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ON MALET'S PROOF THAT EVERY EQUATION HAS ROOTS,
REAL OR IMAGINARY, EQUAL IN NUMBER TO ITS
DEGREE.

BY PROFESSOR W. N. ROSEVEARE, M.A.

(Communicated by Professor Lawrence Crawford.)

(Read June 17, 1914.)

The writer has not found it easy, with the books of reference at his command, to trace the final result of the mathematical search for a proof that "every equation has a root real or imaginary." Cauchy's proof by vectors is of course well known, but the text-books give no purely analytical proof. Burnside and Panton have an instructive note in an Appendix; and the German "Encyklopädie," on pp. 233 *et seq.*, gives summaries of various attempts. In a "Note" of 1870 Clifford suggests a form of proof. As will be shown below this proof is unsound; but Clifford's idea was taken up in 1882 by Malet, who published a sound proof, of which the following paper gives the substance. It seems to be generally accepted that this is the earliest straightforward proof of the "Fundamental Theorem of Algebra." Elliott in Proc. Lond. Math. Soc., xxv., March and April, 1894, discusses Clifford's and Malet's proofs, dismisses both as unsound, and suggests another proof which he confesses that he cannot complete owing to the intractability of a determinant. (To the present writer Elliott's reasons for rejecting the earlier proofs are not convincing.) Clifford's proof ends thus: E_8 (an equation of degree 8) has a quadratic factor if E_{28} has a real root; E_{28} has a quadratic factor if $E_{14 \times 27}$ has a real root; $E_{14 \times 27}$ has a quadratic factor if E_{odd} has a real root. This last condition is true: therefore ... E_{28} has *real* root; $\therefore E_8$ has a quadratic factor. Thus stated the unsoundness is obvious. On the other hand, Malet's proof seems to be sound. The following paper gives the substance of the proof, and develops some of the detailed work by means of the treatment of certain determinants which the writer has suggested in a previous paper.

§ 1. Malet's proof is (in substance):—

Let $f(x) \equiv a_0x^n + a_1x^{n-1} + \dots + a_n$ be an algebraic function of x of degree n , with the coefficients $a_0, a_1, \dots, a_n$ real. We proceed to prove that $f(x) = 0$ always has a root, real or imaginary.

We know by the principle of continuity that if n is odd, $f(x) = 0$ has a real root. If n is even, let $x \equiv \alpha + \beta$.

Then

$$f(x) \equiv A_0\beta^n + A_1\beta^{n-1} + A_2\beta^{n-2} + \dots + A_n,$$

where

$$A_n \equiv f(\alpha),$$

and

$$A_{n-i} \equiv \frac{f^i(\alpha)}{i!} \text{ (the derived functions); } A_0 \equiv a_0.$$

Now try to determine α and β so that the even and odd parts of $f(x)$ are separately 0.

That is

$$\left. \begin{aligned} A_0\beta^n + A_2\beta^{n-2} + \dots + A_{n-2}\beta^2 + A_n &= 0 \\ \beta(A_1\beta^{n-2} + A_3\beta^{n-4} + \dots + A_{n-3}\beta^2 + A_{n-1}) &= 0 \end{aligned} \right\} \dots\dots\dots (i)$$

The possible solution $\beta = 0, A_n \equiv f(\alpha) = 0$ we need not discuss. Other solutions are obtained by eliminating β^2 by the "dialytic" method, which gives an equation for α of degree $n(n-1)/2$. Moreover, the process also gives β^2 as a unique function of any root α of the final equation. Therefore, if a certain equation of degree $n(n-1)/2$ has a root (real or imaginary), $f(x) = 0$ has two roots, $x = \alpha \pm \beta$.

Now, if n is odd $f(x) = 0$ has a real root. Assume n to have a factor 2^v , then $n(n-1)/2$ has 2 only to the degree 2^{v-1} .

Thus $f(x) = 0$, when n contains 2^v , has a root if a certain equation whose degree contains 2^{v-1} has a root. It follows that this second equation has a root if one of degree containing 2^{v-2} has a root, and as this process ultimately leads us to an equation of odd degree which has a (real) root, the proposition is proved.

