with those of the preceding odd order is evidently identically similar to the connexion of the allied functions of any order with the preceding set of Pfaffians.

We will thus suppose that the derived functions of order $2r-1$, derived by my method, have been identified with the r^{th} set of

* Vol. xxx., pp. 263-271.

† See Forsyth, *Theory of Differential Equations*, Pt. 1, pp. 85, 86, and 209-218.

Pfaffians. From these we form the set of allied functions next in order, which will obviously be identical with my derived functions of order $2r$. We have a set of formulæ of the type

$$[012 \dots (2r+1)] = X_1[23 \dots (2r+1)] - X_2[134 \dots (2r+1)] + \dots \\ \dots + (-1)^{2r} X_{2r+1}[123 \dots (2r)]. \quad (1)$$

The formulæ for the derivation of our next set of derived functions are of the type

$$\frac{\partial}{\partial x_1}[023 \dots (2r+2)] - \frac{\partial}{\partial x_2}[0134 \dots (2r+2)] + \dots \\ \dots + (-1)^{2r+1} \frac{\partial}{\partial x_{2r+2}}[0123 \dots (2r+1)] = (r+1)K, \quad (2)$$

in which we have to identify K with

$$[123 \dots (2r+2)].$$

Now, substituting in equation (2) the values of $[023 \dots (2r+2)]$, &c., as given by the equations of the type (1), we obtain

$$(r+1)K = [12][345 \dots (2r+2)] - [13][245 \dots (2r+2)] + \dots \\ \dots + (-1)^{2r} [1(2r+2)][234 \dots (2r+1)] \\ + [23][145 \dots (2r+2)] - [24][135 \dots (2r+2)] + \dots \\ \dots + (-1)^{2r-1} [2(2r+2)][134 \dots (2r+1)] \\ + [34][125 \dots (2r+2)] - [35][1246 \dots (2r+2)] + \dots \\ \dots + (-1)^{2r-2} [3(2r+2)][1245 \dots (2r+1)] \\ + \&c. \quad (3)$$

By means of a well known method of expanding Pfaffians,* we have

$$[123 \dots (2r+2)] = [12][345 \dots (2r+2)] - [13][245 \dots (2r+2)] + \dots \\ \dots + (-1)^{2r} [1(2r+2)][234 \dots (2r+1)], \\ [123 \dots (2r+2)] = [12][345 \dots (2r+2)] + [23][145 \dots (2r+2)] \\ - [24][135 \dots (2r+2)] + \dots \\ \dots + (-1)^{2r-1} [2(2r+2)][134 \dots (2r+1)], \\ [123 \dots (2r+2)] = -[13][245 \dots (2r+2)] + [23][145 \dots (2r+2)] \\ + [34][125 \dots (2r+2)] - [35][1246 \dots (2r+2)] + \dots \\ \dots + (-1)^{2r-2} [3(2r+2)][1245 \dots (2r+1)],$$

* See Forsyth, p. 95.

and so on. Adding together the $2r+2$ equations so formed, and comparing the right-hand side of the result with that of equation (3), we obtain

$$(2r+2) [123 \dots (2r+2)] = 2(r+1) K;$$

and therefore

$$K = [123 \dots (2r+2)].$$

This will provide us with sufficient material for completing the induction.

There are relations among the derived functions of any order. In the case of any set of Pfaffians these relations are of a differential type. In the case of any set of allied functions they are of an algebraic type.

From the expressions for the Pfaffians $[123 \dots (2r)]$, &c., in terms of the preceding set of allied functions, we readily deduce

$$\begin{aligned} \frac{\partial}{\partial x_1} [234 \dots (2r+1)] - \frac{\partial}{\partial x_2} [134 \dots (2r+1)] + \dots \\ \dots + (-1)^{2r} \frac{\partial}{\partial x_{2r+1}} [123 \dots (2r)] = 0. \end{aligned}$$

Similarly, from the expressions for the functions $[0123 \dots (2r-1)]$, &c., in terms of the preceding set of Pfaffians, we obtain

$$\begin{aligned} X_1 [0234 \dots (2r)] - X_2 [0134 \dots (2r)] + \dots \\ \dots + (-1)^{2r-1} X_{2r} [0123 \dots (2r-1)] = 0. \end{aligned}$$

To avoid unnecessary complications, we lay down the rule that our derived functions shall be so constituted that the numbers within the square brackets shall be always in their natural order. Expressions in which this is not the case would be possible, but they would either be the same as the corresponding expressions formed by restoring the natural order of the numbers, or would merely differ from them in sign.

