III.—*On the Extension of Brouncker's Method to the Comparison of Several Magnitudes.* By EDWARD SANG, Esq.

(Read 7th February 1870.)

The discovery of those numbers which shall, either truly or approximately, represent the ratio of two magnitudes, necessarily attracted the attention of the earliest cultivators of exact science. The definition of the equality of ratios given in EUCLID'S compilation clearly exposes the nature of the process used in his time. This process consisted in repeating each of the two magnitudes until some multiple of the one agreed perfectly or nearly with a multiple of the other; the numbers of the repetitions, taken in inverse order, represented the ratio. Thus, if the proposed magnitudes were two straight lines, EUCLID would have opened two pairs of compasses, one to each distance, and, beginning at some point in an indefinite straight line, he would step the two distances along, bringing up that which lagged behind, until he obtained an exact or a close coincidence.

He seems to have assumed that, in the case of incommensurable magnitudes, the further continuation of the process must give still closer approximations; but we do not find any indication of a knowledge of the fact that, in the course of that continuation, we shall certainly come upon coincidences still more close than any which we have already obtained.

This process for finding the numerical expression for a ratio is inconvenient from its bulkiness; it is also unnatural, for the mind, in comparing two unequal magnitudes, is rather inclined to regard them as made up each of so many measures, than to consider how many times the one must be augmented in order that the result may be a multiple of the other; it prefers the direct to the inverse comparison.

LORD BROUNCKER'S method of continued fractions enables us with great rapidity and within the compass of the magnitudes themselves, to determine directly their ratio. It is one of the great landmarks in the progress of the science of numbers.

By one or two slight improvements in the mode of calculation, the *chain* or *continued fraction* became a ready tool in the hands of arithmeticians. It placed in a clear light the whole doctrine of indeterminate equations of the first degree, leaving scarcely anything further to be desired in this branch of the Diophantine analysis.

On applying BROUNCKER's method to two incommensurable quantities of the second degree, it was found that the denominators eventually came to be repeated or circulated indefinitely; and LAGRANGE showed that while every circulating chain-fraction was known to represent the root of a quadratic equation, the roots of all such equations were developable in such a fraction. Hence the conclusion was drawn that the root of no equation of a higher order can possibly be represented by a circulating chain-fraction.

Although, to the mind of BROUNCKER, the continued fraction presented the readiest way of expounding his idea, it is not essential thereto; a much clearer view of the true nature of the process may be obtained without it. The operation consists, essentially, in deducting, as often as possible, the less from the greater; the remainder again from the preceding subtrahend, and so on; in keeping note of the numbers of the subtractions; and in computing from these numbers the value of the magnitudes in terms of the ultimate subtrahend. The chain-fraction is merely one way of representing the final computation. By stopping at the first denominator, then at the second, afterwards at the third, and so on, we obtain a series of fractions alternately too great and too small, but approaching rapidly to each other and to the true expression for the ratio. Now this series may be deduced directly from the equations representing the various subtractions; wherefore, in our subsequent investigations, we may put the idea of the chain-fraction entirely aside, without thereby changing the intrinsic character of the Brounckerian process.

On examining the two series converging to the two roots of a quadratic equation, I observed that the circulating quotients are the same for both, but that their order is inverted. This observation led me to a singular law, which some years ago I submitted to the Society. It is this, that if we continue the formation of the series for one root beyond the non-circulating quotients, obliterate these and the fractions adjoining them, and then, using only the circulators, compute the series backwards, we shall obtain the other root of the equation; so that both roots are given by a, so to speak, two-headed progression.

The periodical recurrence of the quotients enables us to approximate as closely as may be desired to the roots of equations of the second order, with very little labour; and a kind of regret accompanied the conviction that the same facilities cannot be obtained for equations of higher degrees. On considering the arguments on which this conviction rested, it appeared to me that the whole circumstances of the case had not been taken into account; one, and a most influential one, had been concealed under the notation employed, that is, under the scheme of continued fractions. If we assume any two fractions to take the place of two contiguous terms in a Brounckerian progression, and operate upon these in the usual way, that is, by adding to a multiple of each member of the second fraction the corresponding member of the preceding; and

if we continue this operation, using always the same multiplier, or a circulating set of multipliers, the fractions so resulting converge to the root of a quadratic equation. If we should assume *three* fractions, and combine fixed multiples of their members, so as to form a progression of the third order, as we may call it, to what value do the terms of this progression converge?

In a paper read by me some time ago to the Society, it was shown that the convergence in this case is toward the root of a cubic equation; and that the same arrangement may be extended to the still higher orders; as examples of the application of this method, two cases may be cited.

