A Forward Problem Model of Electrical Impedance Tomography with Generalized Finite Element Method

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Abstract—In our study, a Generalized Finite Element Method (GFEM) was applied for forward problem modeling of Electrical Impedance Tomography (EIT). The EIT forward model is normally based on the conventional Finite Element Method (FEM). One of the major problems of complex geometry shape or 3-D EIT is its high demand in computation capability power and memory. High precision both in numerical calculation and in data acquisition is required for obtaining the reconstruction images for a small anomaly. To calculate the forward problem accurately, a conventional FEM model with large number of nodes and elements is required. We proposed the Generalized Finite Element Method (GFEM) for EIT forward problem. With the introduction of GFEM, a smaller number of nodes compared with FEM are needed. In the forward solution it is capable of achieving better or the same accuracy with less computational time and memory. The results demonstrate the efficiency of the GFEM in EIT simulation.

Keywords-electrical impedance tomography; Generalized Finite Element Method (GFEM); forward model

I. INTRODUCTION

Electrical impedance tomography uses electrodes placed on the surface to make measurements and then an image of the electrical conductivity distribution within the body is reconstructed with an algorithm. It is a relatively novel low-cost non-invasive imaging technique that has evolved over the past 30 years. And EIT shows the potential to be of great value in clinical diagnosis[1-2].The inverse problem of EIT is badly ill-posed. Small errors brought by solution of the forward problem can lead large errors in reconstruction. The finite element method is the main approximation approach to solve forward problem.

II. FORMULATION

A. Generalized Finite Element Method

The generalized finite element method comes from manifold method [3] is a developed general method to analyze material response to external and internal changes in stress originally. And now it has been used in electromagnetic computation and analysis [4-5]. In this method, the node is generalized, and so it can have more than two or three generalized degrees of freedom, and those degrees of freedom are not required to have their own definite physical meaning necessarily. At each generalized node, we can take a polynomial or series to define a generalized type of nodal interpolation function.

Let us suppose $\mathbf{S}^h$ is the conventional FEM space, and a Lagrange interpolation function $[\varphi_1\ \varphi_2\ \cdots \varphi_n]^T$ is used, then the field variable $\mathbf{U}^h$ can be written as a summation with the conventional as the FEM: $\mathrm{U}^h = \sum_{i=1}^{N} \vec{u}_i \vec{\varphi}_i$, where the $\vec{u}_i$ (i= 1,2,⋯,N) is the vector of degrees of freedom $[u_i\ v_i]^T$ on the i[th] node which represents the potential variation on the ndoe. When the node is generalized it can have more degrees of freedom, and those degrees of freedom. $\vec{u}_i = \begin{Bmatrix} u_i \\ v_i \end{Bmatrix} = \sum_{j=1}^{m_i} \begin{bmatrix} f_{ij}(x,y) & 0 \\ 0 & f_{ij}(x,y) \end{bmatrix} \begin{Bmatrix} d_{i,2j-1} \\ d_{i,2j} \end{Bmatrix}$. Then we get the following from above, $\mathrm{U}^h = \sum_{i=1}^{N} \sum_{j=1}^{m_i} \begin{bmatrix} f_{ij}(x,y) & 0 \\ 0 & f_{ij}(x,y) \end{bmatrix} \begin{Bmatrix} d_{i,2j-1} \\ d_{i,2j} \end{Bmatrix} \vec{\varphi}_i = \sum_{i=1}^{N} \vec{\varphi}^i \vec{F}^i \vec{D}^i = \sum_{i=1}^{N} \vec{N}^i \vec{D}^i$, where $\vec{N}^{\mathbf{i}}$ is the matrix

of interpolation function has its origin $\vec{\varphi}_i$, and $\vec{D}^i$ is the generalized vector of degrees of freedom with the form of $\vec{\mathbf{D}}^i = \left[d_{i,1} \;\; d_{i,2} \;\; \ldots \;\; d_{i,2mi}\right]^T$.

