where δ_{ij} is the adjoint of a_{ij} in the determinant $| a_{ij} |$. K is thus generated by the $T_{i,\lambda}$ and $Q_{i,j,\lambda}$. Its order and structure follow from that of the general linear homogeneous group* on m indices in the $GF[p^n]$.

35. THEOREM.—*For $p>2$ the largest sub-group common to $H_{m,n,p}$ and $B_{m,n,p}$ (for $\lambda=1$) is $K_{m-1,n,p}$.*

It is clearly that sub-group of $K_{m,n,p}$ which has the invariant $\xi_1+\eta_1$. Hence

$$a_{11}=\delta_{11}=1,\quad a_{1j}=\delta_{1j}=0\quad (j=2,\ldots,m).$$

Writing these relations for the inverse of (1), we have

$$\delta\ =a_{j1}=0\quad (j=2,\ldots,m).$$

The substitutions of the sub-group have therefore the form

$$\left\{\begin{array}{ll}\xi_1'=\xi_1, & \xi_i'=\sum_{j=2}^{m} a_{ij}\xi_j \\ \eta_1'=\eta_1, & \eta_i'=\sum_{j=2}^{m}\delta_{ij}\eta_j\end{array}\right\}\quad (j=2,\ldots,m).$$

On the Influence of Gravity on Elastic Waves, and, in particular, on the Vibrations of an Elastic Globe. By T. J. I'A. BROMWICH, B.A., Fellow of St. John's College, Cambridge.

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This paper contains solutions of four elastic problems, all originally suggested by hypotheses which might modify the velocity of propagation of shocks along the surface of the earth. The first, second, and third deal with gravitational effects; hence, in these three I have assumed the material incompressible in order to avoid the difficulties that arise, even in the statical problem, if the material be compressible (Love's *Elasticity*, Vol. I., Art. 127). In the fourth

* Dickson, *Annals of Mathematics*, pp. 161–183, 1897.

problem I consider the effect of a thin skin, whose elastic constants differ from those of the main body; here gravity does not enter into the problem and the material is supposed compressible.

The first, second, and fourth cases suppose the free surface to be an infinite plane: these are based on a paper by Lord Rayleigh (*Proc. Lond. Math. Soc.*, Vol. XVII.). From the first and second it appears that when the length of waves is short enough for us to regard the earth as plane the effect of gravity must be in all cases small. The fourth solution shows that the effect of the skin must be proportional to its thickness, and hence must be small.

The third problem solves the vibration of a sphere under its own gravity. Here the modification introduced by gravity appears to be considerable, on using the approximate elastic constants of the earth. The method adopted here is practically the same as Prof. Lamb's, to be found in his well-known paper on a vibrating sphere (*Proc. Lond. Math. Soc.*, Vol. XIII.); but I have used a slightly modified form for the analysis, which reduces the labour of manipulation and also gives a more convenient form of the period-equation.

It appears that gravity has no effect if the order of the harmonic disturbance is zero or unity; when this order is 2, I have calculated a number of roots of the period-equation. In particular for a sphere of the size, mass, and gravity of the earth, but with rigidity about that of steel, the gravest free period is 55 minutes; the corresponding period without gravity is 66 minutes. If the rigidity be about that of glass, the periods are 78 and 120 minutes, respectively.

These problems were originally undertaken at the suggestion of Dr. Larmor, to whom I am indebted for many valuable criticisms.

1. *Propagation of Waves under Constant Gravity on the Surface of an Infinite Incompressible Elastic Solid with an Infinite Horizontal Face.*

Following Lord Rayleigh's method (*loc. cit. supra*), we have to make but one modification, viz., the normal traction on the mean free surface has to be just sufficient to support the weight of the harmonic inequality, instead of vanishing. The proof of this statement will be found in Love's *Elasticity* (Vol. I., Art. 173).

To shorten the work, I take the axis of x to be the direction of propagation of the waves; then, if z is vertically upwards, we take all the displacements independent of y and $v=0$. The ordinary equations of elasticity then become

$$\rho \frac{\partial^2 u}{\partial t^2} = (\lambda+\mu)\frac{\partial \Delta}{\partial x} + \mu \nabla^2 u,$$

$$\rho \frac{\partial^2 w}{\partial t^2} = (\lambda+\mu)\frac{\partial \Delta}{\partial z} + \mu \nabla^2 w,$$

$$\Delta = \frac{\partial u}{\partial x} + \frac{\partial w}{\partial z}.$$

But, since the solid is incompressible, Δ, the dilatation, will be zero; however, $\lambda\Delta$ will be finite, and let us put $p_1 = \lambda\Delta$, so that p_1 is a kind of negative hydrostatic pressure. We then have the modified equations

$$\rho \frac{\partial^2 u}{\partial t^2} = \frac{\partial p_1}{\partial x} + \mu \nabla^2 u,$$

$$\rho \frac{\partial^2 w}{\partial t^2} = \frac{\partial p_1}{\partial z} + \mu \nabla^2 w,$$

$$0 = \frac{\partial u}{\partial x} + \frac{\partial w}{\partial z}.$$

Now assume all the displacements to contain the factor $\exp(ipt+ilx)$; so that the wave-length $\lambda' = 2\pi/l$, and the velocity of propagation is l/p. We then have, if

$$\kappa^2 = \rho p^2/\mu,$$

$$(\nabla^2+\kappa^2)\, u = -\frac{1}{\mu}\frac{\partial p_1}{\partial x},$$

$$(\nabla^2+\kappa^2)\, w = -\frac{1}{\mu}\frac{\partial p_1}{\partial z},$$

whence we find $$\nabla^2 p_1 = 0.$$

Thus we take $$p_1/\mu\kappa^2 = (Pe^{-lz} + Qe^{lz}) \exp(ipt+ilx),$$

which is the general solution, if p_1 contains the exponential $\exp(ipt+ilx)$. Now in the solid z ranges from 0 at the mean free surface to $-\infty$; consequently, if l be positive, we must take $P = 0$, so that p_1 may not increase indefinitely with the depth. Whence

$$p_1/\mu\kappa^2 = Qe^{lz} \exp(ipt+ilx).$$

Next, since $\nabla^2 p_1 = 0$, a particular set of values of the displacements will be

$$u = -\frac{1}{\mu\kappa^2}\frac{\partial p_1}{\partial x},$$

$$w = -\frac{1}{\mu\kappa^2}\frac{\partial p_1}{\partial z},$$

to which we must add complementary solutions of the equations

$$(\nabla^2+\kappa^2)\,u = 0,$$

$$(\nabla^2+\kappa^2)\,w = 0,$$

$$\frac{\partial u}{\partial x}+\frac{\partial w}{\partial z} = 0,$$

so that in all we find

$$u = -\frac{1}{\mu\kappa^2}\frac{\partial p_1}{\partial x} + Ae^{sz}\exp i\,(lx+pt),$$

$$w = -\frac{1}{\mu\kappa^2}\frac{\partial p_1}{\partial z} + Be^{sz}\exp i\,(lx+pt),$$

where $ilA+sB = 0$ and $s^2+\kappa^2 = l^2$.

It is assumed that the real part of s is positive in order that u, w may not become infinite at $z = -\infty$; the case when s is purely imaginary will be considered later.

The conditions at $z = 0$ are

$$\lambda\Delta+2\mu\frac{\partial w}{\partial z}+g\rho w = 0,$$

$$\mu\left(\frac{\partial u}{\partial z}+\frac{\partial w}{\partial x}\right) = 0,$$

the first of which makes the normal traction on the mean free surface just support the weight of the harmonic inequality, while the second makes the tangential traction zero. Whence

$$\mu\kappa^2 Q+2\mu\,(sB-l^2Q)+g\rho\,(B-lQ) = 0,$$

$$-2il^2Q+sA+ilB = 0.$$

But $ilA = -sB$, and thus we have

$$-lQ\,(2-\kappa^2/l^2)+(g\rho/\mu l)(B-lQ)+2sB/l = 0,$$

$$2lQ-(2-\kappa^2/l^2)\,B = 0,$$

in the second of which we have put

$$s^2 = l^2 - \kappa^2.$$

Writing now $\kappa^2/l^2 \equiv \zeta$, we find, after eliminating the ratio $B : Q$,

$$(2-\zeta)(2-\zeta+g\rho/\mu l) = 2\,(2\sqrt{1-\zeta}+g\rho/\mu l),$$

or

$$(2-\zeta)^2 - 4\sqrt{1-\zeta} - \zeta\,(g\rho/\mu l) = 0,$$

and here $\sqrt{1-\zeta} = s/l$, and so the real part of $\sqrt{1-\zeta}$ is to be positive.

When $g = 0$, the equation is the same as that found by Rayleigh for an incompressible solid (*loc. cit. supra*).

I now proceed to obtain an estimate of the magnitude of $(g\rho/\mu l)$. In the fifteenth *Brit. Assoc. Report*, "On the Earthquake Phenomena of Japan" (p. 58), we find that approximate values near the earth's surface are, in C.G.S. units,

$$\rho = 3, \quad \mu = (1{\cdot}5)\,10^{11},$$

and the mean value of g is known to be 981 in these units. Now

$$g\rho/\mu l = g\rho\lambda'/2\pi\mu,$$

and with the values above we find roughly

$$2\pi\mu/g\rho = (3{\cdot}204)\,10^8;$$

also in centimetres the earth's mean radius $a = (6{\cdot}37)\,10^8$ nearly. Thus a rough estimate of $g\rho/\mu l$ is $2\lambda'/a$.

Now it is clear that λ'/a must be small in order that we may treat the earth as approximately plane. Consequently the roots of my period-equation cannot differ greatly from those given by Rayleigh. Suppose, then, ζ_0 to be a root of Rayleigh's equation, and now put

$$\zeta = \zeta_0 + \delta\zeta;$$

we have then, approximately,

$$2\delta\zeta\left[(1-\zeta_0)^{-\frac{1}{2}} - (2-\zeta_0)\right] - (g\rho/\mu l)\,\zeta_0 = 0;$$

but

$$4\,(1-\zeta_0)^{\frac{1}{2}} = (2-\zeta_0)^2,$$

so that this becomes

$$2\,(\delta\zeta/\zeta_0)\left[4\,(2-\zeta_0)^{-2} - (2-\zeta_0)\right] = g\rho/\mu l.$$

Lord Rayleigh shows that, of the three values of ζ_0 which differ from zero, only one is a solution of the problem, as the other two make the real part of $\sqrt{1-\zeta_0}\;(= s/l)$ negative, which is inadmissible.

This value of ζ_0 is given by him as 0·91275, but my calculations have led me to 0·91262. Taking this value, we have

$$4(2-\zeta_0)^{-2}-(2-\zeta_0) = 2·2956 \text{ nearly},$$

and so
$$\delta\zeta/\zeta_0 = (0·2178)(g\rho/\mu l) \text{ nearly}.$$

The velocity of propagation $V = p/l = (\mu\zeta/\rho)^{\frac{1}{2}}$; thus, if $V_0 = (\mu\zeta_0/\rho)^{\frac{1}{2}}$, we have, to the same degree of accuracy as before,

$$(V-V_0)/V_0 = \delta\zeta/2\zeta_0 = (0·1089)(g\rho/\mu l) = (0·213)(\lambda'/a),$$

with the values of μ, ρ quoted above. Consequently the ratio $(V-V_0)/V_0$ must be a very small fraction in all cases to which this method of approximation can be applied.

After the above solution had been completed, it was pointed out as a means of verification that the period-equation ought to lead to the known value of the velocity of propagation of short waves on water. Making μ small in the equation, I found as the first approximation to ζ the value $g\rho/\mu l$, giving the velocity $(\zeta\mu/\rho)^{\frac{1}{2}} = (g/l)^{\frac{1}{2}} = (g\lambda'/2\pi)^{\frac{1}{2}}$, the well-known form. But this would clearly make $(1-\zeta)^{\frac{1}{2}}$ imaginary, and thus the terms neglected in ζ would have to be complex, leading to a complex period. Since this is inadmissible, it will be advisable to examine the assumptions made above.

It now appears that when s is purely imaginary the values of u, w may include terms in e^{-sz} as well as those in e^{sz}, both sets being finite at $z = -\infty$. This will introduce a new arbitrary constant; and hence also an additional boundary-condition. To express this condition in the simplest way, take the solid as a slab of thickness $2h_0$, where h_0 will be subsequently made infinite. I shall replace the terms in e^{lz}, e^{sz}, &c., by hyperbolic functions, and take the origin as midway between the two faces of the slab. Thus we have

$$p_1/\mu\kappa^2 = A\cosh(lz) + B\sinh(lz),$$

$$u = -\frac{1}{\mu\kappa^2}\frac{\partial p_1}{\partial x} + X_1\cosh(sz) + X_2\sinh(sz),$$

$$w = -\frac{1}{\mu\kappa^2}\frac{\partial p_1}{\partial z} + Z_1\cosh(sz) + Z_2\sinh(sz).$$

Also
$$\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} = 0,$$

so we find
$$ilX_1 + sZ_2 = 0 \quad\text{and}\quad ilX_2 + sZ_1 = 0,$$

the factor $\exp(ipt + ilx)$ in p_1, u, w having been suppressed for brevity.

I take as the boundary conditions

$$p_1 + 2\mu \frac{\partial w}{\partial z} + g\rho w = 0,$$

$$\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} = 0,$$

at each mean free surface $z = h_0$ and $z = -h_0$. Thus we have

$$(\kappa^2 - 2l^2)\left[A\cosh(lh_0) + B\sinh(lh_0)\right] + 2s\left[Z_1\sinh(sh_0) + Z_2\cosh(sh_0)\right]$$
$$+ (g\rho/\mu)\left[Z_1\cosh(sh_0) + Z_2\sinh(sh_0) - l\left\{A\sinh(lh_0) + B\cosh(lh_0)\right\}\right] = 0,$$

from the first condition; and, from the second,

$$-2il^2\left[A\sinh(lh_0) + B\cosh(lh_0)\right] + s\left[X_1\sinh(sh_0) + X_2\cosh(sh_0)\right]$$
$$+ il\left[Z_1\cosh(sh_0) + Z_2\sinh(sh_0)\right] = 0,$$

together with two similar equations which are the same as these when the sign of h_0 is changed. Substituting in the second of these for X_1, X_2 in terms of Z_1, Z_2, we have

$$-2l^3\left[A\sinh(lh_0) + B\cosh(lh_0)\right]$$
$$+ (2l^2 - \kappa^2)\left[Z_1\cosh(sh_0) + Z_2\sinh(sh_0)\right] = 0.$$

Whence, changing the sign of h_0 and adding and subtracting, we have

$$(\kappa^2 - 2l^2) A\cosh(lh_0) + 2sZ_2\cosh(sh_0)$$
$$+ (g\rho/\mu)\left[Z_1\cosh(sh_0) - lB\cosh(lh_0)\right] = 0,$$

$$(\kappa^2 - 2l^2) B\sinh(lh_0) + 2sZ_1\sinh(sh_0)$$
$$+ (g\rho/\mu)\left[Z_2\sinh(sh_0) - lA\sinh(lh_0)\right] = 0,$$

$$(2l^2 - \kappa^2) Z_1\cosh(sh_0) - 2l^3B\cosh(lh_0) = 0,$$

$$(2l^2 - \kappa^2) Z_2\sinh(sh_0) - 2l^3A\sinh(lh_0) = 0;$$

whence we find, eliminating Z_1, Z_2,

$$-(2l^2 - \kappa^2)^2 A\coth(lh_0) + 4l^3sA\coth(sh_0) + (g\rho/\mu)\, l\kappa^2 B\coth(lh_0) = 0,$$

$$-(2l^2 - \kappa^2)^2 B\tanh(lh_0) + 4l^3sB\tanh(sh_0) + (g\rho/\mu)\, l\kappa^2 A\tanh(lh_0) = 0.$$

Eliminating the ratio $A : B$, we have*

$$\left[(2l^2 - \kappa^2)^2\tanh(lh_0) - 4l^3s\tanh(sh_0)\right]$$
$$\times\left[(2l^2 - \kappa^2)^2\coth(lh_0) - 4l^3s\coth(sh_0)\right] = (g\rho l\kappa^2/\mu)^2.$$

* On putting $g = 0$ this reduces to two period-equations which agree with (38), (47) of Lord Rayleigh's paper "On the Vibrations of an Infinite Plate" (*Proc. Lond. Math. Soc.*, Vol. xx.).

Now, consider the limiting form of this period-equation when h_0 is indefinitely increased; coth (lh_0) approaches the limit unity, so also does coth (sh_0), provided that the real part of s is positive. This was the condition previously imposed on s; we shall hold over for the moment the consideration of the case when s is purely imaginary. Our period-equation is thus

$$\left[(2l^2-\kappa^2)^2-4l^3s\right]^2 = (g\rho l\kappa^2/\mu)^2,$$

and it will be seen that we must choose

$$(2l^2-\kappa^2)^2 = 4l^3s+g\rho l\kappa^2/\mu,$$

in order that $z = +h_0$ may be the surface at which the disturbance is finite. This equation is the same as that found previously.

Next take $s = is'$, where s' is supposed real. Then

$$s \tanh(sh_0) = -s' \tan(s'h_0), \quad \text{and} \quad s \coth(sh_0) = s' \cot(s'h_0);$$

these two expressions do not tend to limits independent of s' as h_0 is increased indefinitely. Thus here the period-equation must involve h_0; but we can obtain an approximate solution when μ is small. In this case κ will be large, provided p, ρ be supposed finite. Our equation will then yield approximately

$$(\kappa^4)^2 = (g\rho l\kappa^2/\mu)^2;$$

whence $$\kappa^2 = g\rho l/\mu, \quad \text{or} \quad p^2 = gl,$$

which gives the velocity of wave propagation

$$p/l = (g/l)^{\frac{1}{2}} = (g\lambda'/2\pi)^{\frac{1}{2}}.$$

This is the well-known result for the velocity of propagation on water of waves whose length is short compared with the depth.