If we begin with the two fractions $\frac{1}{0}, \frac{1}{1}$ and form a progression by adding to the double of each member of the last, the corresponding member of the preceding, we form the well-known progression

$$\frac{1}{0}, \frac{1}{1}, \frac{3}{2}, \frac{7}{5}, \frac{17}{12}, \frac{41}{29}, \frac{99}{70}, \&c.,$$

which converges toward the ratio of the diagonal to the side of a square.

If, beginning with the same pair, we form a progression by taking the sums of the members of the last and of the penult, we obtain

$$\frac{1}{0}, \frac{1}{1}, \frac{2}{1}, \frac{3}{2}, \frac{5}{3}, \frac{8}{5}, \frac{13}{8}, \frac{21}{13}, \frac{34}{21}, \&c.,$$

which converges toward the ratio of the diagonal of a regular pentagon to its side. In this case, the numerator of one fraction becomes the denominator of the succeeding, so that it is unnecessary to write both progressions. These are familiar examples of quadratic roots.

Let us now assume *three* terms, 0, 0, 1, and continue a progression by adding to the double of the last term the difference between the two previous ones, thus—

$$0, 0, 1, 2, 5, 11, 25, 56, 126, 283, 636, 1429, 3211, 7215, 16212, 36428, \\ 81853, 183922, \&c.,$$

and we obtain an approximation to the ratio of the long diagonal to the side of a regular heptagon. Thus, if the side of the heptagon be 283, its longest diagonal is almost exactly 636.

Or again if, assuming the same three terms 0, 0, 1, we form a series by deducting the antepenult from the triple of the last term, thus—

$$0, 0, 1, 3, 9, 26, 75, 216, 622, 1791, 5157, 14849, 42756, 123111, 354484, \&c.,$$

we obtain an approximation to the ratio of the long diagonal of an enneagon to its side.

I have shown that, in progressions of this kind, that is, where the numerator

of the one fraction becomes the denominator of the other, the approximation is toward that root of the equation which is farthest from zero ; and that if the progression be carried backwards, the approximation is then toward the root nearest to zero.

These remarks may suffice to show that this branch of the theory of numbers promises to yield important results. Now, the whole doctrine of quadratic recurrence sprung from the comparison of two magnitudes ; and so the comparison of three magnitudes must be the true foundation on which to build the doctrine of cubic recurrence. I propose, therefore, in the present paper, to discuss the elementary operation by which the ratios of three incommensurable magnitudes may be approximately ascertained.

Let there be three homogeneous quantities, A, B, C, arranged in the order of their magnitudes, and let it be proposed, if possible, to find their common measure.

By repeatedly subtracting the second B from the greatest A, we obtain a remainder less than B; this remainder may or may not be greater than C; if it be greater, we may take C from it until we obtain a remainder D less than the least of the three proposed quantities. In this way we have an equation of the form

$$A = p_1 B + q_1 C + D ,$$

in which p cannot be zero, while q may.

Treating now the three quantities B, C, D, in the same way we have a new equation

$$B = p_2 C + q_2 D + E ,$$

and we may proceed in this way until there be no remainder, or until the remainder be so small as to pass the limits of exactitude demanded by the nature of the case in hand.

We are now able, by means of successive substitutions, to obtain values of A, B, C, in terms of the ultimate remainders ; and our first business is to devise some convenient arrangement for this purpose.

For the sake of giving greater generality to our investigations, let us put the successive equations in the form

$$\begin{aligned} A &= p_1 B + q_1 C + r_1 D , \\ B &= p_2 C + q_2 D + r_2 E , \\ C &= p_3 D + q_3 E + r_3 F , \\ &\dots \dots \dots \\ Q &= p_n R + q_n S + r_n T , \end{aligned}$$

in which $r_1, r_2, \dots$ have been written for the unit of the usual process.

By successive substitutions we arrive at a value of A , in terms of R, S, T , which may be conveniently represented by the equation,

$$A = \phi n \cdot R + \theta n \cdot S + \psi n \cdot T ,$$

and our business becomes to discover the law of formation of the functions ϕn , θn , ψn .

Continuing the operation one step further, we have

$$R = P_{n+1} \cdot S + q_{n+1} \cdot T + r_{n+1} \cdot U ,$$

and substituting this value for R in the preceding equation,

$$A = \{p_{n+1} \cdot \phi n + \theta n\} S + \{q_{n+1} \cdot \phi n + \psi n\} T + r_{n+1} \cdot \phi n U ,$$

wherefore the law of successive formation is contained in the three equations—

$$\phi(n+1) = p_{n+1} \cdot \phi n + \theta n , \quad . \quad . \quad . \quad (1),$$

$$\theta(n+1) = q_{n+1} \cdot \phi n + \psi n , \quad . \quad . \quad . \quad (2),$$