We call the node with more degrees of freedom generalized nodes. When a zero-order generalized nodal interpolation function $\vec{F}^i = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and $\vec{\mathbf{D}}^i = \begin{bmatrix} d_{i,1} \\ d_{i,2} \end{bmatrix}$ is used, the method would be reduced to conventional finite element method $u_i = \begin{Bmatrix} u_i \\ v_i \end{Bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{Bmatrix} d_{i,1} \\ d_{i,2} \end{Bmatrix}$,

With a one-order generalized type of nodal interpolation function $\vec{F}^i = \begin{bmatrix} 1 & 0 & x & 0 & y & 0 \\ 0 & 1 & 0 & x & 0 & y \end{bmatrix}$ and $\vec{\mathrm{D}}^i = \left[d_{i,1} \quad d_{i,2} \quad d_{i,3} \quad d_{i,4} \quad d_{i,5} \quad d_{i,6}\right]^{\mathrm{T}}$, we get

$$\vec{u}_i = \begin{Bmatrix} u_i \\ v_i \end{Bmatrix} = \begin{bmatrix} 1 & 0 & x & 0 & y & 0 \\ 0 & 1 & 0 & x & 0 & y \end{bmatrix} \left[d_{i,1} \quad d_{i,2} \quad d_{i,3} \quad d_{i,4} \quad d_{i,5} \quad d_{i,6}\right]^{\mathrm{T}}$$

With a two-order generalized type of nodal interpolation function

$$\vec{F}^i = \begin{bmatrix} 1 & 0 & x & 0 & y & 0 & xy & 0 & x^2 & 0 & y^2 & 0 \\ 0 & 1 & 0 & x & 0 & y & 0 & xy & 0 & x^2 & 0 & y^2 \end{bmatrix} \quad (i = 1,2,3) \text{ and } \vec{\mathbf{D}}^i = \begin{bmatrix} d_{i,1} \\ \vdots \\ d_{i,12} \end{bmatrix}.$$

B. EIT Forward Problem

A low-frequency EIT forward problem is modeled as (6). The electric field is conservative, and the conduction currents dominant with respect to their displacement counterparts lead to the equation:

$$\nabla \cdot \rho^{-1} \nabla \phi = 0 \qquad \text{in } \mathbf{S^h} \tag{1}$$

where ∇ is the gradient operator, $\nabla\phi$ represents the static electric field; ρ is the resistivity of the body; ϕ is the electric potential; $\mathbf{S^h}$ represents the body to be imaged. Electrodes are modeled with boundary conditions as the complete electrode model [7]. For one triangular element, there are three generalized nodes. Then the field variable $\mathbf{U}^h$ in the element could be written as $\mathbf{U}^h{}_e = \sum_{i=1}^{N} \vec{N}^i{}_e \vec{D}^i{}_e$, $\vec{N}^i{}_e = \vec{\varphi}^i{}_e F^i \quad (i = 1,2,3)$.

C. Method of Weighted Residuals

For EIT problem, it is difficult to derive the governing equations of the GFEM with variational principle. So the method of weighted residuals was implemented to derive the governing equations of the GFEM. The method of weighted residuals is as follows.The field variable is represented as a combination of piecewise polynomial interpolation functions after a discretization: $\tilde{u}^h = \sum_{i=1}^{N} N^i D^i$, There are errors introduced by using the approximating functions. The Laplacian is not equal to zero. So $\tilde{u}^h$ represents a finite approximation of the

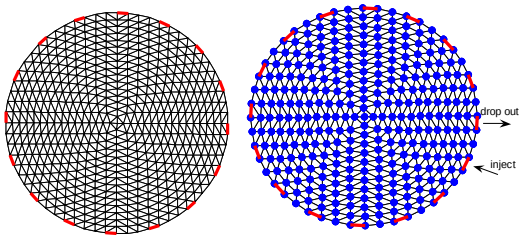
potential. The weak form of the governing equation in the method of weighted residuals could be proceeded by deriving through the multiplication of Laplace's equation 1 by a arbitrary test function v^i and integration over the domain, $\mathbf{S}^h$, $\int_{S^h} v^i[\nabla\cdot(\rho^{-1}\nabla\tilde{u}^h)]dS^h = 0$. The Galerkin method of weighted residuals is used, and using the same interpolation functions $\vec{F}^i$ the arbitrary test function v^i has the same form as the trial function $\tilde{\mathbf{U}}^h$. That is $v^i = \sum_{i=1}^{N} \omega^i \vec{F}^i$, Here ω^i is the coefficients that weigh the interpolation functions $\vec{F}^i$. If we could find the $\tilde{U}^h$, solve equation 16 is solved. Then we get $\int_{S^h} \nabla\cdot(\rho^{-1}\nabla\tilde{u}^h)dS^h = \int_{S^h} \rho^{-1}\nabla\tilde{u}^h\cdot\nabla v^i dS^h$. Boundary conditions could be introduced using Gauss' Theorem: $\int_{S^h} \rho^{-1}\nabla\tilde{u}^h\cdot\nabla v^i dS^h = \int_{\partial S^h} v^i\rho^{-1}\frac{\partial\tilde{u}^h}{\partial\hat{n}}d\Gamma$ (2). The boundary integral is carried out for elements underneath electrodes. The left side of 18 is for the entire mesh. When examined for a single 2D triangular element, k, the left side is $\rho^{-1}{}_k\sum_{i=1}^{3}u^i\sum_{j=1}^{3}\omega^j\int_{E_k}\nabla F^i\cdot\nabla F^j d\Omega$, In terms of the interpolating functions the boundary conditions are: $\int_{\partial S^h} v^i\rho^{-1}\frac{\partial\tilde{u}^h}{\partial\hat{n}}d\Gamma = \rho^{-1}{}_k\sum_{i=1}^{3}\omega^i\sum_{j=1}^{3}u^i\int_{\partial S^h}F^i\nabla F^j\cdot\hat{n}d\Gamma$. For a single element of the GFEM 2 is $\rho^{-1}{}_k\sum_{i=1}^{3}u^i\sum_{j=1}^{3}S_{ik}^k = \rho^{-1}{}_k\sum_{i=1}^{3}\sum_{j=1}^{3}u^i\int_{\partial S^h}F^i\nabla F^j\cdot\hat{n}d\Gamma$. In terms of the entire domain, the left side of 2 will be