§ 2. This (Malet's) proof of the "Fundamental Theorem of Algebra" leaves nothing to be desired; but the following detailed discussion of the work of eliminating β^2 by the dialytic method, including a more direct proof that $f(x) = 0$ has n roots real or imaginary, offers some points of interest.

We shall give the work in detail for $n = 8$ for the sake of brevity; but we shall show that the method is quite general.

To eliminate β^2 from

$$A_0\beta^n + A_2\beta^{n-2} + \dots + A_n = 0$$

and

$$A_1\beta^{n-2} + A_3\beta^{n-4} + \dots + A_{n-1} = 0$$

we multiply both equations by $\beta^2, \beta^4, \dots$, the upper one $\left(\frac{n-2}{2}-1\right)$ times, and the lower $\left(\frac{n}{2}-1\right)$ times, so obtaining $(n-1)$ equations in $\beta^2, \beta^4, \dots, \beta^{2(n-2)}$, from which we get a determinant eliminating β .

Thus for $n=8$

$$\begin{aligned} A_0\beta^{12} + A_2\beta^{10} + A_4\beta^8 + A_6\beta^6 + A_8\beta^4 &= 0 \\ A_0\beta^{10} + A_2\beta^8 + A_4\beta^6 + A_6\beta^4 + A_8\beta^2 &= 0 \\ A_0\beta^8 + A_2\beta^6 + A_4\beta^4 + A_6\beta^2 + A_8 &= 0 \\ A_1\beta^6 + A_3\beta^4 + A_5\beta^2 + A_7 &= 0 \\ A_1\beta^4 + A_3\beta^2 + A_5\beta^0 + A_7\beta^2 &= 0 \\ A_1\beta^{10} + A_3\beta^8 + A_5\beta^6 + A_7\beta^4 &= 0 \\ A_1\beta^{12} + A_3\beta^{10} + A_5\beta^8 + A_7\beta^6 &= 0 \end{aligned}$$

and

$$\therefore \begin{vmatrix} A_0 & A_2 & A_4 & A_6 & A_8 & . & . & . \\ . & A_0 & A_2 & A_4 & A_6 & A_8 & . & . \\ . & . & A_0 & A_2 & A_4 & A_6 & A_8 & . \\ . & . & . & A_1 & A_3 & A_5 & A_7 & . \\ . & . & A_1 & A_3 & A_5 & A_7 & . & . \\ . & A_1 & A_3 & A_5 & A_7 & . & . & . \\ A_1 & A_3 & A_5 & A_7 & . & . & . & . \end{vmatrix} = 0 \equiv D_7.$$

(a dot meaning zero.)

Also, omitting the first and last of the equations, we have 5 equations for the 5 "unknowns," $\beta^2, \beta^4, \dots, \beta^{10}$.

Thus

$$\begin{aligned} -\beta^2 &= \begin{vmatrix} A_0 & A_2 & A_4 & A_6 & . \\ . & A_0 & A_2 & A_4 & A_8 \\ . & . & A_1 & A_3 & A_7 \\ . & A_1 & A_3 & A_5 & . \\ A_1 & A_3 & A_5 & A_7 & . \end{vmatrix} \div \begin{vmatrix} A_0 & A_2 & A_4 & A_6 & A_8 \\ . & A_0 & A_2 & A_4 & A_6 \\ . & . & A_1 & A_3 & A_5 \\ . & A_1 & A_3 & A_5 & A_7 \\ A_1 & A_3 & A_5 & A_7 & . \end{vmatrix} \\ &\equiv \frac{D'_5}{D_5}, \text{ say.} \end{aligned}$$

We proceed to prove that $D_7=0$ has (at least) 4 real roots. We shall use the suffixes of A_0, A_2 , etc., only.