It will be noticed that the equation originally found,

$$(2l^2-\kappa^2)^2 = 4l^3s+g\rho\kappa^2l/\mu,$$

always gives a real value of (κ^2/l^2) which lies between 0 and 1. Apparently we should thus have in all cases a real value of s given by this equation; but when the ratio $(g/l) : (\mu/\rho)$ is greater than unity it will be found that this value of s must be negative in order to satisfy the period-equation, and this must be excluded according to the original conditions. Hence, if $(g/l) > (\mu/\rho)$, *i.e.*, if the velocity of propagation due to gravity alone be greater than that of rotational waves, then the more complicated period-equation just found must be used.

It will be observed that in the physical application originally considered $g\rho/\mu l$ was a small fraction, and consequently this point did not present itself.

2. *The effect on the previous problem due to an Ocean of Depth small compared with the Wave-length.*

For simplicity take the depth as uniform, so that the mean boundaries are two infinite horizontal planes. Neglecting viscosity, the motion in the water is irrotational; let ϕ be the velocity-potential with $\partial\phi/\partial s$ as the velocity in the direction ds.

Retaining the axes and notation of the former problem, we write at once, in the solid,

$$p_1/\mu\kappa^2 = Qe^{lz} \exp(ipt+ilx),$$

$$u = -\frac{1}{\mu\kappa^2}\frac{\partial p_1}{\partial x} + Ae^{sz} \exp(ipt+ilx),$$

$$w = -\frac{1}{\mu\kappa^2}\frac{\partial p_1}{\partial z} + Be^{sz} \exp(ipt+ilx),$$

where $$ilA+sB = 0 \quad \text{and} \quad s^2+\kappa^2-l^2 = 0.$$

To determine ϕ we have $\nabla^2\phi = 0$, and hence

$$\phi = \left[C\cosh(lz)+D\sinh(lz)\right] \exp(ipt+ilx).$$

Next we have at $z = 0$ $$\frac{\partial\phi}{\partial z} = \frac{\partial w}{\partial t},$$

which gives $$lD = ip(B-lQ).$$

At the free surface ($z = h_0$, when undisturbed) the pressure must be constant. Thus

$$g\frac{\partial\phi}{\partial z} + \frac{\partial^2\phi}{\partial t^2} = 0 \quad \text{at} \quad z = h_0,$$

or

$$gl\left[C\sinh(lh_0)+D\cosh(lh_0)\right]-p^2\left[C\cosh(lh_0)+D\sinh(lh_0)\right] = 0.$$

Now lh_0 ($= 2\pi h_0/\lambda'$) is supposed to be small; so, approximately,

$$\sinh(lh_0) = lh_0 \quad \text{and} \quad \cosh(lh_0) = 1;$$

whence $$gl(D+Clh_0)-p^2(C+Dlh_0) = 0.$$

At $z=0$ we have the two conditions

$$\mu\left(\frac{\partial u}{\partial z}+\frac{\partial w}{\partial x}\right)=0,$$

$$p_1+2\mu\frac{\partial w}{\partial z}+g(\rho-\rho')w-\rho'\frac{\partial\phi}{\partial t}=0,$$

where, in the second condition, ρ' is the density of the water; and the effect of the water-pressure has been included. Thus

$$-2il^2Q+sA+ilB=0,$$

$$\mu\kappa^2Q+2\mu(-l^2Q+sB)+g(\rho-\rho')(-lQ+B)-ip\rho'C=0.$$

Now we have

$$C(p^2-gl^2h_0)=Dl(g-p^2h_0)=ip(g-p^2h_0)(B-lQ),$$

which gives, approximately,

$$pC=i(B-lQ)(g-p^2h_0),$$

the terms rejected being of order (gl^2h_0/p^2) in comparison with those retained. Now (gl^2h_0/p^2) is $(g\rho/\mu l)(lh_0)(l^2/\kappa^2)$, and, by what has been already proved in the first section, $(g\rho/\mu l)$ is a small fraction, while lh_0 is also small. We thus have, on substituting for A and C in terms of B, Q,

$$2l^3Q-(s^2+l^2)B=0$$

and $$\mu(\kappa^2-2l^2)Q+2\mu sB+(g\rho-\rho'p^2h_0)(B-lQ)=0.$$

These give

$$(2-\kappa^2/l^2)^2-4s/l-(g\rho/\mu l)(\kappa^2/l^2)+(\rho'/\rho)(lh_0)(\kappa^4/l^4)=0.$$

Writing $\zeta=\kappa^2/l^2$ as before, this becomes

$$(2-\zeta)^2=4\sqrt{1-\zeta}+(g\rho/\mu l)\,\zeta-(lh_0\rho'/\rho)\,\zeta^2.$$

Obviously, if $\rho'/\rho=0$, or if $lh_0=0$, we get back to the period-equation found in the first section. Solving by approximation in the same way, we get

$$2(\delta\zeta/\zeta_0)\left[4(2-\zeta_0)^{-2}-(2-\zeta_0)\right]=(g\rho/\mu l)-(lh_0\rho'/\rho)\,\zeta_0,$$

which yields with $\zeta_0=0{\cdot}91262$

$$(V-V_0)/V_0=\delta\zeta/2\zeta_0=(0{\cdot}109)(g\rho/\mu l)-(0{\cdot}099)(lh_0\rho'/\rho).$$

Expressed in terms of the wave-length, with the same values of μ, ρ as used above,

$$(V-V_0)/V_0=(0{\cdot}213)(\lambda'/a)-(0{\cdot}522)(\rho'/\rho)(h_0/\lambda').$$

3. *The Vibrations of an Incompressible Sphere under its own Gravity.*

We shall neglect the central part of gravity in solving for u, v, w, as its only effect is to introduce into the traction on the mean free surface a term which is equal to the weight of the harmonic inequality (Love's *Elasticity*, Vol. I., Art. 173). But we must retain the gravitational potential of the harmonic inequality, which we denote by V, so that V contains terms of the same order as the displacements.

We then have the differential equations of motion

$$\rho \frac{\partial^2 u}{\partial t^2} = \frac{\partial p_1}{\partial x} + \mu \nabla^2 u + \rho \frac{\partial V}{\partial x},$$

$$\rho \frac{\partial^2 v}{\partial t^2} = \frac{\partial p_1}{\partial y} + \mu \nabla^2 v + \rho \frac{\partial V}{\partial y},$$

$$\rho \frac{\partial^2 w}{\partial t^2} = \frac{\partial p_1}{\partial z} + \mu \nabla^2 w + \rho \frac{\partial V}{\partial z},$$

$$0 = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z},$$

the last equation holding on account of the incompressibility, and in the others $\lambda\Delta = p_1$, a finite quantity. It at once appears that

$$\nabla^2 p_1 = 0 \quad \text{since} \quad \nabla^2 V = 0$$

by properties of the potential.

We thus get a set of particular integrals

$$(u_1, v_1, w_1) = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right) \phi,$$

where $\phi = -(p_1 + \rho V)/\mu\kappa^2$,

and, as before, $\kappa^2 = \rho p^2/\mu$.

The complementary solutions are to satisfy

$$(\nabla^2 + \kappa^2)\, u_2 = 0,$$

$$(\nabla^2 + \kappa^2)\, v_2 = 0,$$

$$(\nabla^2 + \kappa^2)\, w_2 = 0,$$

$$\frac{\partial u_2}{\partial x} + \frac{\partial v_2}{\partial y} + \frac{\partial w_2}{\partial z} = 0.$$

I shall now introduce the hypothesis that the displacements are symmetrical round an axis; this is really perfectly general, for by superposition of such solutions we can get every possible case. We reject the displacements called by Prof. Lamb "those of the first class," in which the displacement is in circles round the axis; and proceed at once to those of the second class, where the displacement is in a meridian plane. In displacements of the first class there is no radial motion; consequently the effect of gravity is nil.

For the future u, v will represent the radial and transverse displacements in the directions of r, θ respectively increasing; the notation is that of three-dimensional polars. Then

$$u = \frac{\partial\phi}{\partial r} - \frac{1}{r^2 \sin\theta}\frac{\partial\psi}{\partial\theta},$$

$$v = \frac{1}{r}\frac{\partial\phi}{\partial\theta} + \frac{1}{r\sin\theta}\frac{\partial\psi}{\partial r},$$

where $$\phi = -(p_1+\rho V)/\mu\kappa^2,$$

as before; and the terms in ψ give the complementary solutions u_2, v_2, w_2 of the previous notation. Here ψ satisfies

$$(D+\kappa^2)\,\psi = 0,$$

where $$D \equiv \frac{\partial^2}{\partial r^2} + \frac{\sin\theta}{r^2}\frac{\partial}{\partial\theta}\left(\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\right),$$

which is the operator usually associated with a Stokes' stream-function.

We then find that typical terms in ϕ, ψ are, if P_n is Legendre's coefficient of degree n in $\cos\theta$,

$$\phi = Ar^n P_n e^{ipt},$$

$$\psi = Br^{n+1}\psi_n(\kappa r)\sin\theta\frac{dP_n}{d\theta}e^{ipt},$$

where

$$\psi_n(x) \equiv \left(-\frac{1}{x}\frac{d}{dx}\right)^n\left(\frac{\sin x}{x}\right)$$

$$\equiv \frac{1}{1.3.5\ldots 2n+1}\left[1 - \frac{x^2}{2.2n+3} + \frac{x^4}{2.4.2n+3.2n+5} - \ldots\right],$$

according to the notation of Prof. Lamb (*Hydrodynamics*, Art. 267).

This value of ψ is at once obvious by remembering that

$$D\psi \equiv \frac{r\sin\theta}{\cos\omega}\nabla^2\left(\frac{\psi\cos\omega}{r\sin\theta}\right),$$

ω being the azimuthal angle of polars.

We then have

$$u = nr^{n-1}\left[A+(n+1)B\psi_n(\kappa r)\right]P_n e^{ipt},$$
$$v = r^{n-1}\left[A+\{(n+1)\psi_n(\kappa r)+\kappa r\psi_n'(\kappa r)\}B\right]\frac{dP_n}{d\theta}e^{ipt}.$$

From this value of u we see at once that V is of the form

$$3gu_0 r^n/(2n+1)a^n,$$

where u_0 is the value of u at $r=a$. Also p_1 satisfies $\nabla^2 p_1 = 0$, and so we put $p_1 = \beta u_0 r^n/a^n$, where β is a constant. To determine β, we have

$$\mu\kappa^2\phi = -(p_1+\rho V),$$

and hence $\mu\kappa^2 Aa + n\left[\beta+3g\rho/(2n+1)\right]\left[A+(n+1)B\psi_n(\kappa a)\right] = 0.$

The equations to be satisfied at the surface are now seen to be

$$p_1 + 2\mu\frac{\partial u}{\partial r} + g\rho u = 0,$$

and
$$r\frac{\partial}{\partial r}\left(\frac{v}{r}\right)+\frac{1}{r}\frac{\partial u}{\partial\theta} = 0.$$

Substituting, we get the two conditions

$$n\left[\beta+g\rho+2\mu(n-1)/a\right]\left[A+(n+1)B\psi_n(\kappa a)\right] + 2\mu\kappa n(n+1)B\psi_n'(\kappa a) = 0$$

and $2(n-1)A + B\left[2(n^2-1)\psi_n(\kappa a) - 2\kappa a\psi_n'(\kappa a) - \kappa^2 a^2\psi_n(\kappa a)\right] = 0,$

in the second of which $\psi_n''(\kappa a)$ has been expressed by $\psi_n(\kappa a)$ and $\psi_n'(\kappa a)$. Substituting for β in the first of these, we find

$$2n(n-1)\left[1+g\rho a/\mu(2n+1)\right]\left[A+(n+1)B\psi_n(\kappa a)\right] - \kappa^2a^2A + 2\kappa an(n+1)B\psi_n'(\kappa a) = 0.$$

For brevity put

$$B\psi_n(\kappa a)\equiv C,\quad \kappa a\equiv x,\quad \psi_n'(\kappa a)/\psi_n(\kappa a)\equiv X,\quad ng\rho a/(2n+1)\mu\equiv\theta;$$

then we have

$$2(n-1)(n+\theta)\left[A+(n+1)C\right] - x^2A + 2n(n+1)XxC = 0,$$

while the second surface-condition becomes

$$2(n-1)A+C\left[2(n^2-1)-2Xx-x^2\right]=0.$$

On eliminating the ratio $A:C$ and rejecting some superfluous factors, these give

$$(n+1)(x+2nX)+\left[n+\theta-x^2/2(n-1)\right](x+2X)=0,$$

which may be written

$$\frac{2}{\kappa a}\frac{\psi_n'(\kappa a)}{\psi_n(\kappa a)}+\frac{(2n+1)+ng\rho a/(2n+1)\,\mu-\kappa^2a^2/2\,(n-1)}{n\,(n+2)+ng\rho a/(2n+1)\,\mu-\kappa^2a^2/2\,(n-1)}=0.$$

By putting $g=0$ we arrive at an equation which is the same as that found by Prof. Lamb (*Proc. Lond. Math. Soc.*, Vol. XIII.), when allowance is made for the fact that the value of $\psi_n(\kappa a)$ which is there adopted is $[1.3.5\,...\,2n+1]$ times the value used above.

From the form of the period-equation above it appears that $n=0$, $n=1$ define modes of vibration which are not affected by gravity.

It is of interest to see that the equation just found reduces to the form given previously when we considered an infinite solid with a plane face. We take a, n as both infinite and the harmonics as sectorials; then $2\pi a/n$ = wave-length $=2\pi/l$ of former work; so $n=al$. We must now investigate the form of ψ_n when both n and the argument are very great. I have not succeeded in finding a known form either of ψ_n or of $J_{n+\frac{1}{2}}$ in this case; accordingly I proceed to determine a form by first principles. We have here that, with $\theta=\pi/2$, $(r^n\psi_n)\,e^{in\omega}=U$ is a solution of

$$(\nabla^2+\kappa^2)\,U=0 \quad \text{and} \quad n\omega=la\omega=lx;$$

so we have $\qquad r^n\psi_n=Ae^{s(r-a)},$

where $\qquad s^2=\kappa^2+l^2,$

and the real part of s is positive, so that ψ_n may not be infinite at $r=0$.*

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* Another method is as follows:—$\psi_n(\kappa r)$ is a solution of

$$\frac{d^2y}{dr^2}+\frac{2(n+1)}{r}\frac{dy}{dr}+\kappa^2y=0.$$

Now write $r=a-z$, and suppose z/a to be small; the equation for y will become

$$\frac{d^2y}{dz^2}-2l\frac{dy}{dz}+\kappa^2y=0,$$

and we find $\qquad \psi_n=Ae^{(l-s)z}.$

Differentiate now with respect to r and put $r = a$; we find

$$\frac{n}{a} + \frac{\kappa\psi_n'(\kappa a)}{\psi_n(\kappa a)} = s,$$

i.e.,

$$\frac{\psi_n'(\kappa a)}{\psi_n(\kappa a)} = \frac{s-l}{\kappa}.$$

Also $$ng\rho a/(2n+1)\mu = ng\rho/2\mu l$$

in the limit; thus the period-equation becomes

$$2(s-l)/\kappa^2 a + (2 + g\rho/2\mu l - \kappa^2/2l^2)/n = 0,$$

which is equivalent to

$$4l(s-l) + \kappa^2(4 + g\rho/\mu l - \kappa^2/l^2) = 0,$$

and with $$\zeta = \kappa^2/l^2,$$

as before, we find $(2-\zeta)^2 = 4\sqrt{1-\zeta} + (g\rho/\mu l)\,\zeta$,

the form already given in Section 1.

An additional verification is afforded by taking μ extremely small; we ought then to find one of the periods the same as that given by Kelvin's formula for a gravitating fluid sphere (*Phil. Trans.*, 1863).

Taking μ as very small, p^2 being kept finite, κ will be very great, and then, after multiplying up, the most important terms in the period-equation contain the factor

$$ng\rho a/(2n+1)\mu - \rho p^2 a^2/2(n-1)\mu,$$

and thus the approximate period-equation may be taken as

$$p^2 = 2n(n-1)g/(2n+1)a,$$

which is Kelvin's formula.

I now proceed to the discussion of the roots of the period-equation. We see that $n = 2$ is the first harmonic which gives any difference from the case without gravity; and for the future this alone will be considered. The equation is

$$\frac{2}{x}\frac{\psi_2'(x)}{\psi_2(x)} + \frac{5 + 2\gamma/5 - x^2/2}{8 + 2\gamma/5 - x^2/2} = 0,$$

where γ denotes $g\rho a/\mu$. This can be reduced to the equivalent form

$$\frac{\tan x}{x} = \frac{24(20+\gamma) - 4(23+\gamma)x^2 + 5x^4}{24(20+\gamma) - 12(21+\gamma)x^2 + (25 + 4\gamma/5)x^4 - x^6},$$

remembering that

$$\psi_2(x) = \left[(3-x^2)\sin x - 3x\cos x\right]/x^5.$$

The second form will be seen to reduce to equation (80) of Prof. Lamb's paper previously quoted, on putting $\gamma = 0$.