With this restriction, supposing n to be the number of independent variables, the number of functions in the various sets of derived functions will be respectively

$$\frac{n(n-1)}{1.2}, \frac{n(n-1)(n-2)}{1.2.3}, \dots, n, 1.$$

The sets are thus finite in number, and always terminate with one containing a single function. If $n = 2m-1$, the final derived function is

$$[0123 \dots (2m-1)],$$

while, if $n = 2m$, we have

$$[123 \dots (2m)]$$

for the final derived function.

3. We will suppose our Pfaffian expression to be of the form

$$X_1 dx_1 + X_2 dx_2 + \dots + X_n dx_n;$$

and will assume, in the first instance, that

$$l_1 da_1 + l_2 da_2 + \dots + l_s da_s$$

may be taken for a normal form, where s is not greater than $\frac{1}{2}n$ or $\frac{1}{2}(n+1)$, according as n is even or odd.

We have therefore

$$U_1 = l_1 d_1 a_1 + l_2 d_1 a_2 + \dots + l_s d_1 a_s,$$

and consequently

$$U_{12} = d_{12}(l_1, a_1) + d_{13}(l_2, a_2) + \dots + d_{1s}(l_s, a_s).$$

Now, forming the next covariant by means of the equation

$$U_{123} = U_1 U_{23} - U_2 U_{13} + U_3 U_{12},$$

we readily obtain

$$U_{123} = \Sigma \{ -l_p d_{123}(l_q, a_p, a_q) + l_q d_{123}(l_p, a_p, a_q) \},$$

where q is greater than p , and where p and q may have all possible values from 1 to s .

For the next covariant we have

$$2U_{1234} = d_1 U_{334} - d_2 U_{184} + d_3 U_{124} - d_4 U_{123}.$$

Thus, from the above expression for U_{123} , it easily follows that

$$2U_{1234} = -2\Sigma d_{1234}(l_p, l_q, a_p, a_q),$$

it being readily verified that all terms containing differentials of the second order disappear. Therefore

$$U_{1234} = -\Sigma d_{1234}(l_p, l_q, a_p, a_q),$$

p and q being supposed in their natural order.

For the next covariant, we have

$$U_{12345} = \Sigma \{ -l_p d_{12345}(l_q, l_r, a_p, a_q, a_r) + l_q d_{12345}(l_p, l_r, a_p, a_q, a_r) \\ - l_r d_{12345}(l_p, l_q, a_p, a_q, a_r) \},$$

p, q, r being supposed to be in their natural order.

In like manner, for the fifth, sixth, and seventh covariants, we have

$$\begin{aligned}
 U_{123450} &= -\Sigma d_{123450} (l_p, l_q, l_r, a_p, a_q, a_r), \\
 U_{1234507} &= \Sigma \{ l_p d_{1234507} (l_q, l_r, l_t, a_p, a_q, a_r, a_t) \\
 &\quad - l_q d_{1234507} (l_p, l_r, l_t, a_p, a_q, a_r, a_t) \\
 &\quad + l_r d_{1234507} (l_p, l_q, l_t, a_p, a_q, a_r, a_t) \\
 &\quad - l_t d_{1234507} (l_p, l_q, l_r, a_p, a_q, a_r, a_t) \}, \\
 U_{12345078} &= \Sigma d_{12345078} (l_p, l_q, l_r, l_t, a_p, a_q, a_r, a_t).
 \end{aligned}$$

These examples are sufficient to reveal the general types of the successive covariants, and it is easily seen that the generality of those types may be established by induction. In the form in which we have written them, however, there arises a distinct peculiarity with regard to sign. Take first the case of those covariants which contain Pfaffians when expressed by the method of my former paper. The first of these is affected with a positive sign, the next two with negative signs, the next two with positive signs, and so on. Next consider the case of those covariants which, when expressed as in my former paper, contain allied functions. The first two of these will have their first terms negative; the next two will have those terms positive; the next two will have them negative; and so on. The question of the actual sign, however, will not be of much consequence for the purposes for which the results are required.