Thus

$$D_7 \equiv \begin{vmatrix} 0 & 2 & 4 & 6 & 8 & . & . \\ . & 0 & 2 & 4 & 6 & 8 & . \\ . & . & 0 & 2 & 4 & 6 & 8 \\ . & . & . & 1 & 3 & 5 & 7 \\ . & . & 1 & 3 & 5 & 7 & . \\ . & 1 & 3 & 5 & 7 & . & . \\ 1 & 3 & 5 & 7 & . & . & . \end{vmatrix} \quad (D_5 \text{ shown screened}).$$

Treat D_7 thus: (8), (6), (5), (7), (.) in brackets denoting the minors of the last column of D_5 (viz. 8657.), multiply the 1st row of D_7 by (8), the 2nd by (6), and the last three by (5), (7), (.) as shown below.

$$\begin{vmatrix} 0 & 2 & 4 & 6 & 8 & . & . \\ . & 0 & 2 & 4 & 6 & 8 & . \\ . & . & 0 & 2 & 4 & 6 & 8 \\ . & . & . & 1 & 3 & 5 & 7 \\ . & . & 1 & 3 & 5 & 7 & . \\ . & 1 & 3 & 5 & 7 & . & . \\ 1 & 3 & 5 & 7 & . & . & . \end{vmatrix} \begin{array}{l} \dots \times (8) \\ \dots \times (6) \\ \\ \dots \times (5) \\ \dots \times (7) \\ \dots \times (.) \end{array}$$

and add columns for a new last row.

The elements of this new row will be seven 5-line determinants, of which the first four columns are those of D_5 , the last one varying and being shown below between verticals.

Thus

$$\begin{array}{cccc|c|c|c|c|c|c} 0 & 2 & 4 & 6 & 0 & 2 & 4 & 6 & 8 & . & . \\ . & 0 & 2 & 4 & . & 0 & 2 & 4 & 6 & 8 & . \\ . & . & 1 & 3 & . & . & 1 & 3 & 5 & 7 & . \\ . & 1 & 3 & 5 & . & 1 & 3 & 5 & 7 & . & . \\ 1 & 3 & 5 & 7 & 1 & 3 & 5 & 7 & . & . & . \end{array}$$

so that the seven values are ., ., ., ., D_5 , D'_5 , .

Noticing that the original last row was multiplied by (.) which

$$= 0 \begin{vmatrix} 0 & 2 & 4 \\ . & 1 & 3 \\ 1 & 3 & 5 \end{vmatrix} \equiv 0 \cdot D_3 \text{ (say), and reading off } D_7 \text{ from the new last row,}$$

$$D_7 \cdot 0 \cdot D_3 = -D'_5 \cdot 0 \cdot D'_5 + D_5 \cdot 0 \cdot D''_5 \text{ say,} \\ \therefore D_7 D_3 = D_5 D''_5 - D'^2_5.$$

Therefore when $D_5 = 0$, D_7 , and D_3 have opposite signs.

Treating D_5 in the same way with minors of D_3 we get

$$D_5 \cdot 0 \cdot D_1 = \begin{vmatrix} 0 & 2 & 4 & 6 & 8 \\ . & \boxed{0} & \boxed{2} & \boxed{4} & 6 \\ . & . & \boxed{1} & 3 & 5 \\ . & 1 & 3 & 5 & 7 \\ . & . & D_3 & D'_3 & D''_3 \end{vmatrix} = D'_3 \cdot 0 \cdot D_3 - 0 \cdot D'_3{}^2 + D_3 \cdot 0 \cdot D_3^* \text{ say.}$$

Therefore, when $D_3 = 0$, D_1 , and D_5 have opposite signs.

The proof will be seen to be general (for even n).

The result may be thus stated:—

Build up a determinant D_{n-1} by means of concentric determinants

$$D_1 \equiv A_1; D_3 \equiv \begin{vmatrix} A_0 & A_2 & A_4 \\ . & A_1 & A_3 \\ A_1 & A_3 & A_5 \end{vmatrix} \text{ etc. Then we have proved that of every}$$

three consecutive determinants the extremes have opposite signs when the middle one is 0.

$$\text{Again when } D_1 \equiv A_1 = 0, D_3 = \begin{vmatrix} 0 & 2 & 4 \\ . & . & 3 \\ . & 3 & 5 \end{vmatrix} \equiv -0 \cdot 3^2.$$

Now 0 is $\frac{1}{n} f^n(\alpha) \equiv a_0$, which we take to be +.