I originally attempted to solve the equation by assuming a value of γ, and then using the method of trial and error. By this means I calculated the roots marked (A) in the table subjoined. But it soon became clear that, to trace the roots systematically, an easier plan would be to evaluate the values of γ corresponding to assumed values of x. To do this I tabulated $\psi_2(x)$, $\psi_2'(x)$, and deduced the values of $2\psi_2'(x)/x\psi_2(x)$ corresponding to values of x, differing by $\pi/10$. The calculation of γ then offers but little difficulty. The periods were deduced for a sphere of the same size as that of the earth, with the same surface-value of gravity, using the constants

$$a = (6{\cdot}37)\,10^8, \quad g = (9{\cdot}80)\,10^2.$$

I proceed to make a few notes on my results. Taking μ about the rigidity of steel, I calculate that $\gamma = 4{\cdot}32$, which gives a period about 55 minutes, as against 66 minutes found by neglecting gravity; and, with μ about the rigidity of glass, $\gamma = 15$ nearly, which gives the gravest period about 78·5 minutes, as against 120 minutes when gravity is neglected. These are the cases of chief physical interest.

A general description of the variation of the roots with γ may make the table clearer. The lowest root is $({\cdot}8485)\,\pi$ when $\gamma = 0$, according to Prof. Lamb; this root increases with γ, until γ becomes ∞, corresponding to a value of x between $(1{\cdot}65)\,\pi$ and $(1{\cdot}70)\,\pi$, the period at the same time increasing to ∞. After this, until $x = (1{\cdot}7420)\,\pi$, the value of γ is increasing from $-\infty$ to 0, which indicates that these values of x cannot occur in any real case. We now come to a series of second roots of the period-equation; here the value of γ at first varies rapidly for small variations of x, and for a value of x between $(2{\cdot}8)\,\pi$ and $(2{\cdot}8257)\,\pi$ becomes ∞; it then changes very rapidly from $-\infty$ to 0. The third, fourth, and fifth roots have the same general properties, but it is remarkable that, as the order of the root increases, so also does the value of γ requisite to produce a given period. Moreover it appears that, in the higher periods, the variation in the period is slight in comparison with the variation in γ; also, as the order of the period increases, so does the range of values of γ for which the period differs but little from 94 minutes. It is in this sense that we must understand the period 94 minutes, as found by Kelvin's result for a gravitating fluid sphere of the same size and gravity as the earth. Of course every value of γ gives rise to an infinity of periods, and the particular case of $\gamma = \infty$, corresponding to a fluid sphere, gives an infinity of infinite periods and a finite period 94 minutes.

The table contains about two-thirds of the periods I have calculated, those not inserted can be interpolated with sufficient accuracy.

TABLE of Periods of a Gravitating Elastic Sphere, with the same Radius and Surface-Gravity as the Earth, tabulated for varying values of $(g\rho a/\mu)$:—

	$g\rho a/\mu$.	$\kappa a/\pi$.	Period in Minutes.	
L	0	0·8485	0	Lowest roots of the period-equation
	3·8	1·0	52·5	
A (1)	4·32	1·019	55	
	6·8	1·1	63·5	
	10·9	1·2	74	
(2)	13·9	1·3	77	
	18·0	1·4	81·5	
	24·6	1·5	89	
	36·5	1·6	102	
	56·2	1·65	141	
(3)				
L	0	1·7420	0	Second roots
A	15	1·794	58	
	27·3	1·9	74	
	40·9	2·1	82	
	53·2	2·3	85·5	
	66·5	2·5	88	
	84·0	2·7	91	
	118·0	2·8	104	

	$g\rho a/\mu$.	$\kappa a/\pi$.	Period in Minutes.	
(3)				
L	0	2·8257	0	Third roots
	84·5	2·9	85	
	96	3·0	88	
	173	3·8	93	
	193·5	3·85	97	
(3)				
L	0	3·8709	0	Fourth
	153	3·9	85	
	274	4·8	93	
(3)				
L	0	4·8974	0	Fifth
	198	4·9	77	
	436	5·9	95	
(3)				
L	0	5·9148	0	Sixth
	425	6·0	92	

REMARKS.

L indicates that the root is taken from Prof. Lamb's paper.

A these roots were found by a different method from the rest.

(1) μ = that of steel.

(2) $(g\rho a/\mu) = 15$ nearly, if μ be that of glass, so the corresponding period is about 78 minutes.

(3) in each of these intervals, the ratio $(g\rho a/\mu)$ changes very rapidly to $+\infty$, $-\infty$, and zero. It thus appears that certain values of $(\kappa a/\pi)$ cannot appear in the solution of this problem, viz., those which make $(g\rho a/\mu)$ negative. For instance, I find that $(\kappa a/\pi) = 1·7$ makes this ratio negative, and so the values from (1·7) to (1·742) cannot appear.

Notation.—$2\pi/p$ = period, $\kappa^2 = \rho p^2/\mu$.

Three sets of curves are given to indicate graphically the results. In Fig. 1, the curves show the relation between $(g\rho a/\mu)$ and the period; they in all cases should go off to infinity, but owing to difficulties of computation it has not been possible to find the asymptotic

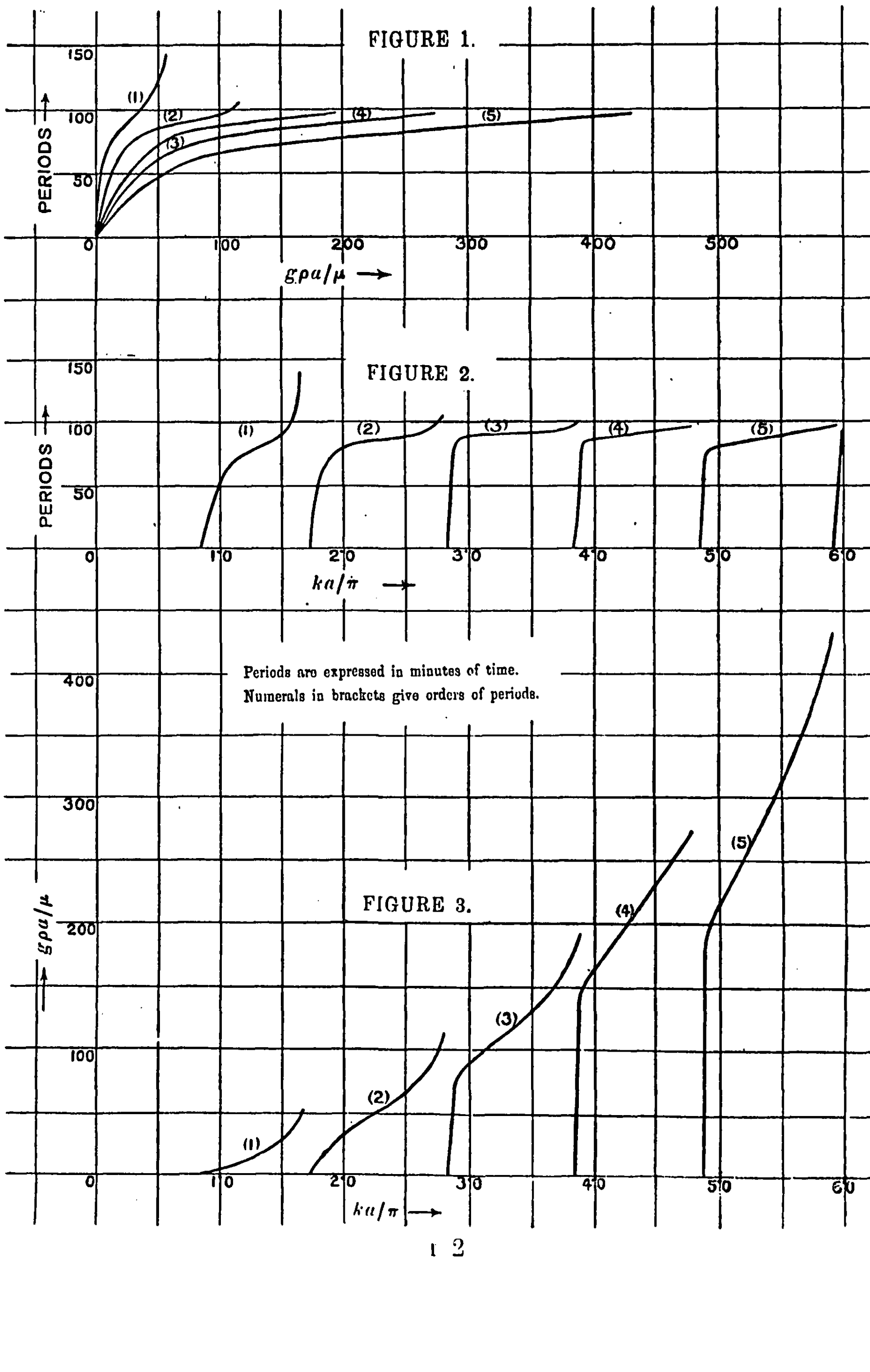
FIGURE 1.
PERIODS →
150
100
50
0
(1)
(2)
(3)
(4)
(5)
100
200
300
400
500
gρa/μ →
FIGURE 2.
PERIODS →
150
100
50
0
(1)
(2)
(3)
(4)
(5)
1·0
2·0
3·0
4·0
5·0
6·0
ka/π →
Periods are expressed in minutes of time.
Numerals in brackets give orders of periods.
FIGURE 3.
gρa/μ →
400
300
200
100
0
(1)
(2)
(3)
(4)
(5)
1·0
2·0
3·0
4·0
5·0
6·0
ka/π →

directions. Moreover curves (4) and (5) pass through the origin, but my calculations do not give the exact shape near the origin, which has been filled in by following the general outline of the first three.

In Figs. 2, 3, the abscissa is $(\kappa a/\pi)$, and the ordinates are the period and $(g\rho a/\mu)$ respectively. Here all the curves go off to positive infinity nearly vertically; and in Fig. 3 they return through negative infinity to the horizontal axis, in a nearly vertical direction. The negative part of the curves is not given, as it can have no physical interpretation, merely arising out of the analytical solutions.

4. *Propagation of Waves in a Thin Shell with Two Infinite Parallel Faces, one of which is rigidly attached to an Infinite Solid.*

The usual equations of small motion of an elastic solid in two dimensions are

$$\left.\begin{aligned} \rho\frac{\partial^2 u}{\partial t^2} &= (\lambda+\mu)\frac{\partial\Delta}{\partial x}+\mu\nabla^2 u \\ \rho\frac{\partial^2 w}{\partial} &= (\lambda+\mu)\frac{\partial\Delta}{\partial z}+\mu\nabla^2 w \end{aligned}\right\},$$

where λ, μ are the elastic constants as defined in Love's *Elasticity*. I take the axis of x to be the direction of propagation of the waves, and that of z perpendicular to the plane boundaries, which will be the two planes $z=0$, $z=h_0$ in equilibrium. I suppose $z=h_0$ to be the free surface, and that h_0 is positive, so that $z=-\infty$ gives the other boundary of the infinite solid.

Also Δ is the dilation and is equal to

$$\frac{\partial u}{\partial x}+\frac{\partial w}{\partial z}.$$

Now assume that u, w both contain the factor $\exp i\,(lx+pt)$; so that $2\pi/l$ is the wave-length, and p/l the velocity of propagation. Then, putting

$$h^2=\rho p^2/(\lambda+2\mu) \quad\text{and}\quad \kappa^2=\rho p^2/\mu,$$

the equations given above will reduce to

$$\left.\begin{aligned} (\nabla^2+\kappa^2)\,u &= \left(1-\frac{\kappa^2}{h^2}\right)\frac{\partial\Delta}{\partial x} \\ (\nabla^2+\kappa^2)\,w &= \left(1-\frac{\kappa^2}{h^2}\right)\frac{\partial\Delta}{\partial y} \end{aligned}\right\},$$

and so $$(\nabla^2+h^2)\,\Delta = 0.$$

Hence we assume that in the infinite solid

$$\Delta/h^2 = Ae^{rz},$$

$$u = -ilAe^{rz}+Xe^{sz},$$

$$w = -rAe^{rz}+Ze^{sz},$$

where $$r^2+h^2 = l^2 = s^2+\kappa^2,$$

and the real parts of r, s must be positive in order that u, w may vanish at $z=-\infty$; the exponential factor $\exp i\,(lx+pt)$ must be understood in all the terms on the right-hand side. From the value of Δ, we have at once $$ilX+sZ = 0.$$

Turning to the shell (whose elastic constants are supposed to be different, say λ', μ', ρ'), it will be seen that we are not restricted to one exponential in z, and for convenience I use two hyperbolic functions.

We may then write, for the displacements in the shell,

$$\Delta'/h'^2 = B\cosh(r'z)+C\sinh(r'z),$$

$$u' = -il\left[B\cosh(r'z)+C\sinh(r'z)\right]+X_1\cosh(s'z)+X_2\sinh(s'z),$$

$$w' = -r'\left[B\sinh(r'z)+C\cosh(r'z)\right]+Z_1\cosh(s'z)+Z_2\sinh(s'z),$$

by using the method of integration given by Lord Rayleigh in his paper (*loc. cit. supra*), where we have put

$$h'^2 = \rho'p^2/(\lambda'+2\mu'),\quad \kappa'^2 = \rho'p^2/\mu',$$

and $$h'^2+r'^2 = l^2 = \kappa'^2+s'^2.$$

Also, since $$\Delta = \frac{\partial u}{\partial x}+\frac{\partial w}{\partial z},$$

we have $$ilX_1+s'Z_2 = 0,\quad ilX_2+s'Z_1 = 0.$$

In virtue of the rigid connexion between the two solids, we have, at $z=0$,

$$u = u' \text{ and } w = w',$$

i.e., $$-ilB+X_1 = -ilA+X,$$

$$-r'C+Z_1 = -rA+Z,$$

Also we have dynamical surface-conditions at $z=0$,

$$\lambda\Delta+2\mu\frac{\partial w}{\partial z}=\lambda'\Delta'+2\mu'\frac{\partial w'}{\partial z},$$

$$\mu\left(\frac{\partial u}{\partial z}+\frac{\partial w}{\partial x}\right)=\mu'\left(\frac{\partial u'}{\partial z}+\frac{\partial w'}{\partial x}\right),$$

and those at $z=h_0$, the free surface, are

$$\lambda'\Delta'+2\mu'\frac{\partial w'}{\partial z}=0,$$

$$\frac{\partial u'}{\partial z}+\frac{\partial w'}{\partial x}=0.$$

The conditions at $z=0$ yield

$$-(2l^2-\kappa^2)A+2sZ=(\mu'/\mu)\left[-(2l^2-\kappa'^2)B+2s'Z_2\right],$$

$$2l^2rA-(2l^2-\kappa^2)Z=(\mu'/\mu)\left[2l^2r'C-(2l^2-\kappa'^2)Z_1\right],$$

in which some reductions have been made by substituting for the X's their values in terms of the Z's.

The conditions at $z=h_0$ give, after expanding and retaining only the first powers of $r'h_0$, $s'h_0$,

$$-(2l^2-\kappa'^2)(B+Cr'h_0)+2s'(Z_2+Z_1s'h_0)=0,$$

$$2l^2r'(C+Br'h_0)-(2l^2-\kappa'^2)(Z_1+Z_2s'h_0)=0,$$

and here again we have substituted for the X's in terms of the Z's.

Thus we have

$$-(2l^2-\kappa'^2)B+2s'Z_2=h_0\left[(2l^2-\kappa'^2)Cr'-2s'^2Z_1\right],$$

$$2l^2r'C-(2l^2-\kappa'^2)Z_1=h_0\left[(2l^2-\kappa'^2)Z_2s'-2l^2r'^2B\right],$$

and, substituting these values in the dynamical conditions at $z=0$, we get

$$-(2l^2-\kappa^2)A+2sZ=(\mu'/\mu)h_0\left[(2l^2-\kappa'^2)Cr'-2s'^2Z_1\right],$$

$$2l^2rA-(2l^2-\kappa^2)Z=(\mu'/\mu)h_0\left[(2l^2-\kappa'^2)s'Z_2-2l^2r'^2B\right].$$

Now it must be observed that we have already rejected squares of h_0, and consequently it will be sufficiently accurate, when reducing the right in the last pair of equations, to entirely reject h_0 in the expressions for Z_1, Z_2.

Thus we take

$$(2l^2-\kappa'^2)\,Cr' -2s'^2 Z_1 = \left[(2l^2-\kappa'^2)^2-4l^2s'^2\right] Cr'/(2l^2-\kappa'^2)$$
$$= Cr'\kappa'^4/(2l^2-\kappa'^2),$$
$$(2l^2-\kappa'^2)\,Z_2 s'-2l^2r'^2 B = \left[(2l^2-\kappa'^2)^2-4l^2r'^2\right] B/2$$
$$= B\left[\kappa'^4-4l^2(\kappa'^2-h'^2)\right]/2;$$

so that $\quad -(2l^2-\kappa^2)A+2sZ = (\mu'/\mu)\,h_0 Cr'\kappa'^4/(2l^2-\kappa'^2),$

$$2l^2rA-(2l^2-\kappa^2)\,Z = (\mu'/\mu)\,h_0 B\left[\kappa'^4-4l^2(\kappa'^2-h'^2)\right]/2.$$

Next we must express C, B in terms of A, Z, and in doing so it will not be necessary to retain h_0, by the argument given before.