Comparing the above expressions for the covariants with those given in my former paper, we may readily deduce forms for the expression of the various orders of derived functions in terms of the quantities contained in a normal form of the given Pfaffian expression. As examples, we have

$$\begin{aligned}
 [12] &= \Sigma \frac{\partial (l_p, a_p)}{\partial (x_1, x_2)}, \\
 [0123] &= \Sigma \left\{ -l_p \frac{\partial (l_q, a_p, a_q)}{\partial (x_1, x_2, x_3)} + l_q \frac{\partial (l_p, a_p, a_q)}{\partial (x_1, x_2, x_3)} \right\}, \\
 [1234] &= -\Sigma \frac{\partial (l_p, l_q, a_p, a_q)}{\partial (x_1, x_2, x_3, x_4)}, \\
 &\quad \&c.
 \end{aligned}$$

One important point to be noticed, in connexion with the type of normal form assumed above, is that, if s be less than $\frac{1}{2}n$, n being even, or less than $\frac{1}{2}(n+1)$, n being odd, then after a certain point

the covariants will become evanescent. The first covariant that does so will be the $2s^{\text{th}}$. In this case we see that all the derived functions belonging to the $2s^{\text{th}}$ and all subsequent orders vanish. We know that this is the necessary and sufficient condition that the normal form may be of the assumed type.*

4. We have discussed only one of the two possible types of normal form, but the results that arise in the case of the other type are readily deduced from those we have obtained in the preceding article. Thus suppose that we have a normal form of the type

$$d\beta + l_1 da_1 + l_2 da_2 + \dots + l_s da_s,$$

where s is less than $\frac{1}{2}n$, n being even, or not greater than $\frac{1}{2}(n-1)$, n being odd.

We have $U_{12} = \Sigma d_{12} (l_p, a_p),$

$$U_{123} = \Sigma \{ d_{123} (\beta, l_p, a_p) \} \\ + \Sigma \{ -l_p d_{123} (l_q, a_p, a_q) + l_q d_{123} (l_p, a_p, a_q) \},$$

$$U_{1234} = -\Sigma d_{1234} (l_p, l_q, a_p, a_q),$$

$$U_{12345} = \Sigma \{ -d_{12345} (\beta, l_p, l_q, a_p, a_q) \} \\ + \Sigma \{ -l_p d_{12345} (l_q, l_r, a_p, a_q, a_r) \\ + l_q d_{12345} (l_p, l_r, a_p, a_q, a_r) \\ - l_r d_{12345} (l_p, l_q, a_p, a_q, a_r) \},$$

and so on.

From these we readily deduce the expressions for the derived functions. We obtain

$$[12] = \Sigma \frac{\partial (l_p, a_p)}{\partial (x_1, x_2)}, \\ [0123] = \Sigma \frac{\partial (\beta, l_p, a_p)}{\partial (x_1, x_2, x_3)} + \Sigma \left\{ -l_p \frac{\partial (l_q, a_p, a_q)}{\partial (x_1, x_2, x_3)} + l_q \frac{\partial (l_p, a_p, a_q)}{\partial (x_1, x_2, x_3)} \right\}, \\ [1234] = -\Sigma \frac{\partial (l_p, l_q, a_p, a_q)}{\partial (x_1, x_2, x_3, x_4)},$$

and so on. It is to be noted that the symbol β occurs in the allied

* See Forsyth, p. 255.