We will prove in subsequent articles that $D_1, D_3, D_5, \dots, D_v$ are of dimensions 1, 6, 15, ..., $v(v+1)/2$ and that the coefficient of the highest power of α is always +. Hence the following scheme of signs and roots (0 on a line indicating a group of an odd number of roots):

It remains to justify the signs of $D_1, D_3, \dots$ when $a = \pm \infty$. We shall then have proved that $D_{n-1} = 0$ has (at least) $\frac{n}{2}$ real roots in a , and that β^2 has one value for each root of a . $\therefore f(x) = 0$ has $\frac{n}{2}$ pairs of roots $(a \pm \beta)$.

Q. E. D.

§ 3. To prove that the highest power of a in D_v has index $v(v+1)/2$ and a + coefficient. Rejecting all but highest powers in $A_0, A_1, \dots$, taking a_0 to be 1, and indicating binomial coefficients by n_i ,

$$D_5 = \begin{vmatrix} 1 & n_2 a^2 & n_4 a^4 & n_6 a^6 & n_8 a^8 \\ . & 1 & n_2 a^2 & n_4 a^4 & n_6 a^6 \\ . & . & n_1 a & n_3 a^3 & n_5 a^5 \\ . & n_1 a & n_3 a^3 & n_5 a^5 & n_7 a^7 \\ n_1 a & n_3 a^3 & n_5 a^5 & n_7 a^7 & n_9 a^9 \end{vmatrix}.$$

Dividing out powers of a in the columns

$$D_5 = a^{1+3+5} \begin{vmatrix} 1 & n_2 a^2 & n_4 a^3 & n_6 a^3 & n_8 a^3 \\ . & 1 & n_3 a & n_4 a & n_6 a \\ . & . & n_1 & n_3 & n_5 \\ . & n_1 a & n_3 a^2 & n_5 a^2 & n_7 a^2 \\ n_1 a & n_3 a^3 & n_5 a^4 & n_7 a^4 & n_9 a^4 \end{vmatrix}.$$

Multiplying the first two columns by a^3, a , we obtain

$$a^{5+(3+1+0+2+4)} \begin{vmatrix} 1 & n_2 & n_4 & n_6 & n_8 \\ . & 1 & n_2 & n_4 & n_6 \\ . & . & n_1 & n_3 & n_5 \\ . & n_1 & n_3 & n_5 & n_7 \\ n_1 & n_3 & n_5 & n_7 & n_9 \end{vmatrix}.$$

The index of a is $5(5+1)/2$, and a similar process gives for the highest index in D_v , $v(v+1)/2$.

§ 4. To prove that the coefficient is +.

$$\text{Consider } D_v \equiv \begin{vmatrix} n_a & n_b & n_c & \dots \\ n_{a-1} & n_{b-1} & n_{c-1} & \dots \\ n_{a-2} & n_{b-2} & n_{c-2} & \dots \\ \dots & \dots & \dots & \dots \\ n_{a-v+1} & n_{b-v+1} & n_{c-v+1} & \dots \end{vmatrix}.$$

Since we know

$$\begin{aligned} n_i + n_{i-1} &\equiv (n+1)_i, \\ n_i + 2n_{i-1} + n_{i-2} &\equiv (n+2)_i, \\ &\dots\dots\dots \end{aligned}$$

and in general

$$n_i + p_1 \cdot n_{i-1} + p_2 \cdot n_{i-2} + \dots + n_{i-p} \equiv (n+p)_i,$$

we can, by grouping rows, express D_v in the form

$$\begin{vmatrix} n_a, & n_b, & \dots \\ (n+1)_a, & (n+1)_b, & \dots \\ \dots\dots\dots \\ (n+v-1)_a, & (n+v-1)_b, & \dots \end{vmatrix}.$$

Which

$$\equiv \frac{(n+v-1)_a \cdot (n+v-1)_b \dots}{(n+v-1)^{v-1} \cdot (n+v-2)^{v-2} \dots} \begin{vmatrix} (n-a+1)(n-a+2)\dots(n+v-1-a), \text{ do. in } b, \dots \\ (n-a+2)\dots(n+v-1-a), \dots\dots\dots \\ \dots\dots\dots \\ 1, & 1, & \dots \end{vmatrix}$$