Now we have $\quad l^2B-s'Z_2 = l^2A-sZ,$

$$r'C-Z_1 = rA-Z,$$

and, rejecting h_0, $\quad 2s'Z_2 = (2l^2-\kappa'^2)\,B,$

$$(2l^2-\kappa'^2)\,Z_1 = 2l^2r'C;$$

thus we find $\quad \kappa'^2B = 2\,(l^2A-sZ),$

$$\kappa'^2r'C = -(2l^2-\kappa'^2)(rA-Z).$$

As a last reduction I now eliminate Z from these values of B, C by substituting in terms of A, still neglecting h_0. Thence

$$\kappa'^2B = \kappa^2A,$$

and $\quad \kappa'^2r'C/(2l^2-\kappa'^2) = \kappa^2rA/(2l^2-\kappa^2).$

Thus our equations connecting A, Z will become

$$-(2l^2-\kappa^2)A+2sZ = (\mu'/\mu)\,h_0\kappa^2\kappa'^2rA/(2l^2-\kappa^2),$$
$$2l^2rA-(2l^2-\kappa^2)\,Z = (\mu'/\mu)\,h\;\kappa^2A\left[\kappa'^4-4l^2(\kappa'^2-h'^2)\right]/2\kappa'^2.$$

Now, eliminating the ratio $A : Z$, we have

$$4l^2rs-(2l^2-\kappa^2)^2 = (\mu'/\mu)\,h_0\left[\kappa^2\kappa'^2(r+s)-4l^2\kappa^2s\,(1-h'^2/\kappa'^2)\right].$$

Writing, as before, $\quad \kappa^2/l^2 = \zeta,$

and $\quad h^2/\kappa^2 = \tau,\quad h'^2/\kappa'^2 = \tau',$

this becomes

$$4\left[(1-\tau\zeta)(1-\zeta)\right]^{\frac{1}{2}}-(2-\zeta)^2$$
$$= lh_0\zeta\left[(\rho'/\rho)\,\zeta\left\{(1-\zeta)^{\frac{1}{2}}+(1-\tau\zeta)^{\frac{1}{2}}\right\}-4(\mu'/\mu)(1-\tau')(1-\zeta)^{\frac{1}{2}}\right];$$

and, to reduce this further, we may insert on the right values found by equating the left to zero; and it will be found that, if

$$(2-\zeta_0)^2 = 4(1-\zeta_0)^{\frac{1}{2}}(1-\tau\zeta_0)^{\frac{1}{2}},$$

then $$\zeta_0\left[(1-\zeta_0)^{\frac{1}{2}}+(1-\tau\zeta_0)^{\frac{1}{2}}\right] = 4(1-\tau)(1-\zeta_0)^{\frac{1}{2}}.$$

Thus $$4(1-\tau\zeta)^{\frac{1}{2}}(1-\zeta)^{\frac{1}{2}}-(2-\zeta)^2$$

$$= 4lh_0\zeta_0(1-\zeta_0)^{\frac{1}{2}}\left[(\rho'/\rho)(1-\tau)-(\mu'/\mu)(1-\tau')\right]$$

is the new form of our equation.

To solve approximately, write

$$\zeta = \zeta_0+\delta\zeta,$$

and then we have

$$-(\delta\zeta/\zeta_0)\left[(1+\tau-2\tau\zeta_0)(1-\tau\zeta_0)^{-\frac{1}{2}}(1-\zeta_0)^{-\frac{1}{2}}-(2-\zeta_0)\right]$$

$$= 2lh_0(1-\zeta_0)^{\frac{1}{2}}\left[(\rho'/\rho)(1-\tau)-(\mu'/\mu)(1-\tau')\right].$$

If, now, V_0 be the velocity of propagation of these waves in the elastic solid when free from the shell, and $V_0+\delta V$ be the velocity of propagation now found, we have

$$V_0^2 = p_0^2/l^2 = \mu\kappa_0^2/\rho l^2 = \zeta_0\mu/\rho,$$

and $$(V_0+\delta V)^2 = (\zeta_0+\delta\zeta)\,\mu/\rho;$$

thus $$2\delta V/V_0 = \delta\zeta/\zeta_0$$

approximately.

Lord Rayleigh has given the appropriate roots of

$$4(1-\tau\zeta)^{\frac{1}{2}}(1-\zeta)^{\frac{1}{2}} = (2-\zeta)^2$$

for four values of τ; and, using these values of ζ_0, I have found roughly

$$\delta V/V_0 = (0{\cdot}13)\,lh_0\left[(\mu'/\mu)(1-\tau')-\rho'/\rho\right], \quad \tau = 0,$$

$$\delta V/V_0 = (0{\cdot}34)\,lh_0\left[(\mu'/\mu)(1-\tau')-2\rho'/3\rho\right], \quad \tau = 1/3,$$

$$\delta V/V_0 = (0{\cdot}70)\,lh_0\left[(\mu'/\mu)(1-\tau')-\rho'/2\rho\right], \quad \tau = 1/2,$$

$$\delta V/V_0 = (2{\cdot}80)\,lh_0\left[(\mu'/\mu)(1-\tau')-\rho'/4\rho\right], \quad \tau = 3/4.$$

It thus appears that the influence of a thin skin on the velocity of propagation of waves of given wave-length can be only slight; hence any application of Lord Rayleigh's results to determine the velocities of earthquake waves cannot be expected to agree at all closely with the values observed until we know something of the elastic constants of the earth at depths comparable with the wave-length.

Some Multiform Solutions of the Partial Differential Equations of Physical Mathematics and their Applications. By H. S. CARSLAW. Received and read November 10th, 1898. Received, in revised form, January 20th, 1899.

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§ 11. Concluding Remarks.

INTRODUCTION.

This paper owes its origin to my work in the University of Göttingen in the Summer Semester of 1897. The study of a memoir by Professor Sommerfeld, then a *Privat-docent* in that University, suggested to me the possibility, by a somewhat similar method, of obtaining multiform solutions of other differential equations of physical mathematics. Their applications are not far to seek. In conversation with Dr. Sommerfeld on the subject, he told me that this field for research had been pointed out by him at the close of his paper communicated on April 10th of that year to this Society, and then in the press. However, as his time was fully occupied with other work, he most generously urged me to take up the investigation, and offered me his help if at any time the obscurities of the subject left me in difficulty. I desire at the outset to express the sense of my gratitude for this great kindness, and for the readiness with which he removed some of the difficulties which faced me at the beginning of my work.

The papers to which I have referred, and to which fuller reference will be made immediately, contain certain multiform solutions of the equations

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \kappa^2 u = 0$$

and

$$\nabla^2 u = 0.$$

The solutions of the first are applied to the two-dimensional problem of the Diffraction and Reflection of Plane Waves of Light incident on an opaque semi-infinite plane bounded by a straight edge. Of this problem Lord Rayleigh had stated some years before, in the article on "Wave Theory" in the *Encyclopædia Britannica*, that its mathematical difficulties were so formidable that no successful attempt had yet been made to solve it; while again, in his *Theory of Sound*,* he has called attention to the claims of such questions involving diffraction.

The solutions of the second equation find their application in such electrical or hydrodynamical problems as deal with this boundary.

The advance made, in this paper, is the determination of corresponding multiform solutions for the equations

$$\nabla^2 u + \kappa^2 u = 0$$

and

$$\frac{\partial u}{\partial t} = \kappa \nabla^2 u,$$

and their application to problems in the theories of sound and conduction of heat. The solutions obtained are exact, and expressed as definite integrals. The work is thus on a different plan from the most important memoirs of Poincaré, "Sur la Polarisation par Diffraction,"† and Lamb, on "The Reflection and Transmission of Electric Waves by a Metallic Grating,"‡ in both of which the results are obtained in series and by approximation.

* *Theory of Sound*, Vol. II., p. 141, 2nd ed.

† *Acta Mathematica*, Bd. XVI., p. 297; Bd. XX., p. 313.

‡ *Proc. Lond. Math. Soc.*, Vol. XXIX., p. 523.

1. *Extension of the Method of Images.*

The method of images, taken from the domain of optics and applied to the solution of certain problems in statical electricity, was soon extended into other branches of applied mathematics. Instances of its application occur in current electricity, hydrodynamics, and the theory of the conduction of heat. The principle of the method is the symmetrical extension* of the problem involved, from the limited to the unlimited space. Thus the question of the point charge between two planes at right angles is solved by the consideration of the infinite space, and charges at the four symmetrical points. This symmetrical extension is obtained by successively reflecting the original space in the bounding planes. By this means the whole space is simply and completely filled up, while the starting point is reproduced in the final reflection. Similarly with the space between two infinite planes meeting at an angle $\frac{1}{3}\pi$. Here six reflections are required before we return to the region from which we started. Fig. 1 shows the position of the poles for this

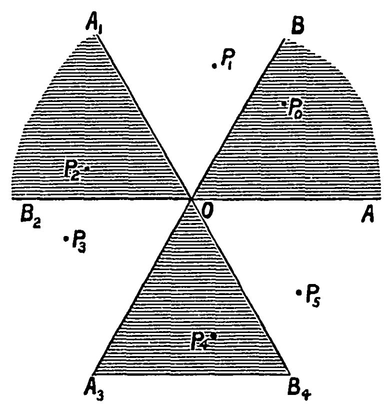

FIG. 1.—Position of charge and images for planes inclined at an angle $\frac{1}{3}\pi$, obtained by successive reflection.

case, the shaded portions being those in which the positive charges are placed.

The result for the angle $\frac{\pi}{m}$ (m a positive integer) follows in the same way.

When we attempt, by this method, to solve the problems in which the angle between the planes is $\frac{n\pi}{m}$ (n, m positive integers), we

* *Analytische Fortsetzung.*

at once meet a difficulty; on reproducing the original space by successive reflection, we have, in the end, more than one pole in the region from which we started. In other words, *the space is not simply filled up, but we are compelled to traverse it n times before we return to our starting point.*

For $\frac{2}{3}\pi$ (Fig. 2) our space is covered two-fold, and we have six reflections. These six are all necessary, as, though the second brings us the complete revolution, the third does not take the starting point back to its original position. The spaces are here shaded, or otherwise, according as the positive or negative charges occur, and we find ourselves with two poles in the region which ought only to possess one.

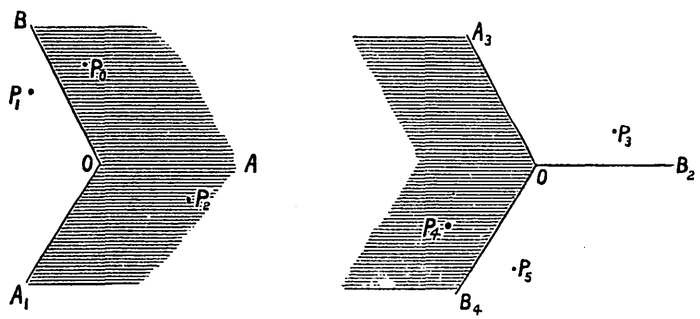

FIG. 2.—Planes inclined at an angle $\frac{2}{3}\pi$.

The method of images, then, seems here to fail.

The first successful attempt to solve any of these problems in mathematical physics appears to have been made in 1894 by Sommerfeld. This was published in a paper, "On the Analytical Theory of the Conduction of Heat,"* *Mathematische Annalen*, Bd. XLV. The ideas there introduced were extended to optics and electricity in a paper in the same journal, Bd. XLVII., "On the Mathematical Theory of Diffraction."† Some of the results of this paper had already been communicated to the Königl. Gesellschaft der Wissenschaften zu Göttingen, and appear in its *Nachrichten* in the communications noted below.‡ The method is somewhat altered, and

* "Zur analytischen Theorie der Wärme-leitung," *Math. Ann.*, Bd. XLV.

† "Mathematische Theorie der Diffraction," *Math. Ann.*, Bd. XLVII. A review of this paper will be found in Voigt's *Kompendium der theor. Physik*, Bd. II., pp. 766–776.

‡ "Zur mathematischen Theorie der Beugungserscheinungen," *Nachrichten von der Königl.-Gesellschaft der Wissenschaften*, Göttingen, 1894. "Zur Integration der partiellen Differential-Gleichung $\nabla^2 u + k^2 u = 0$ auf Riemanns'chen Flächen," ditto, 1895.

brought to bear on potential problems, in a paper "On Multiform Potential in Space"* communicated to this Society.

As used in this last paper, the method may be briefly stated thus. We imagine that we are dealing not with the ordinary space but with a Riemann's space. This is analogous to the Riemann's surface of the theory of functions of a complex variable, and allows us to look upon such many-valued functions in the ordinary space as single-valued in the Riemann's space. In space we shall have "branch-lines"† instead of "branch-points"‡; "branch-membranes"§ for "branch-sections."‖ Every plane section of the Riemann's space will give a Riemann's surface, and the branch-membranes and branch-lines give place to branch-sections and branch-points. We then attempt to find a multiform solution of the differential equation —in this case $\nabla^2 u = 0$—which shall be uniform in the Riemann's space; in other words, our problem, from the pure mathematical point of view, is simply the integration of this partial differential equation in a suitable Riemann's space. Finally, we obtain a function u which has the following properties:—

(i.) *In the Riemann's space outside the branch-lines it is single-valued, finite, and continuous, except in the point P, where it is infinite as $\frac{1}{R}$, R denoting the distance from P to the neighbouring point Q.*

(ii.) *It satisfies the differential equation $\nabla^2 u = 0$ in the whole Riemann's space except in P, and in the branch-lines. In this condition is included the fact that, except in these places, it has finite first and second differential coefficients.*

(iii.) *It vanishes at infinity.*

By taking the images, and considering the space we have to deal with as the Riemann's space, we obtain a potential function with n poles; but, taking the physical space as that given by but one "example"¶ of the Riemann's space, we have the solution of our problem.

For example, take the case solved by Sommerfeld, of the point charge outside a semi-infinite conducting plane at zero potential.

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* "Über verzweigte Potentiale im Raum," *Proc. Lond. Math. Soc.*, Vol. XXVIII.

† *Verzweigungslinien.*

‡ *Verzweigungspunkte.*

§ *Verzweigungsmembranen.*

‖ *Verzweigungsschnitte.*

¶ *Exemplar.*

Here the convenient Riemann's space has the edge of the plane—the axis of z—for branch-line, and the plane itself for branch-membrane. Then, with cylindrical coordinates, we take the range

$$0 < \theta < 2\pi$$

for the physical space; and

$$-2\pi < \theta < 0$$

for the imaginary space, the two building up the twofold Riemann's space.

A solution is found, corresponding to the pole at (r', θ', z'),

$$0 < \theta' < 2\pi,$$

and it is proved that there is only one solution with these properties.

Denoting this by $u(\theta')$,

$$\bar{u} = u(\theta') - u(-\theta')$$

is the required solution of the physical problem.

This paper contains some further extensions of this method.

From the pure mathematical point of view, it deals with the solution on certain Riemann's surfaces, and, in corresponding Riemann's spaces, of the following partial differential equations:—

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \kappa^2 u = 0,$$

$$\nabla^2 u + \kappa^2 u = 0,$$

$$\frac{\partial u}{\partial t} = \kappa \nabla^2 u.$$

From the physical standpoint, it is concerned with problems in which the ordinary image theory fails, and the space concerned has to be looked upon as a Riemann's space (or surface), of which only one example (or sheet) is considered.

2. *Multiform Solution of the Equation* $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} + \kappa^2 u = 0$, *without Infinity.*

The solution discussed in this section forms the subject of the paper on "Diffraction," in *Math. Ann.*, Bd. XLVII., above cited. The results are so important—they solve the problem of the diffraction of electrical waves incident on a semi-infinite plane conducting screen—that it seems worth while to obtain the solution anew, and, in obtaining it, more fully to explain the method hereafter to be employed. Whereas, in these companion papers in *Math. Ann.* and *Gött. Nachrichten*, the solutions of the two-dimensional case are obtained as limiting results from three-dimensional work, just as Bessel's Functions can be deduced from Spherical Harmonics, it is obvious, from the paper on "Potential,"* that nothing hinders the application of its method to the equation

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\kappa^2 u=0.$$

A comparison of the work in this section with that in the paper referred to will show to what an extent the problem is simplified.

In dealing with plane waves we are accustomed to the solution

$$u_0 = e^{ikr\cos(\theta-\theta')}, \tag{1}$$

which represents the disturbance due to waves coming in the direction (θ') from infinity.

If we introduce the complex variable a, and let $f(a)$ stand for any function of a,

$$\int e^{ikr\cos(a-\theta)} f(a)\,da,$$

taken over any path in the a-plane from which infinities are excluded, is also a solution.

Then
$$\frac{1}{2\pi}\int e^{ikr\cos(a-\theta)}\,\frac{e^{ia}}{e^{ia}-e^{i\theta'}}\,da, \tag{2}$$

taken over any circuit in the a-plane, surrounding the point $a=\theta'$ and no other singularity of the integrand, is, by Cauchy's Theorem, the same as u_0; and we have an identical transformation. We may deform this path—provided that in doing so we do not pass over any of the singular points of the integrand.

There is no trouble here about branch-points† because the function to be integrated is uniform.

Since $\cos(a-\theta) = \cos(a-\theta)\cosh b - i\sin(a-\theta)\sinh b$,

when $a = a+ib$, we see that we may deform the path to infinity along the imaginary axis, provided that

for $b=+\infty$, $\sin(a-\theta)$ be negative,

and for $b=-\infty$, $\sin(a-\theta)$ be positive;

* *Cf. Proc. Lond. Math. Soc.*, Vol. XXVIII., p. 429.

† *Verzweigungspunkte.*

for the real part of the exponential is $e^{\kappa r \sin(\alpha-\theta)\sinh b}$, and when $b = +\infty$, $\sinh b = +\infty$, while, when $b = -\infty$, $\sinh b = -\infty$.

Now we may consider, in the first instance, that in the physical space $|\theta-\theta'| < \pi$. This only compels us to make our current co-ordinate θ lie within the range $-(\pi-\theta') < \theta < (\pi+\theta')$.