This equation will possess $2m-1$ independent solutions. For our present purpose, we shall require only m solutions, but these cannot be chosen at random. We have $\frac{1}{2}m(m-1)$ other equations connecting the α 's in pairs. These are to be deduced from the expressions for the set of Pfaffians immediately preceding the last set of allied functions. For these we have

$$[345 \dots (2m)] = \pm \Sigma \frac{\partial (l_1, \dots, l_{r-1}, l_{r+1}, \dots, l_m, \alpha_1, \dots, \alpha_{r-1}, \alpha_{r+1}, \dots, \alpha_m)}{\partial (x_3, x_4, x_5, \dots, x_{2m})},$$

$$[245 \dots (2m)] = \pm \Sigma \frac{\partial (l_1, \dots, l_{r-1}, l_{r+1}, \dots, l_m, \alpha_1, \dots, \alpha_{r-1}, \alpha_{r+1}, \dots, \alpha_m)}{\partial (x_2, x_4, x_5, \dots, x_{2m})},$$

... ..

$$[145 \dots (2m)] = \pm \Sigma \frac{\partial (l_1, \dots, l_{r-1}, l_{r+1}, \dots, l_m, \alpha_1, \dots, \alpha_{r-1}, \alpha_{r+1}, \dots, \alpha_m)}{\partial (x_1, x_4, x_5, \dots, x_{2m})},$$

and so on. Thus we obtain, as the general type of above-mentioned $\frac{1}{2}m(m-1)$ equations, the equation

$$\Sigma \pm [123 \dots (p-1)(p+1) \dots (q-1)(q+1) \dots (2m)] \frac{\partial (\alpha, \alpha')}{\partial (x_p, x_q)} = 0; \quad (5)$$

it being understood that any particular term under the Σ is affected with a positive or a negative sign according as the number of displacements required to reduce the series of numbers

$$p, q, 1, 2, 3, \dots, p-1, p+1, \dots, q-1, q+1, \dots, 2m$$

to their proper numerical order is even or odd.

The m equations of the type (4) combined with the $\frac{1}{2}m(m-1)$ equations of the type (5) make up a set of $\frac{1}{2}m(m+1)$ equations. These are the equations obtained by Clebsch.* They flow naturally from our results, which also suggest the natural method of generalization.

To utilize these results for the integration of the equation, we select, in the first place, any solution of (4). We then substitute this for α' in equation (5). We have then to obtain a common integral of (4) and the equation so formed. Having determined this

* See Forsyth, p. 214.

tions immediately preceding the last set of Pfaffians. We have

$$[0345 \dots (2m+1)]$$

$$= \pm \Sigma \left\{ (-1)^{r-1} l_r \frac{\partial (l_1, \dots, l_{r-1}, l_{r+1}, \dots, l_m, a_1, a_2, \dots, a_m)}{\partial (x_3, x_4, x_5, \dots, x_{2m+1})} \right. \\ \left. + (-1)^{m+r+1} \frac{\partial (\beta, l_1, \dots, l_m, a_1, \dots, a_{r-1}, a_{r+1}, \dots, a_m)}{\partial (x_3, x_4, x_5, \dots, x_{2m+1})} \right\},$$

$$[0245 \dots (2m+1)]$$

$$= \pm \Sigma \left\{ (-1)^{r-1} l_r \frac{\partial (l_1, \dots, l_{r-1}, l_{r+1}, \dots, l_m, a_1, a_2, \dots, a_m)}{\partial (x_2, x_4, x_5, \dots, x_{2m+1})} \right. \\ \left. + (-1)^{m+r+1} \frac{\partial (\beta, l_1, \dots, l_m, a_1, \dots, a_{r-1}, a_{r+1}, \dots, a_m)}{\partial (x_2, x_4, x_5, \dots, x_{2m+1})} \right\},$$

... ..

$$[0145 \dots (2m+1)]$$

$$= \pm \Sigma \left\{ (-1)^{r-1} l_r \frac{\partial (l_1, \dots, l_{r-1}, l_{r+1}, \dots, l_m, a_1, a_2, \dots, a_m)}{\partial (x_1, x_4, x_5, \dots, x_{2m+1})} \right. \\ \left. + (-1)^{m+r+1} \frac{\partial (\beta, l_1, \dots, l_m, a_1, \dots, a_{r-1}, a_{r+1}, \dots, a_m)}{\partial (x_1, x_4, x_5, \dots, x_{2m+1})} \right\},$$

and so on. Thus we obtain

$$\Sigma \pm [0123 \dots (p-1)(p+1) \dots (q-1)(q+1) \dots (2m+1)] \frac{\partial (a, a')}{\partial (x_p, x_q)} = 0, \quad (7)$$

where it is understood that the sign of any particular term is determined by the same rule as that laid down in the case of equation (5).

