A New Specifying Method for Stress and Strain in an Elastic Solid. By Lord Kelvin.

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The method for specifying stress and strain hitherto followed by all writers on elasticity has the great disadvantage that it essentially requires the strain to be infinitely small. As a notational method it has the inconvenience that the specifying elements are of two essentially different kinds (in the notation of Thomson and Tait e, f, g , simple elongations; a, b, c , shearings). Both these faults are avoided if we take the six lengths of the six edges of a tetrahedron of the solid, or, what amounts to the same, though less simple, the three pairs of face-diagonals of a hexahedron,* as the specifying elements. This I have thought of for the last thirty years, but not till a few weeks ago have I seen how to make it conveniently practicable, especially for application to the generalised dynamics of a crystal.

§ 1. We shall suppose the solid to be a homogeneous crystal of any possible character. Cut from it a tetrahedron ABCD of any shape and orientation. Let the three non-intersecting pairs (AB, CD), (BC, AD), (CA, BD) of its six edges be denoted by

$$(3p, 3p'), \quad (3q, 3q'), \quad (3r, 3r') \quad . \quad . \quad . \quad (1).$$

This notation gives

$$(p, p'), \quad (q, q'), \quad (r, r') \quad . \quad . \quad . \quad . \quad (2)$$

for the six edges of a tetrahedron, similar to ABCD, formed by taking for its corners ($\alpha, \beta, \gamma, \delta$) the centres of gravity† of the four

* This name, signifying a figure bounded by three pairs of parallel planes, is admitted in crystallography; but the longer and less expressive "parallelepiped" is too frequently used instead of it by mathematical writers and teachers. A hexahedron, with its angles acute and obtuse, is what is commonly called, both in pure mathematics and crystallography, a rhombohedron. A right angled hexahedron is a brick, for which no Greek or other learned name is hitherto to the front in usage. A rectangular equilateral hexahedron is a cube.

† For brevity I shall henceforth call the centre of gravity of a triangle, or of a tetrahedron, simply *its centre*.

triangular faces BCD, CDA, DAB, ABC respectively, so that we have $p = \alpha\beta$, $q = \beta\gamma$, $r = \gamma\alpha$, $p' = \gamma\delta$, $q' = \alpha\delta$, $r' = \beta\delta$. Consider now, in advance, the amounts of work done by the six pairs of balancing forces constituting the six stress-components described in § 2, when the strain-components vary; for example, the balancing pulls P, parallel to AB, when $\alpha\beta$ increases from p to $p + dp$, all the other five lengths q, r, p', q', r' remaining constant. For the reckoning of work we may suppose the opposite forces, P, to be applied at α and β , instead of being equally distributed over the faces ADC, BDC. Hence the work which they do is Pdp ; and other five pairs of balancing pulls, Q, R, P', Q', R', do no work.

§ 2. Parallel to the edge AB apply to the faces ADC, BDC equal and opposite pulls, P, equally distributed over them. These two balancing pulls we shall call a stress or a stress-component. Similarly, parallel to each of the five other edges apply balancing pulls on the pair of faces cutting it. Thus we have in all six stress-components parallel to the six edges of the tetrahedron, denoted as follows:—

$$(P, P') \quad (Q, Q') \quad (R, R') \quad . \quad . \quad . \quad (3);$$

and we suppose that these forces, applied as they are to the four faces of the solid, are balanced in virtue of the mutual forces between its particles, when its edges are of the lengths specified as in (1). Let $p_0, p'_0, g_0, g'_0, r_0, r'_0$ be the values of the specifying elements when no forces are applied to the faces. Thus the differences from these values, of the six lengths shown in formula (2), represent the strain of the substance when under the stress represented by (3).

§ 3. Let w be the work done when pulls upon the faces, each commencing at zero, are gradually increased to the values shown in (3). In the course of this process we have

$$dw = Pdp + P'dp' + Qdq + Q'dq' + Rdr + R'dr' \quad . \quad . \quad (4).$$

Hence if we suppose w expressed as a function of p, p', q, q', r, r' , we have

$$\frac{dw}{dp} = P, \quad \frac{dw}{dp'} = P', \quad \frac{dw}{dq} = Q, \quad \frac{dw}{dq'} = Q', \quad \frac{dw}{dr} = R, \quad \frac{dw}{dr'} = R' \quad . \quad (4).$$

This completes the foundation of the molar dynamics of an

elastic solid of the most general possible kind according to Green's theory, expressed in terms of the new mode of specifying stresses and strains, without restriction to infinitely small strains.

§ 4. To understand thoroughly the state of strain specified by (1) or (2), let the tetrahedron of reference, $A_0B_0C_0D_0$, for the condition of zero strain and stress, be equilateral (that is to say, according to the notation of § 2 (1) let $\frac{1}{3}$ of each edge = $p_0 = q_0 = r_0 = p'_0 = q'_0 = r'_0$). In $A_0B_0C_0D_0$ inscribe a spherical surface touching each of the six edges. Its centre must be at K_0 , the centre of the tetrahedron; and the points of contact must be the middle points of the edges. Alter the solid by homogeneous strain* to the condition (p, q, r, p', q', r') in which $A_0B_0C_0D_0$ becomes ABCD. The inscribed spherical surface becomes an ellipsoid having its centre at K, the centre of ABCD, and touching its six edges at their middle points.† This ellipsoid shows fully and clearly the state of strain specified by p, q, r, p', q', r' . It is what is called the "strain ellipsoid."‡

§ 5. Two ways of finding the ellipsoid touching the six edges of a tetrahedron are obvious. (1) Through AB and CD draw planes respectively parallel to CD and AB; and deal similarly with the two other pairs of non-intersecting edges. The three pairs of parallel planes thus found constitute a hexahedron which contains the required ellipsoid touching the six faces at their centres; or (2) draw AK, BK, CK, DK, and produce to equal distances KA', KB', KC', KD' beyond K. We thus find four points, A', B', C', D', which, with A, B, C, D, are the eight corners of the hexahedron which we found by construction (1). A circumscribed hexahedron being thus given, the principal axes of the ellipsoid, and their orientation, are found by the solution of a cubic equation.