In Fig. 4 the shaded portions represent the parts of the α-plane where our path may reach infinity. The curve drawn is a possible deformation of the original circuit round $\alpha = \theta'$.

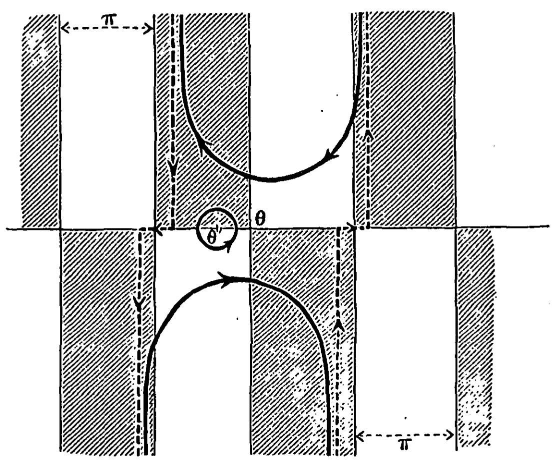

FIG. 3.—Breadth of strip, π; deformation of circuit round $\alpha = \theta'$; $|\theta-\theta'| < \pi$; $n = 1$.

The breadth of the strips is π. The parts of the path made up of straight lines, dotted in the figure, are separated by 2π. This enables us to leave these out of account, owing to the periodicity by 2π of the integrand, and the fact that the corresponding parts are described in opposite directions. The curved parts are to be asymptotic to these lines. It will be easily seen that any other path, starting to the left of the circuit round θ' and ending at distance 2π on the right, will be deformable into this.

These two *curved branches* we call, after Sommerfeld, the path (A) corresponding to the value of θ.

We have here proved that

$$\frac{1}{2\pi}\int e^{ikr\cos(\alpha-\theta)}\frac{e^{i\alpha}}{e^{i\alpha}-e^{i\theta'}}\,d\alpha,$$

* *Verzweigungspunkte.*

over the path (A), is equal to

$$e^{i\kappa r\cos(\theta-\theta')},$$

and this solution is uniform.

We now proceed to the Multiform Solution.

Consider the function defined by

$$u = \frac{1}{2\pi n}\int e^{ikr\cos(\alpha-\theta)}\frac{e^{i\alpha/n}}{e^{i\alpha/n}-e^{i\theta'/n}}\,d\alpha, \tag{3}$$

the integral being taken over the path (A), in the α-plane, which corresponds to the value of the current coordinate θ.

(i.) *This function is a solution of our equation,* since every element of the integrand is a solution, and we have excluded the possibility of infinite values. Also, when $n = 1$, it takes the form

$$u = u_0 = e^{ikr\cos(\theta-\theta')}.$$

(ii.) *The function is multiform, and of period* $2n\pi$, in the ordinary sense; but on the n-sheeted Riemann's surface with the origin as branch-point, and the line $\theta = -(\pi-\theta')$ as branch-section, it is uniform.

To prove this we must again have recourse to Fig. 3.

When we put for θ, $\theta+2\pi$, the alteration on the path (A) is simply to move it parallel to the axis of imaginary quantities through a distance 2π. Thus a change in θ of $2n\pi$, or n revolutions round the axis of z, moves the path (A) along the real axis of α through $2n\pi$.

Now the integrand is periodic in θ and of period $2n\pi$; therefore the values assigned at each point of the path for $\theta+2n\pi$ are the same as those at corresponding points for θ. Thus the value of u for the point (r, θ) is the same as for the point $(r, \theta+2n\pi)$.

(iii.) *It is finite and continuous for all real finite values of* r.

That the function is continuous follows from the fact that a slight change in θ only displaces through an infinitesimal amount the path of the integration, and only alters the integrand infinitesimally. That it is finite follows from the way in which we have chosen the path.

(iv.) *Further, at infinity in the first sheet, i.e.* ($r = \infty$, $|\theta-\theta'| < \pi$), $u = u_0$, *and, in the other sheets,* $u = 0$.

In speaking of the different sheets of the Riemann's surface, we only mean that at each complete revolution on passing over $\theta = -(\pi-\theta')$, or $\pi+\theta$, $3\pi+\theta'$, &c., we are passing from one sheet to the other.

To prove the proposition it is sufficient to note that the paths (A), corresponding to points on the second, third, &c., sheets, may be deformed to the rectilinear portions alone, as no pole of the integrand lies in the portion of the a-plane enclosed. These portions lie wholly in the shaded parts of the plane, and therefore, when $r = \infty$, vanish. On the other hand, for points at infinity on the first sheet, $u = u_0$, since, in addition to the rectilinear portion, our path (A) gives a circuit round the pole $a = \theta'$. This is plain from Fig. 4.

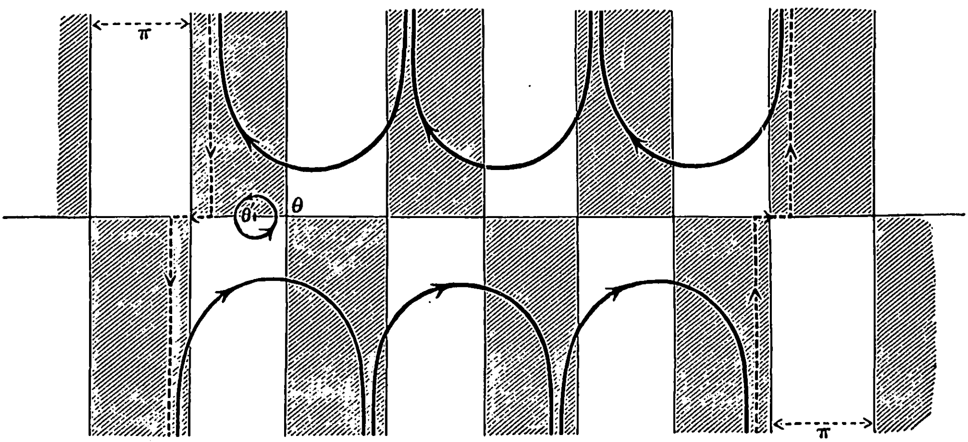

Fig. 4.—Breadth of strip, π; deformation of circuit round $a = \theta'$;
$|\theta - \theta'| < \pi$; $n = 3$.

(v.) *If* $u_1, u_2, u_3, \ldots, u_n$ *be the values of* u *at underlying points on the Riemann's surface—in other words, at the points* (r, θ), $(r, \theta+2\pi)$, *&c.*—

$$u_1 + u_2 + \ldots + u_n = u_0.$$

To prove this we have only to give the accompanying figure containing the paths corresponding to $u_1, u_2, \ldots, u_n$ for $n = 3$. These paths may be joined at $b = \pm\infty$, and we may introduce the rectilinear portions separated by $2n\pi$ (*i.e.*, 6π) without altering the sum $u_1 + u_2 + \ldots + u_n$. The integral over the completed path, by Cauchy's theorem, is the same as u_0, the only pole enclosed being at $a = \theta'$.

To sum up, we have found a function u which has the following properties :—

(i.) *It is a solution of our differential equation*

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \kappa^2 u = 0.$$

(ii.) *It is uniform on the n-sheeted Riemann's surface; or, in other words, periodic of period $2n\pi$ in θ.*

(iii.) *It is finite and continuous for all real values of r.*

(iv.) *It is equal to u_0 at infinity on the first sheet, i.e., when* $|\theta-\theta'| < \pi$ *and* $r=\infty$, $u = e^{i\kappa r\cos(\theta-\theta')}$; *on the other sheets it is zero at infinity, i.e., when* $\pi < |\theta-\theta'| < 3\pi$, $3\pi < |\theta-\theta'| < 5\pi$, ..., $(2n-3)\pi < |\theta-\theta'| < (2n-1)\pi$, *and* $r=\infty$, $u=0$.

(v.) *The n values at the corresponding points on the n sheets satisfy the condition* $u_1+u_2+\ldots+u_n = u_0$.

Calculation of the Value of u for n = 2.

It would be possible to calculate the value of u for a point on any one of the sheets and for any value of n. However, the chief interest of the problem lies in the case $n=2$.

Consider any value of θ', and suppose that we wish to find the values of u at underlying points on the Riemann's surface. We thus allow θ to move from $-(\pi-\theta')$ to $(3\pi+\theta')$. On the first sheet

$$-(\pi-\theta') < \theta < (\pi+\theta');$$

on the second

$$(\pi+\theta') < \theta < (3\pi+\theta').$$

Let the values of u at corresponding points be denoted by u_1 and u_2.

Then

$$u_1+u_2 = u_0.$$

Also, u_2 is easily evaluated. We replace the two curved portions of the path (A) by the rectilinear parts, and these in turn by the lines $a = \theta+\pi$ and $\theta+3\pi$, taken in opposite directions. Thus

$$u_2 = \frac{1}{4\pi}\int_{-\infty}^{\infty} e^{-i\kappa r\cosh b}\left(\frac{1}{1-e^{\frac{1}{2}[i(\theta'-\theta-ib-\pi)]}} - \frac{1}{1-e^{\frac{1}{2}[i(\theta'-\theta-ib-3\pi)]}}\right) idb$$

$$= \frac{1}{4\pi}\int_{-\infty}^{\infty} e^{-i\kappa r\cosh}\left(\frac{1}{1+ie^{\frac{1}{2}[i(\theta'-\theta-ib)]}} - \frac{1}{1-ie^{\frac{1}{2}[i(\theta'-\theta-ib)]}}\right) idb$$

$$= \frac{1}{4\pi}\int_{-\infty}^{\infty} e^{-i\kappa r\cosh b}\,\frac{1}{\cos\frac{1}{2}(\theta'-\theta-ib)}\,db$$

$$= \frac{1}{\pi}\cos\tfrac{1}{2}(\theta-\theta')\int_{0}^{\infty} e^{-i\kappa r\cosh b}\,\frac{\cosh\frac{1}{2}b}{\cosh b+\cos(\theta-\theta')}\,db; \qquad (4)$$

therefore

$$\frac{u_2}{u_0} = \frac{1}{\pi} \cos \tfrac{1}{2} (\theta - \theta') \int_0^\infty e^{-i\kappa r [\cosh b + \cos (\theta - \theta')]} \frac{\cosh \frac{1}{2} b}{\cos b + \cos (\theta - \theta')} db$$

$$= X,\text{* say};$$

therefore

$$\frac{\partial X}{\partial r} = -\frac{i\kappa}{\pi} \cos \tfrac{1}{2} (\theta - \theta') \, e^{-2i\kappa r \cos^2 \frac{1}{2} (\theta - \theta')} \int_0^\infty e^{-2i\kappa r \sinh^2 \frac{1}{2} b} \cosh \tfrac{1}{2} b \, db$$

$$= -\frac{i\kappa}{\pi} \cos \tfrac{1}{2} (\theta - \theta') \, e^{-2i\kappa r \cos^2 \frac{1}{2} (\theta - \theta')} \sqrt{\frac{\pi}{2i\kappa r}};$$

therefore

$$\frac{\partial X}{\partial r} = -\frac{1}{\sqrt{\pi}} e^{\frac{1}{4} i \pi} \frac{\partial}{\partial r} \left[\int_0^{\sqrt{2\kappa r} \cos \frac{1}{2} (\theta - \theta')} e^{-i\lambda^2} d\lambda \right];$$

therefore

$$X = -\frac{1}{\sqrt{\pi}} e^{\frac{1}{4} i \pi} \int_0^{\sqrt{2\kappa r} \cos \frac{1}{2} (\theta - \theta')} e^{-i\lambda^2} d\lambda + X_0, \tag{5}$$

where X_0 is the value of X for $r = 0$.

This is easily found to be $\frac{1}{2}$, *i.e.*,

$$\frac{e^{\frac{1}{4} i \pi}}{\sqrt{\pi}} \int_0^\infty e^{-i\lambda^2} d\lambda, \quad \text{or} \quad \frac{e^{\frac{1}{4} i \pi}}{\sqrt{\pi}} \int_{-\infty}^0 e^{-i\lambda^2} d\lambda.$$

Hence
$$u_2 = u_0 \frac{e^{\frac{1}{4} i \pi}}{\sqrt{\pi}} \int_{-\infty}^{-T} e^{-i\lambda^2} d\lambda, \tag{6}$$

and
$$u_1 = u_0 \frac{e^{\frac{1}{4} i \pi}}{\sqrt{\pi}} \int_{-\infty}^{+T} e^{-i\lambda^2} d\lambda, \tag{7}$$

where
$$T = \sqrt{2\kappa r} \cos \tfrac{1}{2} (\theta - \theta').$$

It is to be noticed that the value u_2 is that at the point $(r, \theta + 2\pi)$ in the second sheet, since u_1 is found for the point (r, θ). Reducing this to the current coordinates, we have on the second sheet, at (r, θ),

$$u = u_0 \frac{e^{\frac{1}{4} i \pi}}{\sqrt{\pi}} \int_{-\infty}^{+T} e^{-i\lambda^2} d\lambda,$$

the same form as for u at the point (r, θ) on the first sheet.

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* *Cf. Math. Ann.*, Bd. XLVII., p. 358.

Thus we have found that

$$u = \frac{e^{i[\kappa r \cos(\theta-\theta') + \frac{1}{4}\pi]}}{\sqrt{\pi}} \int_{-\infty}^{T} e^{-i\lambda^2} d\lambda, \tag{8}$$

where $T = \sqrt{2\kappa r} \cos \frac{1}{2} (\theta - \theta')$,

is a finite and continuous solution of the equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \kappa^2 u = 0,$$

which is periodic in θ and of period 4π; and that at $r = \infty$, when $|\theta - \theta'| < \pi$, it takes the form $e^{i\kappa r \cos(\theta-\theta')}$, while, when

$$\pi < |\theta - \theta'| < 3\pi,$$

it is zero.

Since our solutions are reduced to the same form and are of period 4π, we are able to remove the condition that θ lies between $-(\pi - \theta')$ and $(3\pi + \theta')$, and take the more convenient range from -2π to $+2\pi$. In considering the value at infinity of the function we shall still need to note in which sheet of the surface the point lies; in other words, whether, for the required values of θ and θ', $\cos \frac{1}{2} (\theta - \theta')$ is positive or negative.

This is the solution found by another method by Sommerfeld, in his paper on "Diffraction."

3. *Application to the Theory of Sound.—The Problem of the Diffraction of Plane Waves of Sound incident on a Thin Semi-infinite Rigid Plane bounded by a Straight Edge.*

Taking ϕ for the velocity potential of the medium in which the velocity of sound is V, we know that it satisfies the equation

$$\frac{\partial^2 \phi}{\partial t^2} = V^2 \nabla^2 \phi. \tag{9}$$

To solve this in the case of periodic motion we may assume

$$\phi = \text{real part of } (u \,.\, e^{2i(\pi/\tau)t}), \tag{10}$$

and we find for u the equation

$$\nabla^2 u + \kappa^2 u = 0,$$

where $$\kappa^2 = \frac{4\pi^2}{\tau^2 V^2}.$$

Thus our equation for two-dimensional motion takes the form of that of last section. The solution found is applicable to the case in which we have plane waves of sound coming from the direction $\theta = \theta'$, and incident on the plane, which we take as $\theta = 0$ $(0 < r < \infty)$.

This problem is fully discussed in Sommerfeld's paper.* The waves there are supposed to be electro-magnetic or optical. The solution is obtained by adding† the multiform solutions of period 4π for waves from the directions (θ') and $(-\theta')$; *i.e.*,

$$\bar{u} = \frac{e^{i(\frac{1}{4}\pi)}}{\sqrt{\pi}} \left(e^{i\kappa r \cos(\theta - \theta')} \int_{-\infty}^{\sqrt{2\kappa r}\cos\frac{1}{2}(\theta-\theta')} e^{-i\lambda^2} d\lambda + e^{i\kappa r \cos(\theta+\theta')} \int_{-\infty}^{\sqrt{2\kappa r}\cos\frac{1}{2}(\theta+\theta')} e^{-i\lambda^2} d\lambda \right), \tag{11}$$

where the physical space is taken as given by

$$0 < \theta < 2\pi,$$

and within it $\bar{u}$ satisfies all the conditions.

Sommerfeld finds approximations for the results, when r is great. He proves that the space has to be considered in five sections: namely,

(i.) That from $\theta = 0$ to a parabola with the line $(\pi - \theta')$ as axis, the pole for focus, and extremely small parameter;
(ii.) The area enclosed by this parabola;
(iii.) The area between this parabola and a similar one at $\pi + \theta'$;
(iv.) The area enclosed by this curve; and, lastly,
(v.) That between this curve and $\theta = 2\pi$.

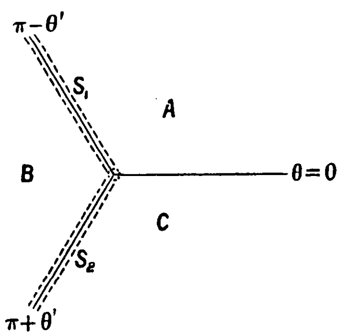

Fig. 5.

* *Math. Ann.*, Bd. XLVII., pp. 368, 369.

† In the sound problem we consider only the case in which the two solutions are added. In the optical there is also a physical interpretation of the results obtained by subtraction.