The $\frac{1}{2}m(m+1)$ equations formed by writing down all equations of the types (6) and (7) enable us to determine the a 's by a method exactly similar to that adopted in the preceding case. It only remains to determine β .

The m equations connecting β with each of the a 's are of the type (7). In fact, we have

$$\Sigma \pm [0123 \dots (p-1)(p+1) \dots (q-1)(q+1) \dots (2m+1)] \frac{\partial (\beta, a)}{\partial (x_p, x_q)} = 0. \quad (8)$$

The equation, containing β only, differs slightly from the type (6). We have

$$\begin{aligned} [234 \dots (2m+1)] \frac{\partial \beta}{\partial x_1} - [134 \dots (2m+1)] \frac{\partial \beta}{\partial x_2} + \dots \\ \dots + (-1)^{2m} [123 \dots (2m)] \frac{\partial \beta}{\partial x_{2m+1}} = \pm \frac{\partial (\beta, l_1, \dots, l_m, a_1, \dots, a_m)}{\partial (x_1, x_2, x_3, \dots, x_{2m+1})} \\ = [0123 \dots (2m+1)]. \quad (9) \end{aligned}$$

7. We now proceed to discuss the case of a conditioned system in any number of variables. This naturally divides itself into two cases, according to the type of normal form suitable. We shall take first the case in which we have

$$l_1 da_1 + l_2 da_2 + \dots + l_s da_s$$

for a normal form, s being less than $\frac{1}{2}n$ when n is even, and less than $\frac{1}{2}(n+1)$ when n is odd. Then the derived functions of the $2s^{\text{th}}$ and all following orders vanish. Thus the last set of derived functions that do not vanish are the s^{th} set of Pfaffians. Expressing these in terms of the quantities involved in the normal form, we have

$$\begin{aligned} [123 \dots (2s)] &= \pm \frac{\partial (l_1, l_2, \dots, l_s, a_1, a_2, \dots, a_s)}{\partial (x_1, x_2, x_3, \dots, x_{2s})}, \\ [234 \dots (2s+1)] &= \pm \frac{\partial (l_1, l_2, \dots, l_s, a_1, a_2, \dots, a_s)}{\partial (x_2, x_3, x_4, \dots, x_{2s+1})}, \\ &\quad \&c., \quad \&c., \end{aligned}$$

there being in all
$$2s! \frac{n!}{(n-2s)!}$$

equations of this type. From these we might deduce a set of equations on the model of (4) and (6). As, however, it is necessary to fall back upon the $(s-1)^{\text{th}}$ set of Pfaffians to obtain the equations connecting the a 's in pairs, we will follow strictly the analogy of Article 5, and make use of the $(s-1)^{\text{th}}$ set of allied functions to obtain the required equations. We have

$$\begin{aligned} [0123 \dots (2s-1)] &= \pm \Sigma (-1)^{r-1} l_r \frac{\partial (l_1, \dots, l_{r-1}, l_{r+1}, \dots, l_s, a_1, a_2, \dots, a_s)}{\partial (x_1, x_2, x_3, \dots, x_{2s-1})}, \\ [0234 \dots (2s)] &= \pm \Sigma (-1)^{r-1} l_r \frac{\partial (l_1, \dots, l_{r-1}, l_{r+1}, \dots, l_s, a_1, a_2, \dots, a_s)}{\partial (x_2, x_3, x_4, \dots, x_{2s})}, \\ &\quad \&c., \quad \&c., \end{aligned}$$

there being in all $\frac{n!}{(2s-1)!(n-2s+1)!}$

equations of this type.