§ 6. Another way of finding the strain-ellipsoid, which is in some respects simpler, and which has the advantage that in its construction it does not take us outside the boundary of our

* Thomson and Tait's *Natural Philosophy*, § 155; *Elements*, § 136.

† Thus we have an interesting theorem in the geometry of the tetrahedron:—If an ellipsoid touching the edges of a tetrahedron has its centre at the centre of the tetrahedron, the points of contact are at the middles of the edges.

‡ Thomson and Tait's *Natural Philosophy*, § 160; *Elements*, § 141.

fundamental tetrahedron, is as follows:—In the equilateral tetrahedron $A_0B_0C_0D_0$ describe, from its centre K_0 , a spherical surface touching any three of its faces. It touches these faces at their centres; and it also touches the fourth face, and at its centre. Hence, if we solve the determinate, one-solutional, problem to draw an ellipsoid touching at their centres any three of the four faces of any tetrahedron $ABCD$, and having its centre at K , this ellipsoid touches at its centre the fourth face of the tetrahedron; and it is the strain ellipsoid for the homogeneous strain by which an equilateral tetrahedron of solid is altered to the figure $ABCD$.

§ 7. To bring our new method of specifying strain and stress into relation with the ordinary method for infinitesimal strains and the corresponding stresses:—Let λ denote the length of each edge of the equilateral tetrahedron of reference, $A_0B_0C_0D_0$; and let h be the edge of the cube of which A_0, B_0, C_0, D_0 are four corners (this cube being the hexahedron found by applying either of the constructions of § 5 to the tetrahedron $A_0B_0C_0D_0$). The twelve face-diagonals of this cube are each equal to λ , and therefore $\lambda = h\sqrt{2}$. Let now the cube be infinitesimally strained so that its edges become $h(1+e)$, $h(1+f)$, $h(1+g)$; and so that the angles in its three pairs of faces are altered from right angles to acute and obtuse angles differing respectively by a , b , c from right angles. This is the strain (e, f, g, a, b, c) in the notation of Thomson and Tait referred to in the introductory paragraph above. By the infinitesimal geometry of the affair, we easily find the corresponding alterations of the face diagonals, which according to our present notation are $(p-1)\lambda$, $(p'-1)\lambda$, $(q-1)\lambda$, etc., and thus we have as follows:—

$$\left. \begin{aligned} p-1 &= \frac{1}{2}(f+g+a) \\ p'-1 &= \frac{1}{2}(f+g-a) \\ q-1 &= \frac{1}{2}(g+e+b) \\ q'-1 &= \frac{1}{2}(g+e-b) \\ r-1 &= \frac{1}{2}(e+f+c) \\ r'-1 &= \frac{1}{2}(e+f-c) \end{aligned} \right\} \dots \dots \dots (5)$$

for the relation between the two specifications of any infinitesimal strain. Adding these, and denoting $e+f+g$ by s , we find

$$p+p'+q+q'+r+r'-6=2s \dots \dots \dots (6).$$

And solving for a, b, c, e, f, g , in terms of p, q, r, p', q', r' , we have

$$\left. \begin{aligned} a &= p - p'; & b &= q - q'; & c &= r - r'; \\ e &= s - p - p' + 2; & f &= s - q - q' + 2; & g &= s - r - r' + 2; \end{aligned} \right\} \quad (7).$$

§ 8. The work required to produce an infinitesimal strain e, f, g, a, b, c , in a homogeneous solid of cubic crystalline symmetry is expressed by the following formula:—

$$2w = \mathbf{A}(e^2 + f^2 + g^2) + 2\mathbf{B}(fg + ge + ef) + n(a^2 + b^2 + c^2) \quad (8).$$

This may be conveniently modified by putting

$$k = \frac{1}{3}(\mathbf{A} + 2\mathbf{B}); \quad n_1 = \frac{1}{2}(\mathbf{A} - \mathbf{B}) \quad \dots \quad (9),$$

where k denotes the bulk modulus and n_1, n the two rigidity-moduluses. With this notation (8) becomes

$$2w = k(e + f + g)^2 + \frac{2}{3}n_1[(f - g)^2 + (g - e)^2 + (e - f)^2] + n(a^2 + b^2 + c^2) \dots (10).$$

The rigidity relative to shearings parallel to the pairs of planes of the cube, or, which is the same thing, changes of the angles of the corners of the square faces from right angles to acute or obtuse angles, is n_1 . The rigidity relative to changes of the angles between the diagonals of the faces from right angles to acute or obtuse angles is n . The compressibility modulus is k . Using now (7) in (10) we have

$$\begin{aligned} 2w &= ks^2 + \frac{2}{3}n_1[(q + q' - r - r')^2 + (r + r' - p - p')^2 + (p + p' - q - q')^2] \\ &\quad + n[(p - p')^2 + (q - q')^2 + (r - r')^2] \quad \dots \quad (11). \end{aligned}$$

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