He finds, when r is very great, that in these divisions A, B, C the following approximations hold:

$$\text{(A)}\quad \phi = \cos 2\pi\left(\frac{r}{\lambda}\cos(\theta-\theta')+\frac{t}{\tau}\right)+\cos 2\pi\left(\frac{r}{\lambda}\cos(\theta+\theta')+\frac{t}{\tau}\right)$$
$$-\frac{1}{4\pi}\left\{\cos\left[2\pi\left(\frac{r}{\lambda}-\frac{t}{\tau}\right)+\frac{\pi}{4}\right]\sqrt{\frac{\lambda}{r}}\left(\frac{1}{\cos\frac{1}{2}(\theta+\theta')}+\frac{1}{\cos\frac{1}{2}(\theta-\theta')}\right)\right\}. \quad (12)$$

$$\text{(B)}\quad \phi = \cos 2\pi\left(\frac{r}{\lambda}\cos(\theta-\theta')+\frac{t}{\tau}\right)$$
$$-\frac{1}{4\pi}\left\{\cos\left[2\pi\left(\frac{r}{\lambda}-\frac{t}{\tau}\right)+\frac{\pi}{4}\right]\sqrt{\frac{\lambda}{r}}\left(\frac{1}{\cos\frac{1}{2}(\theta+\theta')}+\frac{1}{\cos\frac{1}{2}(\theta-\theta')}\right)\right\}. \quad (13)$$

$$\text{(C)}\quad \phi = -\frac{1}{4\pi}\left\{\cos\left[2\pi\left(\frac{r}{\lambda}-\frac{t}{\tau}\right)+\frac{\pi}{4}\right]\sqrt{\frac{\lambda}{r}}\left(\frac{1}{\cos\frac{1}{2}(\theta+\theta')}+\frac{1}{\cos\frac{1}{2}(\theta-\theta')}\right)\right\}. \quad (14)$$

In S_1 and S_2 we have to refer to the integrals.

These results throw light on the physical problem and illustrate the fact that the continued presence of the incident gives rise to reflected and diffracted waves.*

It is interesting to note that there is in the solution, as might be expected, infinite velocity at the sharp edge $r = 0$. This is evident from the value of u in the integral form, and the velocity components will be found to contain $\frac{1}{\sqrt{r}}$.

4. *Multiform Solution of the Equation $\nabla^2 u+\kappa^2 u = 0$, with an Infinity at a Point at a Finite Distance from the Origin.*

In the last two sections we have treated of a finite multiform solution of this equation in two dimensions which may be applied to the problem of plane waves incident on a thin rigid semi-infinite plane bounded by a straight edge. From the physical standpoint we ought now to examine the case of a source of sound, or a vibratory source of any kind, in two dimensions with the same obstacle. We should have the same differential equation to solve, and our solution would need to be of period 4π in θ, and finite and continuous for finite values of r, except at the point where the source is situated, where it

---

* See the remarks on these results, *Math. Ann.*, Bd. XLVII., pp. 369, 370.

must be infinite as $\log x$, when $x = 0$. However, from the pure mathematical point of view, the three-dimensional case is much the simpler. No introduction of Bessel's Functions of the Second Kind is necessary. We shall examine this now, and return to the two-dimensional later.

To speak analytically, we desire a solution of the equation $\nabla^2 u + \kappa^2 u = 0$, with the following properties:—

(i.) *In our n-fold Riemann's space with the axis of z as branch-line, and the plane $\theta = 0$ as branch-membrane, it is to be uniform; in other words, it is to be periodic in θ and of period $2n\pi$.*

(ii.) *It is to be infinite as $\frac{e^{-i\kappa R}}{R}$, when $R = 0$, at the point (r', θ', z') in the first example, where R stands for the distance from (r', θ', z') to the neighbouring point.*

(iii.) *It is to be finite and continuous for all real finite values of r in all the examples, except at the above-mentioned point.*

(iv.) *It is to be zero at infinity.*

The method of obtaining such a solution is perfectly analogous to that employed in § 2, and in Sommerfeld's paper on "Potential." Starting from the solution

$$u_0 = \frac{e^{-ik\sqrt{r^2+r'^2+(z-z')^2-2rr'\cos(\theta-\theta')}}}{\sqrt{r^2+r'^2+(z-z')^2-2rr'\cos(\theta-\theta')}}, \tag{15}$$

we proceed to the integral

$$\frac{1}{2\pi}\int \frac{e^{-ik\sqrt{2rr'[\cosh\alpha_1-\cos(\alpha-\theta)]}}}{\sqrt{2rr'[\cosh\alpha_1-\cos(\alpha-\theta)]}}\,\frac{e^{i\alpha}}{e^{i\alpha}-e^{i\theta'}}\,d\alpha \tag{16}$$

taken round a circuit in the α-plane enclosing $\alpha = \theta'$, and no other singularity, or branch-point, of the integrand.

We have now to deal with branch-points, because the radical sign has brought a multiform function of α into our integrand; further, we have written $\cosh\alpha_1$ for $\frac{r^2+r'^2+(z-z')^2}{2rr'}$.

With the above restrictions, this integral, by Cauchy's Theorem, is the same as u_0.

We can deform the path of integration in the α-plane without affecting the value of the integral, provided that we do not deform it over any of the singular points or branch-points of the function integrated; in this condition is contained the restriction from deforming our path to points where the function would be infinite. Also, since we are dealing with a multiform function of the complex variable a, we must fix the value to be assigned to the function—in other words, the sign of the root—at a particular point of the path, and see that the values we assign to it at all points of the deformed path are those belonging to the "branch" of the function we are following. If we make sure of these things, we may treat the integrand as single-valued, and apply to it Cauchy's Theorem and its extensions. This requires only the definiteness and continuity of the function to be integrated.

Since we are dealing primarily with the ordinary space, we may suppose $|\theta-\theta'| < \pi$, which means that, in the first instance, we think of θ as varying from $-(\pi-\theta')$ to $(\pi+\theta')$, a full range of 2π.

The singularities of the integrand are given by

$$a = 2m\pi + \theta', \quad a = 2m\pi+\theta \pm ia_1$$

(m, any integer), and the latter are branch-points.

The simplest method of determining the continuity of the values of $\sqrt{2rr'[\cosh a_1 - \cos(a-\theta)]}$, which we shall denote by R, is obtained from the consideration of the conformal representation of the a-plane on the R-plane.

Starting with

$$R = +\sqrt{2rr'(\cosh a_1 + \cosh\infty)} = +\infty, \quad \text{for } a = \theta-\pi+i\infty,$$

we proceed through

$$R = +\sqrt{2rr'(\cosh a_1 + \cosh b)}, \quad \text{for } a = \theta-\pi+ib,$$

$$R = +\sqrt{2rr'(\cosh a_1 + 1)}, \quad \text{for } a = \theta-\pi,$$

$$R = +\sqrt{2rr'(\cosh a_1 - 1)}, \quad \text{for } a = \theta,$$

$$R = +\sqrt{2rr'(\cosh a_1 - \cosh b)}, \quad \text{for } a = \theta+ib \ (b<a_1);$$

and, if we took $a = \theta+ia_1$, we should find $R = 0$.*

However, since $a = \theta+ia_1$ is a branch-point, we suppose that a small circuit is described from the point $a = \theta+ib$ $(b<a_1)$ back to the neighbouring point. This alters the branch of the function, and gives us there

$$R = -\sqrt{2rr'(\cosh a_1 - \cosh b)}.$$

* *Cf.* Sommerfeld, *Math. Ann.*, Bd. XLVII., p. 352.

Then, proceeding through the set of values $\alpha = \theta,\ \theta+\pi,\ \theta+\pi+ib,$ $\theta+\pi+i\infty$, we find that in the R-plane the point describes the negative part of the axis of real quantities. Thus the path (p, q, r, s, t, u, v) in the α-plane of Fig. 6 corresponds to the real axis in the R-plane. We should find a similar correspondence from the image of this path in the real axis of the α-plane, and the position taken up by either when instead of θ we have $\theta \pm 2m\pi$. Thus we see that on crossing directly, *i.e.*, without the loop, from one side to the other of any part of these lines in the α-plane, we cross from one side to the other of the real axis of the R-plane, and that without a jump; in other words, we pass from a value of R with an infinitesimal positive or negative imaginary part to one with an infinitesimal negative or positive imaginary part.

To return to the integral (16),

$$u_0 = \frac{1}{2\pi} \int \frac{e^{-i\kappa R}}{R} \frac{e^{i\alpha}}{e^{i\alpha} - e^{i\theta'}} \, d\alpha.$$

We deform our path as in Fig. 6, which must now be explained. It has already been shown that the path (p, q, r, s, t, u, v), and its

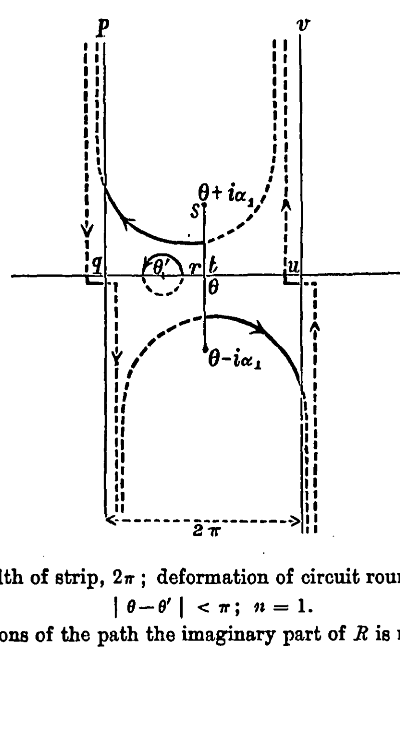

Fig. 6.—Breadth of strip, 2π; deformation of circuit round $\alpha = \theta'$; $|\theta - \theta'| < \pi$; $n = 1$.

In the dotted portions of the path the imaginary part of R is negative.

image in the real axis of α, correspond to the real axis of R; also, that as we pass, in the α-plane, from one side to the other of any part of these broken lines, we pass, in the R-plane, from one side to the other of the axis of real quantities.

Now, from the term $e^{-i\kappa R}$ in our integrand, we must, if we wish to deform the α-path to $\alpha = a \pm i\infty$, ensure that the value of R there has a negative imaginary part.

Starting with the value of R, with positive imaginary part, at a point in the upper part of the figure, our elementary circuit round $\alpha = \theta'$ may be deformed into that composed of the thickly-drawn and dotted lines. The dotted parts denote the portions of the path where the imaginary part of R is negative, and we have made sure that it is negative, by starting with a value of R with positive imaginary part, and remembering that a single crossing of the real axis of R causes the sign of the imaginary part to change. The only places where infinities could arise lie in these portions; so the deformation is permissible.

Making the restriction that the rectilinear portions, those parallel to the imaginary axis, are distant 2π from one another, these portions of our path may be neglected owing to the periodicity of the integrand in α by 2π, and we are left with the identical transformation of u_0 to the integral

$$\frac{1}{2\pi}\int \frac{e^{-i\kappa R}}{R}\,\frac{e^{i\alpha}}{e^{i\alpha}-e^{i\theta'}}\,d\alpha,$$

taken over the two curved portions in the α-plane, which we again denote by the path (A).

So far we have had no reason to think of $(\theta-\theta')$ as not contained in $|\,\theta-\theta'\,| < \pi$.

Proceed now to the Multiform Solution.

Consider the function defined by

$$u = \frac{1}{2n\pi}\int \frac{e^{-i\kappa R}}{R}\,\frac{e^{i\alpha/n}}{e^{i\alpha/n}-e^{i\theta'/n}}\,d\alpha, \tag{17}$$

the integral being taken over the path (A), corresponding to the current coordinate θ.

This function satisfies the differential equation, since every element of the integral is a solution, and we have excluded infinities. Also the same kind of reasoning that was used in § 2 shows that in the Riemann's space with which we are dealing it is uniform; or, in other words, that it is periodic in θ, and of period $2n\pi$. It also shows that when $|\theta-\theta'| < \pi$, and (r, θ, z) approaches (r', θ', z'), the function takes the value $\left(\frac{e^{-i\kappa R}}{R}\right)_{R=0}$; that at the underlying points there is no pole, and that at infinity, in all the "examples," the function vanishes. For all these points, and for the general proposition that

$$u_1+u_2+\ldots+u_n = u_0,$$

it is sufficient simply to refer to Fig. 7, drawn for $n = 3$.

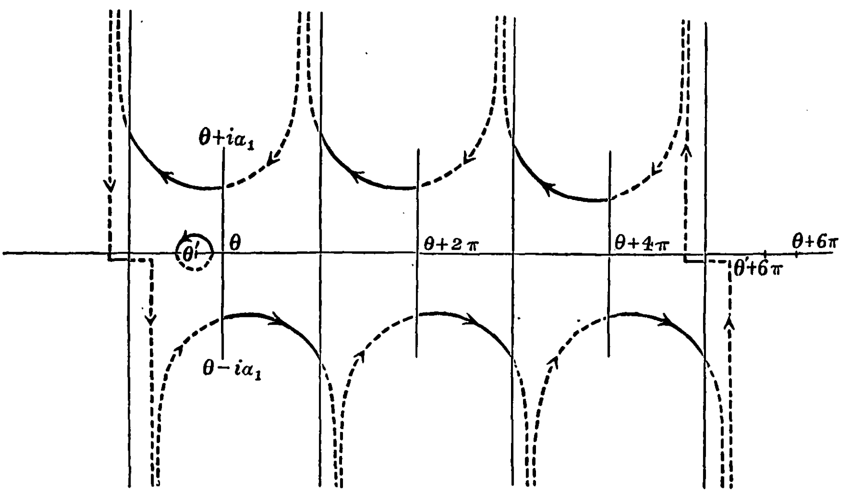

FIG. 7.—Breadth of strip, 2π; deformation of path round $a = \theta'$; $|\theta-\theta'| < \pi$; $n = 3$.
In the dotted portions of the path the imaginary part of R is negative.

To sum up, the function u defined by (17), taken over the proper path (A), corresponding to the θ involved, has the following properties:—

(i.) *It satisfies the equation* $\nabla^2 u+\kappa^2 u = 0$.

(ii.) *It is uniform in the n-fold Riemann's space considered; in other words, it is periodic in* θ *and of period* $2n\pi$.

(iii.) *For all finite values of* (r, θ, z) *it is finite and continuous, unless in the point* (r', θ', z'), *where it possesses a simple pole.*

(iv.) *It vanishes at infinity in all the examples of the Riemann's space.*

(v.) *The n values at corresponding points satisfy the condition*

$$u_1+u_2+\ldots+u_n = u_0.$$

Evaluation of u for $n = 2$.

There would be no difficulty in evaluating u for any value of n. We should need to break up the range into n parts

$$|\theta-\theta'| < \pi, \quad \pi < |\theta-\theta'| < 3\pi, \text{ \&c.},$$

and we should obtain integrals for the function in each of these divisions.

It is important for the physical application to find these values for $n = 2$.

We have $$u_1+u_2 = u_0.$$

Also, as before, we are able to deform the path of u_2 into the two lines $\theta+\pi$, $\theta+3\pi$, and we find

$$u_2 = \frac{1}{4\pi}\int_{-\infty}^{\infty} \frac{e^{-i\kappa\sqrt{2rr'(\cosh a_1+\cosh b)}}}{\sqrt{2rr'(\cosh a_1+\cosh b)}} \frac{1}{\cos\frac{1}{2}(\theta-\theta'+ib)}\,db, \tag{18}$$

i.e.,

$$u_2 = \frac{1}{\pi}\cos\tfrac{1}{2}(\theta-\theta')\int_0^{\infty} \frac{e^{-i\kappa\sqrt{2rr'(\cosh a_1+\cosh b)}}}{\sqrt{2rr'(\cosh a_1+\cosh b)}} \frac{\cosh\frac{1}{2}b}{\cos(\theta-\theta')+\cosh b}\,db. \tag{19}$$

This is the value of u at $\theta+2\pi$ when $|\theta-\theta'| < \pi$, so that, in the second example, we have, for u at (θ),

$$u = -\frac{1}{\pi}\cos\tfrac{1}{2}(\theta-\theta')\int_0^{\infty} \frac{e^{-i\kappa\sqrt{2rr'(\cosh a_1+\cosh b)}}}{\sqrt{2rr'(\cosh a_1+\cosh b)}} \frac{\cosh\frac{1}{2}b}{\cos(\theta-\theta')+\cosh b}\,db, \tag{20}$$

while in the first

$$u = u_0-\frac{1}{\pi}\cos\tfrac{1}{2}(\theta-\theta')\int_0^{\infty} \frac{e^{-i\kappa\sqrt{2rr'(\cosh a_1+\cosh b)}}}{\sqrt{2rr'(\cosh a_1+\cosh b)}} \frac{\cosh\frac{1}{2}b}{\cos(\theta-\theta')+\cosh b}\,db. \tag{21}$$

At first it would appear that there is a discontinuity here at the passage from one space to the other. The following consideration shows that this is not so.

At $\theta = \pi+\theta'-\epsilon$ (ϵ a small positive quantity)

$$u = u_0-\frac{1}{\pi}\sin\frac{\epsilon}{2}\int_0^{\infty} \frac{e^{-i\kappa\sqrt{2rr'(\cosh a_1+\cosh b)}}}{\sqrt{2rr'(\cosh a_1+\cosh b)}} \frac{\cosh\frac{1}{2}b}{-\cos\epsilon+\cosh b}\,db. \tag{22}$$

At $\theta = \pi + \theta' + \epsilon$,

$$u = \frac{1}{\pi}\sin\frac{\epsilon}{2}\int_0^\infty \frac{e^{-i\kappa\sqrt{2rr'(\cosh a_1+\cosh b)}}}{\sqrt{2rr'(\cosh a_1+\cosh b)}}\,\frac{\cosh\frac{1}{2}b}{-\cos\epsilon+\cosh b}\,db. \qquad (23)$$

There must be a discontinuity unless

$$\mathrm{Lt}_{\epsilon=0}\left(\frac{1}{\pi}\sin\frac{\epsilon}{2}\int_0^\infty \frac{e^{-i\kappa\sqrt{2rr'(\cosh a_1+\cosh b)}}}{\sqrt{2rr'(\cosh a_1+\cosh b)}}\,\frac{\cosh\frac{1}{2}b}{-\cos\epsilon+\cosh b}\,db\right) = \frac{1}{2}\,\frac{e^{i\kappa\sqrt{2rr'(\cosh a_1+1)}}}{\sqrt{2rr'(\cosh a_1+1)}}.\text{*}$$

---

* For the following discussion of this integral I am indebted to Prof. Gibson, of the Technical College, Glasgow.