Proceeding to form equations from these on the model of (4) and (6), we obtain a set of the type

$$\begin{aligned} [0234 \dots (2s)] \frac{\partial a}{\partial x_1} - [0134 \dots (2s)] \frac{\partial a}{\partial x_2} + \dots \\ \dots + (-1)^{2s-1} [0123 \dots (2s-1)] \frac{\partial a}{\partial x_{2s}} = 0. \end{aligned}$$

The total number of equations of this type that can be written down will be the same as the number of sets, each containing $2s$ numbers, that can be selected from the series

$$1, 2, 3, \dots, n,$$

viz., $\frac{n!}{2s!(n-2s)!}.$

These equations will not, however, be all independent. We can express our set of allied functions in the form

$$\begin{aligned} [0123 \dots (2s-1)] &= \pm l_1 \frac{\partial (l_2/l_1, l_3/l_1, \dots, l_s/l_1, a_1, a_2, \dots, a_s)}{\partial (x_1, x_2, x_3, \dots, x_{2s-1})}, \\ [0234 \dots (2s)] &= \pm l_1^s \frac{\partial (l_2/l_1, l_3/l_1, \dots, l_s/l_1, a_1, a_2, \dots, a_s)}{\partial (x_2, x_3, x_4, \dots, x_{2s})}, \\ &\&c., \quad \&c., \end{aligned}$$

as can be readily proved in the following manner. We have

$$\begin{aligned} [0123 \dots (2s-1)] \\ = \pm l_1 l_2 \dots l_s (-1)^{r-1} \frac{\partial (\log l_1, \dots, \log l_{r-1}, \log l_{r+1}, \dots, \log l_s, a_1, \dots, a_s)}{\partial (x_1, x_2, x_3, \dots, x_{2s-1})}. \end{aligned}$$

Now

$$\begin{aligned} \frac{\partial (\log l_2, \log l_3, \dots, \log l_s, a_1, \dots, a_s)}{\partial (x_1, x_2, x_3, \dots, x_{2s-1})} - \frac{\partial (\log l_1, \log l_3, \dots, \log l_s, a_1, \dots, a_s)}{\partial (x_1, x_2, x_3, \dots, x_{2s-1})} \\ = \frac{\partial (\log l_2/l_1, \log l_3, \dots, \log l_s, a_1, \dots, a_s)}{\partial (x_1, x_2, x_3, \dots, x_{2s-1})}. \end{aligned}$$

Further,

$$\begin{aligned}
 & \frac{\partial (\log l_2/l_1, \log l_3, \dots, \log l_s, a_1, \dots, a_s)}{\partial (x_1, x_2, x_3, \dots, x_{2s-1})} \dots \\
 & \quad + \frac{\partial (\log l_1, \log l_2, \log l_3, \dots, \log l_s, a_1, \dots, a_s)}{\partial (x_1, x_2, x_3, \dots, x_{2s-1})} \\
 & = \frac{\partial (\log l_2/l_1, \log l_3, \dots, \log l_s, a_1, \dots, a_s)}{\partial (x_1, x_2, x_3, \dots, x_{2s-1})} \\
 & \quad - \frac{\partial (\log l_2/l_1, \log l_1, \log l_3, \dots, \log l_s, a_1, \dots, a_s)}{\partial (x_1, x_2, x_3, \dots, x_{2s-1})} \\
 & = \frac{\partial (\log l_2/l_1, \log l_3/l_1, \log l_4, \dots, \log l_s, a_1, \dots, a_s)}{\partial (x_1, x_2, x_3, \dots, x_{2s-1})}.
 \end{aligned}$$

Proceeding in this manner, we eventually obtain

$$\begin{aligned}
 & [0123 \dots (2s-1)] \\
 & = \pm l_1 l_2 \dots l_s \Sigma \frac{\partial (\log l_2/l_1, \log l_3/l_1, \dots, \log l_s/l_1, a_1, \dots, a_s)}{\partial (x_1, x_2, x_3, \dots, x_{2s-1})} \\
 & = \pm l_1^s \Sigma \frac{\partial (l_2/l_1, l_3/l_1, \dots, l_s/l_1, a_1, a_2, \dots, a_s)}{\partial (x_1, x_2, x_3, \dots, x_{2s-1})}.
 \end{aligned}$$

Considering that our set of allied functions can be expressed in this form, and taking account of known properties of determinants, we see that the said functions are connected by relations of the types

$$\begin{aligned}
 & [0234 \dots (2s)][0145 \dots (2s+1)] - [0134 \dots (2s)][0245 \dots (2s+1)] \\
 & \quad + [01245 \dots (2s)][0345 \dots (2s+1)] \\
 & = 0, \\
 & [0234 \dots (2s)][0156 \dots (2s+2)] - [0134 \dots (2s)][0256 \dots (2s+2)] \\
 & + [01245 \dots (2s)][0356 \dots (2s+2)] - [012356 \dots (2s)][0456 \dots (2s+2)] \\
 & = 0, \\
 & \quad \&c., \quad \&c.
 \end{aligned}$$