$$\psi(n+1) = r_{n+1} \cdot \phi n , \quad . \quad . \quad . \quad (3).$$

Now these forms hold good for every value of n , wherefore

$$\psi n = r_n \cdot \phi(n-1) ,$$

and consequently

$$\theta(n+1) = \phi_{n+1} \cdot \phi n + r_n \cdot \phi(n-1) ,$$

whence

$$\theta n = q_n \cdot \phi(n-1) + r_{n-1} \cdot \phi(n-2) ,$$

so that the equations (1), (2), (3) become

$$\phi(n+1) = p_{n+1} \cdot \phi n + q_n \cdot \phi(n-1) + r_{n-1} \cdot \phi(n-2) , \quad (1),$$

$$\theta(n+1) = q_{n+1} \cdot \phi n + r_n \cdot \phi(n-1) , \quad . \quad . \quad . \quad (2),$$

$$\psi(n+1) = r_{n+1} \cdot \phi n , \quad . \quad . \quad . \quad . \quad . \quad . \quad (3).$$

By means of these formulæ we can construct the series of functions ϕn independently of the others, and thence we can readily deduce the progression θn ; as for the third progression ψn it is, in the usual case of $r = 1$, a mere transcript of the progression ϕn moved one step back. The arrangement of the work is very simple, and may be best studied from a numerical example.

The arrangement of the intervals in music has to follow the natural subdivision of a vibrating column, and so must be made, primarily, to suit the ratios $1 : 2$, $1 : 3$, $1 : 5$, and their compounds. If, then, it be proposed to tune a musical instrument so as to permit of transposition from one key to another, the ratio represented by the smallest interval on it must be contained exactly, or very nearly, in each of these three ratios. Therefore the arrangement of the gamut on an instrument of equable temperament must be obtained by a com-

parison of the logarithms of the three prime numbers 5, 3, and 2. These logarithms are incommensurable, and so it is impossible to tune a keyed instrument perfectly. The comparison of these three logarithms furnishes a convenient instance of the application of our method.

Putting, for shortness' sake, $A = \log 5$, $B = \log 3$, $C = \log 2$, we obtain the following equations :—

$$\begin{aligned}
 \cdot 69897 \quad 00043 &= A = 1.B + 0.C + D \\
 \cdot 47712 \quad 12547 &= B = 1.C + 0.D + E \\
 \cdot 30102 \quad 99956 &= C = 1.D + 0.E + F \\
 \cdot 22184 \quad 87496 &= D = 1.E + 0.F + G \\
 \cdot 17609 \quad 12590 &= E = 2.F + 0.G + H \\
 7918 \quad 12460 &= F = 1.G + 1.H + J \\
 4575 \quad 74905 &= G = 2.H + 0.J + K \\
 1772 \quad 87669 &= H = 1.J + 0.K + L \\
 1569 \quad 49885 &= J = 1.K + 2.L + M \\
 1029 \quad 99566 &= K = 5.L + 0.M + N \\
 203 \quad 37784 &= L = 1.M + 5.N + P \\
 132 \quad 74750 &= M = 10.N + 0.P + Q \\
 13 \quad 10644 &= N = 2.P + 1.Q + R \\
 5 \quad 09810 &= P \\
 1 \quad 68303 &= Q \\
 1 \quad 22720 &= R ;
 \end{aligned}$$

and, in order to compute from these the successive approximations, we may write the three sets of coefficients, p , q , r , in three horizontal lines, as in the subjoined scheme:—

r	1	1	1	1	1	1	1	1	1	1	1	1	1	
q	0	0	0	0	0	1	0	0	2	0	5	0	1	
p	1	1	1	1	2	1	2	1	1	5	1	10	2	
A	1	1	1	2	3	7	9	28	35	44	318	353	5164	10638
B		1	1	1	2	5	6	19	24	30	217	241	3525	7267
C			1	1	1	3	4	12	15	19	137	152	2224	4585

Having written *unit* beneath the first p , to serve as the beginning of the series A, we multiply that unit by p_1 to get 1, the second value of A_2 , which value we write beneath p_2 . We now take the products $A_2 p_2$, $A_1 q_1$, the sum of which gives us $A_3 = 1$. Thereafter we take the sum of $A_3 p_3$, $A_2 q_2$, $A_1 r_1$, to obtain $A_4 = 2$. In this example the first five q 's happen to be zeroes, and so the middle terms of the expressions are wanting; the middle term is first found in the expression for A_8 , which is $A_7 \cdot p_7 + A_6 \cdot q_6 + A_5 \cdot r_5 = 18 + 7 + 3$;

and again we have it in the value of A_{11} , which is $A_{10} \cdot p_{10} + A_9 \cdot q_9 + A_8 \cdot r_8 = 220 + 70 + 28$.

