3. On Compensation Pendulums of two Pieces. By Edward Sang, Esq.

Three or four years ago, wishing to make some experiments on time-keepers, and desiring for that purpose to have a compensation pendulum fitted to my clock, I turned my attention to pendulums composed of two pieces, having different rates of expansion.

Such compensation pendulums are now pretty common; the idea which has led to their construction being this,—that if the pendulum rod be made of material having a very small rate of expansion, and if the bob or weight be of a substance much more expansive, then the upward expansion of the one may be made to compensate for the downward expansion of the other, and so the going of the clock may remain unaffected by changes of temperature.

On designing the arrangement, and on proceeding to make the calculation, according to the received expansions of deal and lead, I was much surprised to find the solution of the resulting equation to be impossible; and the surprise was the greater, because I had understood that of clocks constructed with such pendulums, the compensation had been verified by actual observations. There is not sufficient disparity between the expansibilities of deal and lead.

A very slight glance at the subject is sufficient to show, that if it be impossible to construct such a pendulum with a deal rod of great tenuity, and with an enclosing cylinder of lead also of inappreciable diameter, it must be impossible to make one with the parts having appreciable diameters; since the lateral expansion of the parts tends to augment the length of the corresponding simple pendulum.

Let us then suppose an exceedingly thin rod AB, of some slightly expansive substance, suspended by the end A, and having on the lower end B a small projection or seat, on which the end of a cylinder BC of heavy and expansive metal rests. Then neglecting the weight of the inner rod, and the radius

Fig. 1.

of the cylinder, and putting $AB = b$, $AC = c$, and l for the length of the corresponding simple pendulum, we have

$$\frac{3}{2}l = \frac{b^2 + bc + c^2}{b + c} \quad . \quad . \quad . \quad . \quad . \quad . \quad (1.)$$

Let now β be the rate of expansion of the supporting rod AB , γ that of the cylinder BC , we have for a change of temperature t .

$$b' = b(1 + \beta t), \quad c' = c(1 + \gamma t).$$

and therefore, in order that there be no change in the value of b , we must have

$$\frac{b^2 + bc + c^2}{b + c} = \frac{3}{2}l = \frac{b'^2 + b'c' + c'^2}{b' + c'} = \frac{(3b^2 + 3bc)\beta + (2c^2 - bc - b^2)\gamma}{2b\beta - b\gamma + c\gamma}$$

whence the equation

$$\beta \cdot b^3 + (4\beta - 2\gamma)b^2c + (\beta + \gamma)bc^2 + \gamma c^3 = 0, \quad . \quad . \quad . \quad (2.)$$

by help of which the ratio of b to c may be found. Now, unless γ exceed the double of β , all the terms of this equation have the same sign, and therefore there can be no positive root; so that, on this account, there can be no such pendulum as we are thinking of, unless the one rate of expansion be more than double of the other. In order to discover the least possible disparity of expansion, let us put $b = cx$, $\gamma = n\beta$, and our equation becomes

$$x^3 + (4 - 2n)x^2 + (1 + n)x + n = 0, \quad . \quad . \quad . \quad (3.)$$

When our solution just begins to be possible, this equation must have two equal roots, and therefore must have a divisor common to it and its derivative

$$3x^2 + (8 - 4n)x + (1 + n) = 0, \quad . \quad . \quad . \quad (4.)$$

On eliminating x from the equations (3) and (4), we obtain an equation of the fifth degree; of which $n = 2$ is one solution, and on dividing there results the biquadratic

$$12n^4 - 80n^3 + 123n^2 - 60n + 4 = 0, \quad . \quad . \quad . \quad (5.)$$

which has two possible roots, viz.,

$$n = 0.07873 \quad 06113 \quad 75,$$

and

$$n = 4.71418 \quad 14416 \quad 13.$$

The first of these can have no application to our present problem

and the second, almost exactly $4\frac{1}{2}$, exceeds the ratio of the expansions of any two available solid substances. Thus we conclude that pendulums constructed of two pieces after the manner shown in figure 1, cannot be compensated.

The equation of condition (3) being of an odd degree, must have always one root possible. It is true that this is merely a numerical possibility, which may or may not be represented by a mechanical arrangement; yet it may be worth while to examine into it. Now, when n is positive, the equation (3), must have one of its roots negative, and thus the point B, corresponding to that root, must be the above point of suspension.

Let us suppose, then, a tube $abba$ of glass or some substance having a very slight expansion, to proceed upwards from the axis of motion SS , and from the upper end of the tube let a heavy rod, BC , of more expansive material depend, and the arrangement will represent, mechanically, the negative value of x obtained from the solution of the equation (3).

On assuming for n a number of successive values, and thence computing the corresponding roots of the equation (3), as also the values of a , b , and c , necessary to give a pendulum vibrating isochronously with a simple pendulum whose length is unit, and on representing the results geometrically, we obtain the diagram given in figure 3.

In this figure, the distances measured along the horizontal line OS indicate the ratio of the expansibilities of the two substances, OI standing for that of the suspender; Ol or SL is the length of the corresponding simple pendulum; and the curved lines marked $B_1 B_2 B_3$; $C_1 C_2 C_3$ define the values of b and c , corresponding to the three roots of the equation.

For example, if the ratios of the rates of expansion were 9.2 , let us take the point s at that distance along OS , and draw through it a vertical line, crossing the six curves at the points $b_1 c_3 c_2, c_1 b_2 b_3$; then $sb_1 c_1$ shows the proportions for a pendulum constructed in the manner shown in figure 2. $sb_2 c_2$ shows that which is intended in the actual execution of clocks, while $sb_3 c_3$ is another possible

Fig. 2.