Thus, the number of independent equations will be $n-2s+1$, and the

With the aid of the above expressions we can form a set of equations, similar in form to (5), of which the following may be taken as a type:—

$$\Sigma \pm [123 \dots (p-1)(p+1) \dots (q-1)(q+1) \dots (2s)] \frac{\partial (a, a')}{\partial (x_p, x_q)} = 0. \quad (11)$$

In this equation the combination (p, q) is to be replaced by every possible selection of two numbers out of the series

$$1, 2, 3, \dots, 2s;$$

and the determination of the sign of any particular term under the Σ is to be made according to the rule laid down in connexion with equation (5). The total number of equations of this type that can be formed will be the same as the number of sets of $2s$ numbers that can be selected from the series

$$1, 2, 3, \dots, n,$$

viz.,
$$\frac{n!}{2s! (n-2s)!}.$$

The number of quantities of the form

$$\frac{\partial (a, a')}{\partial (x_p, x_q)}$$

that occur in the whole set of equations is $\frac{1}{2}n(n-1)$. Now

$$\frac{n!}{2s! (n-2s)!} > \frac{1}{2}n(n-1),$$

except for the single case $s = 1$. It is in general much greater. Thus it is evident that the members of our set of equations are not all algebraically independent. The number of independent equations cannot be greater than

$$\frac{1}{2}n(n-1) - 1,$$

and is probably less.

This last set of equations will be satisfied by the l 's, and consequently by the quantities

$$\frac{l_2}{l_1}, \frac{l_3}{l_1}, \dots, \frac{l_s}{l_1}.$$

Thus our complete set of equations, including equations (10) and

the equations of type (11), will be satisfied by these quantities in addition to the α 's. Consequently their general solutions will be of the type

$$\varphi \left(\frac{l_2}{l_1}, \frac{l_3}{l_1}, \dots, \frac{l_s}{l_1}, \alpha_1, \alpha_2, \dots, \alpha_s \right),$$

and therefore the argument used by Forsyth to justify the sufficiency of the equations in the case discussed in Article 5 may also be used here.

8. It now only remains to obtain the necessary equations for the case in which the normal form is

$$d\beta + l_1 da_1 + l_2 da_2 + \dots + l_s da_s,$$

s being less than $\frac{1}{2}n$ if n be even, and less than $\frac{1}{2}(n-1)$ if n be odd. The last set of derived functions that does not become evanescent will be the $2s^{\text{th}}$, *i.e.*, the s^{th} set of allied functions. These may be expressed as follows:—

$$\begin{aligned} [0123 \dots (2s+1)] &= \pm \frac{\partial (\beta, l_1, \dots, l_s, \alpha_1, \dots, \alpha_s)}{\partial (x_1, x_2, x_3, \dots, x_{2s+1})}, \\ [0234 \dots (2s+2)] &= \pm \frac{\partial (\beta, l_1, \dots, l_s, \alpha_1, \dots, \alpha_s)}{\partial (x_2, x_3, x_4, \dots, x_{2s+2})}, \\ &\quad \&c., \quad \&c., \end{aligned}$$

there being in all

$$\frac{n!}{(2s+1)!(n-2s-1)!}$$

equations of this form. We could deduce from these a set of equations that are satisfied by the α 's, but we shall find it most convenient to follow the analogy of our former work. We must accordingly write down the expressions for the s^{th} set of Pfaffians. We have

$$\begin{aligned} [123 \dots (2s)] &= \pm \frac{\partial (l_1, \dots, l_s, \alpha_1, \dots, \alpha_s)}{\partial (x_1, x_2, x_3, \dots, x_{2s})}, \\ [234 \dots (2s+1)] &= \pm \frac{\partial (l_1, \dots, l_s, \alpha_1, \dots, \alpha_s)}{\partial (x_2, x_3, x_4, \dots, x_{2s+1})}, \\ &\quad \&c., \quad \&c., \end{aligned}$$

there being in all

$$\frac{n!}{2s!(n-2s)!}$$

equations of this type. Thus we see that the equations satisfied by

the α 's will be of the type

$$\begin{aligned} [234 \dots (2s+1)] \frac{\partial \alpha}{\partial x_1} - [134 \dots (2s+1)] \frac{\partial \alpha}{\partial x_2} + \dots \\ \dots + (-1)^{2s} [123 \dots (2s)] \frac{\partial \alpha}{\partial x_{2s+1}} = 0. \quad (12) \end{aligned}$$