This determinant is (i) symmetrical in $(n-a)$, $(n-b)$, ...; (ii) of order $(v-1) + (v-2) + \dots + 0$. By (i) it is divisible by the product of differences $(b-a) \cdot (c-a)(c-b) \dots$. This product is of order $1+2+\dots+(v-1)$, therefore any other factor is independent of $(n-a)$, $(n-b)$, ... and can be obtained by taking special values, $n-a=0$, $n-b=-1$, $n-c=-2$, ... $n-\text{last} = -v+1$. All the elements above the leading diagonal become 0, and the diagonal is $\frac{v-1}{v-1} \frac{v-2}{v-2} \dots \frac{1}{1}$.

The product of differences becomes $\frac{v-1}{v-1} \frac{v-2}{v-2} \dots 1$, therefore the numerical factor is 1, and we obtain

$$\begin{vmatrix} n_a, & n_b, & \dots \\ n_{a-1}, & n_{b-1}, & \dots \\ \dots\dots\dots \\ n_{a-v+1}, & n_{b-v+1}, & \dots \end{vmatrix} \equiv \frac{(n+v-1)_a (n+v-1)_b \dots}{(n+v-1)^{v-1} (n+v-2)^{v-2} \dots} \text{prod. of diff.}$$

$$\text{Now, the original } D_v \equiv \begin{vmatrix} n_0, & n_2, & \dots \\ \cdot, & n_0, & \dots \\ \dots\dots\dots \\ \cdot, & n_1, & \dots \\ n_1, & n_3, & \dots \end{vmatrix}$$

can be reduced to the more general form by bringing the last row to the top, the last row but one to the third, and the last but two fifth, etc., and putting a for 1, b for 3, c for 5, ...

Therefore

D_v becomes $(-)^{\overline{v-1}+\overline{v-3}+\dots} \frac{(n+v-1)_1(n+v-1)_3\dots(n+v-1)_{2v-1}}{(n+v-1)^{v-1}(n+v-2)^{v-1}\dots}$ prod. of diff.

which, since $\overline{v-1}, \overline{v-3} \dots$ are even, and $2v-1 < n+v-1$, is $+$.

Thus the signs assumed when $\alpha = \pm\infty$ are justified, and the proposition is fully proved.

§ 5. NOTE.—It can also be proved directly from the foregoing that β is a pure imaginary when $f(x)$ does not vanish for real values of x .

If we write D_7 in the equivalent form

$$\begin{vmatrix} 8 & 6 & 4 & 2 & 0 & . & . \\ . & 8 & 6 & 4 & 2 & 0 & . \\ . & . & 8 & 6 & 4 & 2 & 0 \\ . & . & . & 7 & 5 & 3 & 1 \\ . & . & 7 & 5 & 3 & 1 & . \\ . & 7 & 5 & 3 & 1 & . & . \\ 7 & 5 & 3 & 1 & . & . & . \end{vmatrix}.$$

the concentric determinants will be different.

The new D_5 will be seen to be equal to

$$\begin{vmatrix} 2 & 4 & 6 & 8 & . \\ 0 & 2 & 4 & 6 & 8 \\ . & 1 & 3 & 5 & 7 \\ 1 & 3 & 5 & 7 & . \\ 3 & 5 & 7 & . & . \end{vmatrix}.$$

D_1 is now 7, which $\equiv f'(a)$; and when 7 (of odd degree) = 0, $D_3 = -8 \cdot 5^2$, *i.e.* D_3 has the opposite sign to $f(a)$. Therefore, if $f(a)$ is always $+$ the scheme of signs is the same as in the previous work in § 2 (end).

It follows that when $D_7 = 0$, the new D_5 has the same sign as the old D_5 ; but the simultaneous equations which gave $\beta^2, \beta^4, \dots \beta^{10}$ show that $\frac{\text{new } D_5}{\text{old } D_5} = \frac{-\beta^{10}}{\beta^5}$, therefore β^{10} is $-ve$; therefore β^2 is $-ve$, and β is imaginary.