By the substitution used in the text we reduce the expression to

$$\int_0^\infty \phi(x)\frac{dx}{x^2+1},$$

where
$$\phi(x) = \frac{e^{-i\kappa\sqrt{4rr'(\cosh^2\frac{1}{2}a_1+x^2\sin^2\frac{1}{2}\epsilon)}}}{\sqrt{4rr'(\cosh^2\frac{1}{2}a_1+x^2\sin^2\frac{1}{2}\epsilon)}}.$$

Now choose m so that $\tan^{-1} m \overline{\overline{<}} \frac{1}{2}\pi - \epsilon_1$.

Since the integral is convergent, we can choose m, n $(m<n)$ so large that

$$\int_m^n \phi(x)\frac{dx}{x^2+1} < \epsilon_2.$$

If the previous value of m is not large enough to secure this, let it be increased till it does satisfy this condition.

Then $\tan^{-1} m \overline{\overline{<}} \frac{1}{2}\pi - \epsilon_1$, would hold *a fortiori*. Hence we have

$$\int_0^n \phi(x)\frac{dx}{x^2+1} = \int_0^m \phi(x)\frac{dx}{x^2+1} + \int_m^n \phi(x)\frac{dx}{x^2+1}$$
$$= \phi(0)\tan^{-1} m + \int_0^m [\phi(x)-\phi(0)]\frac{dx}{x^2+1} + \int_m^n \phi(x)\frac{dx}{x^2+1}.$$

But we may choose ϵ so that $|\phi(x)-\phi(0)| < \epsilon_3$. This involves that $x^2\sin^2\frac{1}{2}\epsilon$ be very small. Then

$$\int_0^n \phi(x)\frac{dx}{x^2+1} - \tfrac{1}{2}\pi\,\phi(0) \overline{\overline{<}} -\epsilon_1\,\phi(0) + \epsilon_2 + \epsilon_3(\tfrac{1}{2}\pi - \epsilon_1).$$

But
$$\int_0^\infty \phi(x)\frac{dx}{x^2+1} - \int_0^n \phi(x)\frac{dx}{x^2+1}$$

is less than any assignable quantity; therefore we have found that

$$\int_0^\infty \phi(x)\frac{dx}{x^2+1} = \tfrac{1}{2}\pi\,\phi(0).$$

It is obvious that m may be taken at once large enough to satisfy all the conditions required for m and n.

Finally, ϵ may be chosen such that $|\phi(x)-\phi(0)| < \epsilon_3$, $0 \leqq x \leqq m$.

Thus the $\mathrm{limit}_{\epsilon=0}$ of $\sin\frac{1}{2}\epsilon\int_0^\infty \frac{e^{-i\kappa\sqrt{2rr'(\cosh a_1+\cosh b)}}}{\sqrt{2rr'(\cosh a_1+\cosh b)}}\,\frac{\cosh\frac{1}{2}b}{-\cos\epsilon+\cosh b}\,db$ is clearly $\frac{1}{2}\pi\,\phi(0)$.

As a rough proof of this identity, which the physical interpretation of the problem renders necessary, we might adduce the following:—

Put $\sinh \frac{1}{2}b = x \sin \frac{1}{2}\epsilon$, which is possible, as we want the limit of the integral when $\epsilon = 0$, not the value when $\epsilon = 0$.

We then find

$$\sin\frac{\epsilon}{2}\int_0^\infty \frac{e^{-i\kappa\sqrt{2rr'(\cosh a_1+\cosh b)}}}{\sqrt{2rr'(\cosh a_1+\cosh b)}}\,\frac{\cosh\frac{1}{2}b}{-\cos\epsilon+\cosh b}\,db$$

$$=\int_0^\infty \frac{e^{-i\kappa\sqrt{4rr'(\cosh^2\frac{1}{2}a_1+x^2\sin^2\frac{1}{2}\epsilon)}}}{\sqrt{4rr'(\cosh^2\frac{1}{2}a_1+x^2\sin^2\frac{1}{2}\epsilon)}}\,\frac{dx}{x^2+1}.$$

If now we let x approach infinity in such a way that always, in the limit, $x^2\sin^2\frac{1}{2}\epsilon$ may be neglected, this gives in the limit

$$\sin\frac{\epsilon}{2}\int_0^\infty \frac{e^{-i\kappa\sqrt{2rr'(\cosh a_1+\cosh b)}}}{\sqrt{2rr'(\cosh a_1+\cosh b)}}\,\frac{\cosh\frac{1}{2}b}{-\cos\epsilon+\cos(\theta-\theta')}\,db$$

$$=\frac{\pi}{2}\,\frac{e^{-i\kappa\sqrt{2rr'(\cosh a_1+1)}}}{\sqrt{2rr_1(\cosh a_1+1)}}. \qquad (24)$$

Therefore we find that, at the division* between the spaces, the two values of u take the same value $\frac{1}{2}u_0$. We should find a corresponding coincidence at the branch-line in § 2, and in those which follow.

Our solution, then, is one which, in the region contained in the two complete revolutions of θ, from $-(\pi-\theta')$ to $(3\pi+\theta')$, has but one pole, and that at (r', θ', z'). From the periodicity of u by 4π, we may now remove this restriction on the range of θ, and may take it from -2π to $+2\pi$, provided we are careful to use the proper values to be assigned to u at the different points.

It is easy to see that our function has to be considered in three divisions:

$$-2\pi<\theta<-(\pi-\theta'),$$

$$-(\pi-\theta')<\theta<(\pi+\theta'),$$

$$(\pi+\theta')<\theta<2\pi.$$

---

* This is the branch-membrane (*Verzweigungsmembran*).

In the first

$$u = -\frac{1}{\pi}\cos\tfrac{1}{2}(\theta-\theta')\int_0^\infty \frac{e^{-i\kappa\sqrt{2rr'(\cosh a_1+\cosh b)}}}{\sqrt{2rr'(\cosh a_1+\cosh b)}}\,\frac{\cosh\frac{1}{2}b}{\cosh b+\cos(\theta-\theta')}\,db. \tag{25}$$

In the second

$$u = u_0-\frac{1}{\pi}\cos\tfrac{1}{2}(\theta-\theta')\int_0^\infty \frac{e^{-i\kappa\sqrt{2rr'(\cosh a_1+\cosh b)}}}{\sqrt{2rr'(\cosh a_1+\cosh b)}}\,\frac{\cosh\frac{1}{2}b}{\cosh b+\cos(\theta-\theta')}\,db. \tag{26}$$

In the third it takes the same form as in the first.

5. *Application to the Theory of Sound.—The Problem of a Source of Sound in an Infinite Medium containing a Fixed Thin Rigid Semi-infinite Plane bounded by a Straight Edge.*

Using cylindrical coordinates, take the plane as given by $\theta=0$, its edge by the axis of z, and the position of the source by the coordinates $(r', \theta', 0)$.

Then our physical problem quickly reduces to the solution of the equation

$$\nabla^2 u+\kappa^2 u=0,$$

under the following conditions:—

(i.) $0<\theta<2\pi$; *u is to be finite and continuous for finite values of (r, z) except at the point $(r', \theta', 0)$, where it is to take the form $\frac{e^{-i\kappa R}}{R}$, when $R=0$.*

(ii.) *It is to be zero at infinity.*

(iii.) $\frac{1}{r}\frac{\partial u}{\partial \theta}$ *is to vanish at $\theta=0$ and $\theta=2\pi$.*

To obtain this solution we have only to take into consideration the two-fold Riemann's space, with the axis of z as branch-line, and the plane $\theta=0$ as branch-membrane.

We put poles at $(r', \theta', 0)$ and $(r', -\theta', 0)$, and take the physical space as defined by

$$0<\theta<2\pi.$$

Thus $$\bar{u}=u(\theta')+u(-\theta') \tag{27}$$

satisfies all the conditions of the problem.

As remarked above, care has to be taken to choose the proper values for u in this region. The complete revolution is divided into three portions. From $\theta = 0$ to $\theta = (\pi - \theta')$ both values are those in the first space, namely, those given by u_1. From $\theta = (\pi - \theta')$ to $(\pi + \theta')$, we take u_1 for $u(\theta')$, and u_2 for $u(-\theta')$. From $\theta = (\pi + \theta')$ to 2π, both values are in the second space.

Hence our solution takes the following forms :—

$$
\begin{aligned}
\text{(A)}\quad u = {} & \frac{e^{-i\kappa\sqrt{2rr'[\cosh a_1 - \cos(\theta-\theta')]}}}{\sqrt{2rr'[\cosh a_1 - \cos(\theta-\theta')]}} \\
& - \frac{1}{\pi}\cos\tfrac{1}{2}(\theta-\theta')\int_0^\infty \frac{e^{-i\kappa\sqrt{2rr'(\cosh a_1+\cosh b)}}}{\sqrt{2rr'(\cosh a_1+\cosh b)}}\,\frac{\cosh\frac{1}{2}b}{\cos(\theta-\theta')+\cosh b}\,db \\
& + \frac{e^{-i\kappa\sqrt{2rr'[\cosh a_1 - \cos(\theta+\theta')]}}}{\sqrt{2rr'[\cosh a_1 - \cos(\theta+\theta')]}} \\
& - \frac{1}{\pi}\cos\tfrac{1}{2}(\theta+\theta')\int_0^\infty \frac{e^{-i\kappa\sqrt{2rr'(\cosh a_1+\cosh b)}}}{\sqrt{2rr'(\cosh a_1+\cosh b)}}\,\frac{\cosh\frac{1}{2}b}{\cos(\theta+\theta')+\cosh b}\,db\,; \qquad (28)
\end{aligned}
$$

$$
\begin{aligned}
\text{(B)}\quad u = {} & \frac{e^{-i\kappa\sqrt{2rr'[\cosh a_1 - \cos(\theta-\theta')]}}}{\sqrt{2rr'[\cosh a_1 - \cos(\theta-\theta')]}} \\
& - \frac{1}{\pi}\cos\tfrac{1}{2}(\theta-\theta')\int_0^\infty \frac{e^{-i\kappa\sqrt{2rr'(\cosh a_1+\cosh b)}}}{\sqrt{2rr'(\cosh a_1+\cosh b)}}\,\frac{\cosh\frac{1}{2}b}{\cos(\theta-\theta')+\cosh b}\,db \\
& - \frac{1}{\pi}\cos\tfrac{1}{2}(\theta+\theta')\int_0^\infty \frac{e^{-i\kappa\sqrt{2rr'(\cosh a_1+\cosh b)}}}{\sqrt{2rr'(\cosh a_1+\cosh b)}}\,\frac{\cosh\frac{1}{2}b}{\cos(\theta+\theta')+\cosh b}\,db\,; \qquad (29)
\end{aligned}
$$

$$
\begin{aligned}
\text{(C)}\quad u = {} & - \frac{1}{\pi}\cos\tfrac{1}{2}(\theta-\theta')\int_0^\infty \frac{e^{-i\kappa\sqrt{2rr'(\cosh a_1+\cosh b)}}}{\sqrt{2rr'(\cosh a_1+\cosh b)}}\,\frac{\cosh\frac{1}{2}b}{\cos(\theta-\theta')+\cosh b}\,db \\
& - \frac{1}{\pi}\cos\tfrac{1}{2}(\theta+\theta')\int_0^\infty \frac{e^{-i\kappa\sqrt{2rr'(\cosh a_1+\cosh b)}}}{\sqrt{2rr(\cosh a_1+\cosh b)}}\,\frac{\cosh\frac{1}{2}b}{\cos(\theta+\theta')+\cosh b}\,db. \qquad (30)
\end{aligned}
$$

It is easy to see that, at $\theta = 0$ and $\theta = 2\pi$,

$$\frac{\partial u}{\partial \theta} = 0, \quad 0 < r < \infty.$$

The only information we have at the outset from these integrals is that at $(\pi - \theta')$ the part of the disturbance due to the image is the same as if we had at $(2\pi - \theta')$ a source of half the strength. This is deduced from our work above on the continuity of the two expressions. Similarly, that at $(\pi + \theta')$ the effect of the disturbance due to the source at θ' seems diminished to half, while we have the addition of terms which, from the analogy with what happens in the two-dimensional problem, we might suppose due to sources distributed along the edge of the obstacle. On this analogy the terms in (C) will give the diffracted sound waves.

We can, however, find approximations to the values of our function in the different regions of Fig. 8 as we move away from the origin

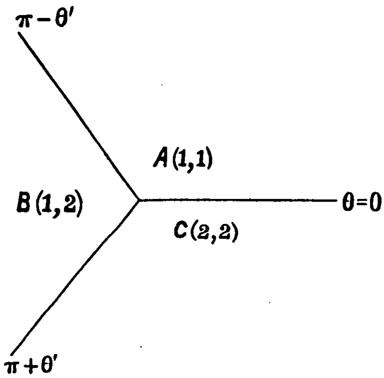

FIG. 8.

and off to infinity. These approximations mark out for us, in some degree, the circumstances of the motion.

Approximation to the Value of u at Infinity.

To obtain this approximation it is necessary to examine the integral

$$\cos\tfrac{1}{2}(\theta-\theta')\int_0^\infty \frac{e^{-i\kappa\sqrt{2rr'(\cosh a_1+\cosh b)}}}{\sqrt{2rr'(\cosh a_1+\cosh b)}}\,\frac{\cosh\frac{1}{2}b}{\cos(\theta-\theta')+\cosh b}\,db.$$

Substitute $$\sinh\tfrac{1}{2}b = x\cos\tfrac{1}{2}(\theta-\theta'),$$

and we obtain

$$\int_0^\infty \frac{e^{-i\kappa\sqrt{4rr'[\cosh^2\frac{1}{2}a_1+\cos^2\frac{1}{2}(\theta-\theta')x^2]}}}{\sqrt{4rr'[\cosh^2\frac{1}{2}a_1+\cos^2\frac{1}{2}(\theta-\theta')x^2]}}\,\frac{dx}{x^2+1},$$

i.e., $$\int_0^\infty \phi(x)\frac{dx}{x^2+1},$$

where $$\phi(x)=\frac{e^{-i\kappa\sqrt{(r+r')^2+z^2+4rr'\cos^2\frac{1}{2}(\theta-\theta')x^2}}}{\sqrt{(r+r')^2+z^2+4rr'\cos^2\frac{1}{2}(\theta-\theta')x^2}}.$$

Now consider
$$\int_0^m \phi(x)\frac{dx}{x^2+1},$$

$$\int_0^m \phi(x)\frac{dx}{x^2+1} = \phi(0)\int_0^m \frac{dx}{x^2+1} + \int_0^m \left[\phi(x)-\phi(0)\right]\frac{dx}{x^2+1}.$$

Let us choose the infinity of m so that when r, z are infinite $\phi(x)-\phi(0)$ may be taken less than any assignable quantity (this involves rm^2 being negligible in comparison with r^2+z^2). Then we have

$$\mathrm{Lt}_{\substack{m=\infty\\ r=\infty\\ z=\infty}} \int_0^m \phi(x)\frac{dx}{x^2+1} = \tfrac{1}{2}\pi\,\phi(0). \qquad (31)$$

The same result follows from the term

$$\cos\tfrac{1}{2}(\theta-\theta')\int_0^\infty \frac{e^{-i\kappa\sqrt{2rr'(\cosh a_1+\cosh b)}}}{\sqrt{2rr'(\cosh a_1+\cosh b)}}\,\frac{\cosh\frac{1}{2}b}{\cos(\theta+\theta')+\cosh b}\,db.$$

Therefore we see that, when we proceed to a great distance from the pole, the disturbance in (A) is the same as that due to a source at $(r', \theta', 0)$, another at $(r', -\theta', 0)$, and a sink at the pole. In (B) we have the remarkable fact that to our approximation the two latter portions of our expression for u disappear; since $\cos\frac{1}{2}(\theta+\theta')$ is negative, and our integral

$$\cos\tfrac{1}{2}(\theta+\theta')\int_0^\infty \frac{e^{-i\kappa\sqrt{2rr'(\cosh a_1+\cosh b)}}}{\sqrt{2rr'(\cosh a_1+\cosh b)}}\,\frac{\cosh\frac{1}{2}b}{\cos(\theta+\theta')+\cosh b}\,db$$

becomes, on substituting $\sinh\frac{1}{2}b = -x\cos\frac{1}{2}(\theta+\theta')$,

$$-\int_0^\infty \phi(x)\frac{dx}{x^2+1}.$$

Thus we are left with the part of the disturbance due to the original source alone. In (C) we trace our disturbance to a source of the same strength at the pole.

6. *The Corresponding Problem in Two Dimensions.—Multiform Solutions of the Equation* $\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\kappa^2 u = 0$, *with Infinity.*

Although the results in the two-dimensional case are not obtained in a workable form owing to the necessity for introducing Bessel's Functions into the integrals, it is interesting to examine the question from the pure mathematical point of view. We shall obtain results which contain the solution of the problem when we have a source of sound in two dimensions in space bounded by the semi-infinite plane with a straight edge.