$$\int\rho^{-1}\nabla\tilde{u}^h\cdot\nabla F^i dV = \rho^{-1}{}_k\sum_{i=1}^{3}\sum_{j=1}^{3}u^i\int_{\partial S^h}F^i\nabla F^j\cdot\hat{n}d\Gamma \qquad i,j=1,\cdots,N.$$

III. RESULT AND DISCUSSION

To validate the results of GFEM, a circle forward model with 16 electrodes, is used. The circle has radius of 1 m. The 16 electrodes attached on its boundary as shown in Fig. 1. In Fig. 1. (a) is the zero-order GFEM model that is a conventional FEM model which contains 545 nodes, 1024 elements; and (b) is the one-order and two-order GFEM model which contains 313 generlized nodes, 576 elements.

We set the contact impedances of the electrodes to 0.01 $\Omega\cdot\mathrm{m}^2$ in the simulation. The adjacent pair current patterns and adjacent measurements protocol are used. In Fig. 2, the normalized voltage values of electrodes results are shown. Dividing the maximum value of voltage on electrode in the same current pattern, the normalized voltage is obtained.



a. zero-order GFEM b one/two-order GFEM model

Fig. 1. Two GFEM models

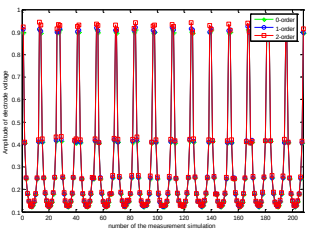
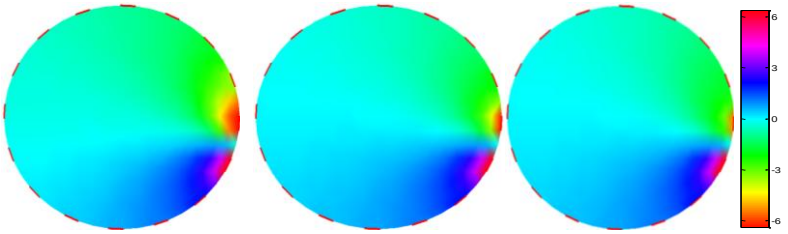


Fig. 2. Normalized voltage values of electrodes measurement

The simulation result is obtained using the zero, one and two-order GFEM basis functions, The results show that the three orders agree very well with the norm error less than 0.0026.



(a) (b) (c)

Fig. 3. Computed voltage with different orders GFEM for one current pattern

(a)zero-order, (b) one-order, (c) two order

The computed voltage with different orders GFEM for one pattern is demonstrated in Fig. 3. We could obtain the same even better computed results with less nodes when the one-order or two-order GFEM is used.

IV. CONCLUSION

The generalized finite element method has been developed and validated for the EIT forward model. Numerical simulation results show that GFEM is able to achieve the same or better accuracy with conventional FEM. In summary, we have shown the GFEM is an efficient and promising method in forward problem solution for electrical impedance tomography.

V. ACKNOWLEDGEMENT

This work was supported in part by the National Natural Science Foundation of China under Grant No. 50937005 and No. 51077040, the Natural Science Foundation of Hebei province, CHINA under Grant No.E2011202026 and No. E2009000085, the Science and Technology Research Project for Universities of Education Department, Hebei Province, CHINA under Grant No. Z2010125.

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