The number of algebraically independent equations of this type will be $n-2s$, and we may choose a set similar to equations (10) of the preceding article.

The equations to be satisfied by β will be of the type

$$\begin{aligned} [234 \dots (2s+1)] \frac{\partial \beta}{\partial x_1} - [134 \dots (2s+1)] \frac{\partial \beta}{\partial x_2} + \dots \\ \dots + (-1)^{2s} [123 \dots (2s)] \frac{\partial \beta}{\partial x_{2s+1}} = [0123 \dots (2s+1)]. \quad (13) \end{aligned}$$

The number of equations of this type that are algebraically independent will be $n-2s$.

In order to form the remaining equations, we have to write down the expressions for the $(s-1)^{\text{th}}$ set of allied functions. We have

$$\begin{aligned} [0123 \dots (2s-1)] \\ = \pm \sum \left\{ (-1)^{r-1} l_r \frac{\partial (l_1, \dots, l_{r-1}, l_{r+1}, \dots, l_s, a_1, a_2, \dots, a_s)}{\partial (x_1, x_2, x_3, \dots, x_{2s-1})} \right. \\ \left. + (-1)^{s+r+1} \frac{\partial (\beta, l_1, \dots, l_s, a_1, \dots, a_{r-1}, a_{r+1}, \dots, a_s)}{\partial (x_1, x_2, x_3, \dots, x_{2s-1})} \right\}, \\ [023 \dots (2s)]. \\ = \pm \sum \left\{ (-1)^{r-1} l_r \frac{\partial (l_1, \dots, l_{r-1}, l_{r+1}, \dots, l_s, a_1, a_2, \dots, a_s)}{\partial (x_2, x_3, x_4, \dots, x_{2s})} \right. \\ \left. + (-1)^{s+r+1} \frac{\partial (\beta, l_1, \dots, l_s, a_1, \dots, a_{r-1}, a_{r+1}, \dots, a_s)}{\partial (x_2, x_3, x_4, \dots, x_{2s})} \right\}, \\ \&c., \quad \&c., \end{aligned}$$

there being in all

$$\frac{n!}{(2s-1)! (n-2s+1)!}$$

equations of this type. From them we readily deduce the remaining

equations. We have a set of the type

$$\Sigma \pm [0123 \dots (p-1)(p+1) \dots (q-1)(q+1) \dots (2s+1)] \frac{\partial (a, a')}{\partial (x_p, x_q)} = 0, \quad (14)$$

and a set of the type

$$\Sigma \pm [0123 \dots (p-1)(p+1) \dots (q-1)(q+1) \dots (2s+1)] \frac{\partial (\beta, \alpha)}{\partial (x_p, x_q)} = 0. \quad (15)$$

This accomplishes our present purpose, which is the determination of the forms of certain equations known to exist. The discussion opens up some interesting questions with regard to the general properties of the derived functions of a Pfaffian expression, but these must be left for future investigation.

On Ampère's Equation $Rr + 2Ss + Tt + U(rt - s^2) = V$. By A. C.

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1. The methods given hitherto for solving the above equation do not apply in general, but only if certain conditions are satisfied by the coefficients R, S, T, U, V . I propose in this article to investigate a modification of the ordinary Monge and Ampère method which applies to all cases.

It must be remembered that no method can be given which will in all cases give an integral in finite terms. It seems almost certain further that no method will ever be found for reducing the solution of Ampère's equation in general to that of equations of the first order with one dependent variable. Even if such a method were found, it would not enable us to solve the equation in all cases, for there is no general method of solving such equations of the first order, except by continued approximation. The ordinary methods, those of Lagrange, Charpit, and Jacobi, depend on inspection. The