The series for B is formed exactly in the same way, only that its commencement is delayed one step ; in other words, B_1 is held as zero, and B_2 is made *unit*.

Similarly for C, C_1, C_2 are held as zeroes, and C_3 is made *unit*. We may in the same way find series for D, E, F, and so on.

The eighth set of values give 28, 19, 12 as nearly proportional to the logarithms of 5, 3, and 2. Assuming these as sufficiently near for the purposes of musicians, we must divide the interval corresponding to the ratio 1 : 2, called by them an *octave*, into 12 parts, to which the name semitone is given. In this way, counting in semitones, we have $\log 2 = 12$, $\log 3 = 19$, $\log 5 = 28$; whence $\log \frac{3}{2} = 7$, $\log \frac{5}{4} = 4$, and so on ; whence the arrangement of the gamut is at once obtained. According to these values, we should have $\log \frac{9}{8} = 2$, and $\log \frac{10}{9} = 2$, wherefore the degree of precision obtained by these numbers is not sufficient to discriminate between the tone major and the tone minor.

In order to obtain a closer approximation, we must proceed further along the series. Now it is important to keep to the nomenclature and arrangement of semitones ; wherefore we search among the series C for some member divisible by 12 ; no one of those above given is so divisible, and therefore we look for some compound of two of them which may be a multiple of 12. Thus $C_{10} + C_{11} = 156$, so that, still counting in semitones, we have $\log 5 = \frac{362}{13} = 27 \frac{11}{13}$, $\log 3 = \frac{247}{13} = 19$, $\log 2 = \frac{156}{13} = 12$. From these values the logarithm of the tone major represented by the ratio $\frac{9}{8}$ is still 2, but that of the tone minor represented by $\frac{10}{9}$ is $1 \frac{11}{13}$; in the same way the corrected values of the other musical intervals may be obtained.

By putting A, B, C to represent the periodic times of three astronomical phenomena, we may ascertain the intervals between their simultaneous recurrence. Thus, if we put A for the time of revolution of the moon's node, B and C for the earth's and moon's periodic times, we shall obtain directly the law of recurrence of eclipses.

If we take three contiguous sets of values, and thence compute the succeeding set, we obtain

$$\begin{array}{llll} A_n & A_{n+1} & A_{n+2} & A_{n+3} = p_{n+2} \cdot A_{n+2} + q_{n+1} \cdot A_{n+1} + r_n A_n \\ B_n & B_{n+1} & B_{n+2} & B_{n+3} = p_{n+2} \cdot B_{n+2} + q_{n+1} \cdot B_{n+1} + r_n B_n \\ C_n & C_{n+1} & C_{n+2} & C_{n+3} = p_{n+2} \cdot C_{n+2} + q_{n+1} \cdot C_{n+1} + r_n C_n ; \end{array}$$

If we compare the last three values of A , which are in this case A_{10} , A_9 , A_8 ; we observe that the order of the quotients must necessarily be, $A_{10} = p_9 A_9 + q_8 A_8 + r_7 A_7$; that is to say, the computation must take the form

1	1	1	1	1	1	1				
0	2	0	0	0	0	4	2			
3	11	3	1	1	2	8	9	3		
1	3	33	106	109	142	390	3229	30763	99137	<i>a</i>
	1	11	35	36	47	129	1068	10175	32790	<i>b</i>
		1	3	3	4	11	91	867	2794	<i>c</i>
			1	1	1	3	25	238	767	<i>d</i>
				1	1	2	17	162	522	<i>e</i>
					1	2	16	153	493	<i>f</i>
						1	8	76	245	<i>g</i>
							1	9	29	<i>h</i>
								1	3	<i>i</i>
									1	<i>k</i>

Hence, if the values of q were written above and between the contiguous values of p , and similarly with those of r , as in the subjoined scheme, the computation carried from left to right leads to the ultimate values of A , B , C ; when carried from right to left it leads to those of A_n , A_{n-1} , and so on; but in each case it gives the same aggregate group of numbers; with a difference merely in position; and hence, whenever the numbers

		1	1	1	1	1	1	1	1	1		
	2		4	0	0	0	0	0	2	0		
3		9		8	2	1	1	3	11	3		

$p_0, p_1, p_2, \dots, q_0, q_1, q_2, \dots$, &c., are arranged symmetrically, the series of values $A_0, A_1, A_2, \dots$ is identic with A_1, B_1, C_1 read inversely.

The continuation of the same process to the case of a greater number of magnitudes is so obvious as to stand in no need of farther illustration. The application of this process to problems involving the higher powers of numbers may be expected, as the Brounckerian process has already done for squares, to throw considerable light upon that difficult branch of the Theory of Numbers.