We propose, then, to discuss the solution of the equation

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\kappa^2 u=0,$$

which is at (x', y') infinite as $\log(R)$, when $R=0$.

Following the method already illustrated, we proceed from the simplest uniform solution of our equation with an infinity as required.

For this case, *i.e.*, in the physical interpretation, when we have a symmetrical disturbance diverging from the pole in an infinite space, our solution is given by Rayleigh, *Theory of Sound*, Vol. II., § 341, where that problem is fully discussed. The solution may be written in either of the two following ways :—

$$Y_0(\kappa r)=\left(\gamma+\log\frac{i\kappa r}{2}\right)\left(1-\frac{\kappa^2 r^2}{2^2}+\frac{\kappa^4 r^4}{2^2 . 4^2}-\&c.\right)+\frac{\kappa^2 r^2}{2^2}S_1-\frac{\kappa^4 r^4}{2^2 . 4^2}S_2+\&c., \tag{32}$$

where $\gamma=$ Euler's constant, and

$$S_m=\frac{1}{1}+\frac{1}{2}+\frac{1}{3}+\ldots+\frac{1}{m};$$

or $$Y_0(\kappa r)=-\left(\frac{\pi}{2i\kappa r}\right)^{\frac{1}{2}}e^{-i\kappa r}\left(1-\frac{1^2}{1 . 8i\kappa r}+\frac{1^2 . 3^2}{1 . 2(8i\kappa r)^2}-\ldots\right). \tag{33}$$

Referring to Gray and Mathews' *Treatise on Bessel's Functions*, p. 22, (50), we find the proof that this value of Y_0 may be written as

$$\left(\gamma+\log\frac{i\kappa r}{2}\right)J_0+4\left(\frac{J_2}{2}-\frac{J_4}{4}+\ldots\right), \tag{34}$$

and we see that this is related to the solution used by J. J. Thomson in his "Recent Researches" (the sign of C being corrected), and by Sommerfeld in *Math. Ann.*, Bd. XLVII., p. 327, by the equation

$$Y_0(x)=-U_0(x)=\frac{i\pi}{2}J_0(x)-K_0(x). \tag{35}$$

Suppose the pole at (r', θ'), and we must change our solution to $Y_0(\kappa R)$, where

$$R=\sqrt{r^2+r'^2-2rr'\cos(\theta-\theta')}.$$

Introduce, as before, the complex variable α, and we have the identical transformation

$$Y_0(\kappa R) = \frac{1}{2\pi}\int Y_0(\kappa R')\,\frac{e^{i\alpha}}{e^{i\alpha}-e^{i\theta'}}\,d\alpha \tag{36}$$

$$\left[R'^2 = r^2+r'^2-2rr'\cos(\alpha-\theta),\ \text{putting } \alpha \text{ for } \theta' \text{ above}\right],$$

the integral being taken round a small circuit in the α-plane enclosing $\alpha = \theta'$, and no other singularity or branch-point of the integrand.

Before discussing the possible deformations of our path we must examine these critical points.

From the equation

$$Y_0(\kappa R') = \left(\gamma+\log\frac{i\kappa R'}{2}\right)\left(1-\frac{\kappa^2R'^2}{2^2}+\frac{\kappa^4R'^4}{2^2.4^2}-\&c.\right)$$
$$+\frac{\kappa^2R'^2}{2^2}S_1-\frac{\kappa^4R'^4}{2^2.4^2}S_2+\&c.,$$

it is evident that the branch-points are given by those of $R'=0$, *i.e.*, by $\alpha = \theta+2m\pi \pm i\alpha_1$, where $\cosh\alpha_1 = \dfrac{r^2+r'^2}{2rr'}$.

In considering the behaviour of $Y_0(\kappa R')$ at infinity we take the second of the forms given. It follows that a condition necessary for a possible deformation of the path to $\alpha = a \pm ib$ ($b = \infty$) is that the imaginary part of R' there be negative. Hence our work is absolutely analogous to that in the former section. We are able to deform the path as given in Fig. 6, and to take as our multiform solution

$$u = \frac{1}{2n\pi}\int Y_0(\kappa R')\,\frac{e^{i\alpha/n}}{e^{i\alpha/n}-e^{i\theta'/n}}\,d\alpha \tag{37}$$

over the path (A) corresponding to the current coordinate (θ).

By means of a discussion similar to that on pp. 137–140, we should find that this function has the following properties :—

(i.) *It satisfies the differential equation*

$$\frac{\partial^2u}{\partial x^2}+\frac{\partial^2u}{\partial y^2}+\kappa^2u = 0.$$

(ii.) *It is uniform on our n-sheeted Riemann's surface ; in other words, it is periodic in* θ, *and of period* $2n\pi$.

(iii.) *It is finite and continuous for all finite values of* (x, y), *except in the point* (x', y'), *where it has a simple pole.*

(iv.) *It vanishes at infinity.*

(v.) *The values at the* n *corresponding points of the Riemann's surface satisfy the equation*

$$u_1 + u_2 + \ldots + u_n = Y_0(\kappa R),$$

where
$$R = \sqrt{r^2 + r'^2 - 2rr' \cos(\theta - \theta')}.$$

Just as before, we could obtain integrals giving the values of u for any assigned integer n.

For the application to the physical problem of a line source parallel to a semi-infinite rigid thin plane, we require the value for $n = 2$, so that the period of the function may be 4π.

We obtain the following expressions for u on the first and second sheets respectively:—

$$u_1 = Y_0\left[\kappa\sqrt{r^2 + r'^2 - 2rr'\cos(\theta - \theta')}\right]$$
$$-\frac{1}{\pi}\cos\tfrac{1}{2}(\theta - \theta')\int_0^\infty Y_0\left[\kappa\sqrt{2rr'(\cosh a_1 + \cosh b)}\right]\frac{\cosh\frac{1}{2}b}{\cos(\theta - \theta') + \cosh b}\,db, \quad (38)$$

and

$$u_2 = -\frac{1}{\pi}\cos\tfrac{1}{2}(\theta - \theta')\int_0^\infty Y_0\left[\kappa\sqrt{2rr'(\cosh a_1 + \cosh b)}\right]\frac{\cosh\frac{1}{2}b}{\cos(\theta - \theta') + \cosh b}\,db. \quad (39)$$

7. *Application to the Theory of Sound.—The Problem in Two Dimensions of a Source outside a Semi-infinite Thin Rigid Plane bounded by a Straight Edge.*

Taking the physical space as defined by $0 < \theta < 2\pi$, the source as at (r', θ'), and the obstacle as $\theta = 0$, $0 < r < \infty$, we obtain the required solution from the function found in the last section.

This solution is $\quad \bar{u} = u(\theta') + u(-\theta'),$

and in evaluating it we have to break up the area into the three portions

$$0 < \theta < \pi - \theta', \quad (A)$$

$$\pi - \theta' < \theta < \pi + \theta', \quad (B)$$

$$\pi + \theta' < \theta < 2\pi. \quad (C)$$

In these we have the following results:—

$$\text{(A)}\quad u = Y_0(\kappa R) - \frac{1}{\pi}\cos\tfrac{1}{2}(\theta-\theta')\int_0^\infty Y_0\left[\kappa\sqrt{2rr'(\cosh\alpha_1+\cosh b)}\right]\frac{\cosh\frac{1}{2}b}{\cos(\theta-\theta')+\cosh b}\,db$$

$$+ Y_0\left[\kappa\sqrt{r^2+r'^2-2rr'\cos(\theta+\theta')}\right]$$

$$-\frac{1}{\pi}\cos\tfrac{1}{2}(\theta+\theta')\int_0^\infty Y_0\left[\kappa\sqrt{2rr'(\cosh\alpha_1+\cosh b)}\right]\frac{\cosh\frac{1}{2}b}{\cos(\theta+\theta')+\cosh b}\,db, \quad (40)$$

$$\text{(B)}\quad u = Y_0(\kappa R) - \frac{1}{\pi}\cos\tfrac{1}{2}(\theta-\theta')\int_0^\infty Y_0\left[\kappa\sqrt{2rr'(\cosh\alpha_1+\cosh b)}\right]\frac{\cosh\frac{1}{2}b}{\cos(\theta-\theta')+\cosh b}\,db$$

$$-\frac{1}{\pi}\cos\tfrac{1}{2}(\theta+\theta')\int_0^\infty Y_0\left[\kappa\sqrt{2rr'(\cosh\alpha_1+\cosh b)}\right]\frac{\cosh\frac{1}{2}b}{\cos(\theta+\theta)+\cosh b}\,db, \quad (41)$$

(C) u = the last two expressions of (B). (42)

Hence, from analogy with what we have found above, we may say that in (A) there exist incident, reflected, and diffracted waves; in (B) incident and diffracted; in (C) diffracted, only; and that they are represented by the respective parts of the above expressions.

8. *Multiform Solution of the Partial Differential Equation of the Theory of the Conduction of Heat in a Body of Uniform Conductivity.—Two-Dimensional Case.*

So far we have been considering the equation which meets us in oscillatory motion, be it in the vibrations of sound, light, or electricity. It is a much simpler problem, though perhaps not so interesting, to examine the corresponding solutions of the equation which forms the basis of the mathematical theory of the conduction of heat, namely,

$$\frac{\partial u}{\partial t} = \kappa\nabla^2 u.$$

As in the potential theory use has been made of the particular solution $\dfrac{1}{r}$ and in that of sound of $\dfrac{e^{-i\kappa r}}{r}$ and $e^{i\kappa r\cos(\theta-\theta')}$, so here we start from the distribution of temperature in an infinite solid of uniform conductivity, due to a unit quantity of heat, placed at the time $t=0$ at the point (x', y', z') and left to diffuse.

The temperature at (x, y, z) at time t is given by

$$u = \frac{1}{2^3(\pi\kappa t)^{\frac{3}{2}}}\,e^{-[(x-x')^2+(y-y')^2+(z-z')^2]/4\kappa t} \quad (43)$$

This synthetical method of dealing with the subject has been used by Kelvin,* Hobson,† Bryan,‡ and Sommerfeld.§

In this section, and in those which follow, I propose to find solutions suitable for the application of this method to cases where the ordinary image theory fails; that is, to those where we must imagine not the ordinary, but a Riemann's, space to be that in which we desire a solution of the equation.

We begin with the two-dimensional problem, and start from the solution

$$u_0 = \frac{1}{t} e^{-[(x-x')^2+(y-y')^2]/4\kappa t} = \frac{1}{t} e^{-[r^2+r'^2-2rr'\cos(\theta-\theta')]/4\kappa t}, \tag{44}$$

which differs by a constant multiplier from the temperature due to a unit source of heat.

Introduce the complex variable α, and we have the identical transformation

$$u_0 = \frac{1}{2\pi} \int \frac{e^{-[r^2+r'^2-2rr'\cos(\alpha-\theta)]/4\kappa t}}{t} \cdot \frac{e^{i\alpha}}{e^{i\alpha}-e^{i\theta'}} d\alpha, \tag{45}$$

the integral being taken over a path in the α-plane, enclosing $\alpha = \theta'$, and no other singularity of the integrand.

On these conditions

$$u_0 = \frac{1}{2\pi t} e^{-(r^2+r'^2)/4\kappa t} \int e^{rr'/2\kappa t \cos(\alpha-\theta)} \frac{e^{i\alpha}}{e^{i\alpha}-e^{i\theta'}} d\alpha. \tag{46}$$

The only ways in which singularities can occur are from the poles $\alpha = 2m\pi+\theta'$, and the infinities of $e^{rr'/2\kappa t \cos(\alpha-\theta)}$. On putting $\alpha = a+ib$, since

$$\cos(\alpha-\theta) = \cos(a-\theta)\cosh b - i\sin(a-\theta)\sinh b,$$

we see that, when $b = \pm\infty$, $\cos(a-\theta)$ must be negative, or an infinite value will be given to the integrand.

Hence, in deforming the path to $b = \pm\infty$, we must take care to have α in such a region that $\cos(a-\theta)$ be negative.

The shaded portions of Fig. 9 represent such parts of the α-plane, and, taking $|\theta-\theta'| < \pi$, the circuit round $\alpha = \theta'$ may be deformed

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* *Math. and Physical Papers*, Vol. II., LXXII., "Compendium of the Fourier Mathematics."

† *Proc. Lond. Math. Soc.*, Vol. XIX., "Synthetic Solutions in the Theory of Heat."

‡ *Proc. Lond. Math. Soc.*, Vol. XIX., "An Application of the Method of Images to the Theory of Heat."

§ *Math. Ann.*, Bd. XLV., "Zur analytischen Theorie der Wärme-leitung."

into that there given;* the new path being composed of two curved parts extending to infinity and two rectilinear parts.

These rectilinear parts—dotted in the figure—are supposed drawn at distance 2π from one another, and therefore the portions of the integral contributed by these, taken in opposite directions, disappear owing to the periodicity of the integrand. We are left with the integral over the two curved portions, which we, as before, denote by the integral over the path (A). It is to be noted that, as the function

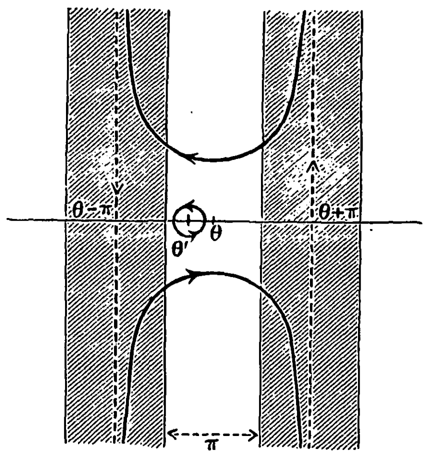

FIG. 9.—Breadth of strip, π; deformation of circuit round $\alpha = \theta'$;
$|\theta-\theta'| < \pi$; $n = 1$.
In the shaded portions $\cos(\alpha-\theta)$ has a negative real part.

to be integrated is uniform and has no branch-points, the question of the deformation of the path is much simpler here than in the former problems.

We now obtain the Multiform Solution.

Consider the integral

$$u = \frac{1}{2n\pi}\,\frac{e^{-(r^2+r'^2)/4\kappa t}}{t}\int e^{rr'/2\kappa t\cos(\alpha-\theta)}\,\frac{e^{i\alpha/n}}{e^{i\alpha/n}-e^{i\theta'/n}}\,d\alpha, \tag{47}$$

taken over the path (A), corresponding to the value of the current coordinate θ; we have given up the restriction

$$|\theta-\theta'| < \pi.$$

* It is not necessary that θ' lie on the unshaded portion. It must lie, in the first instance, between the two lines $\alpha = \theta\pm\pi$.

This function u is a solution of our differential equation, as every element of the integral is a solution, and infinite values are excluded from the path.

It is also periodic in θ and of period $2n\pi$; or, in other words, on the n-sheeted Riemann's surface, with the line from the origin to infinity in the direction $(\pi+\theta')$ as branch-section, the function is uniform.

The proof of this is exactly similar to that of the former sections. Changing the value of θ by $2n\pi$ simply moves the path through a distance $2n\pi$. The value of the integrand at each point of the new path is the same as the value at the corresponding point of the old, because of its periodicity by $2n\pi$ in a. Hence the above result.

When $t = 0$, the value of u vanishes, unless at the point (r', θ'), where it takes the form

$$\mathrm{Lt}_{\substack{r=r'\\ \theta=\theta'\\ t=0}} \left(\frac{e^{-[r^2+r'^2-2rr'\cos(\theta-\theta')]/4\kappa t}}{t}\right).$$

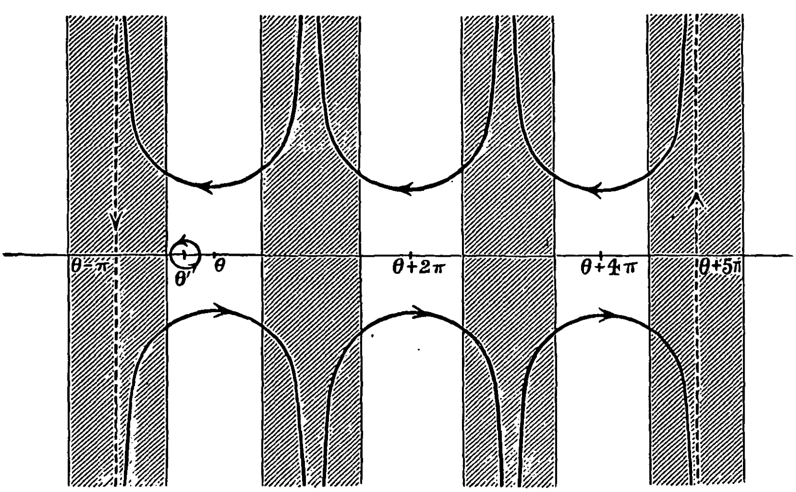

FIG. 10.—Breadth of strip, π; deformation of circuit round $a = \theta'$; $|\theta-\theta'| < \pi$; $n = 3$.

To prove this it is simplest to consider Fig. 10, where we have taken $n = 3$, and have drawn the curves for a point θ on the first sheet, *i.e.*, when $|\theta-\theta'| < \pi$, and for the underlying points on the other two sheets, *i.e.*, for the points $\theta+2\pi$, $\theta+4\pi$. For points on the second and third sheets our path (A) can be replaced by the rectilinear path over the two lines distant by 2π (dotted in figure); and these, being completely in the shaded portion, vanish when $t = 0$, since every element of the integrand then vanishes. For points on the first sheet we have, in addition to the straight lines, to take the circuit round the pole $\alpha = \theta'$; and hence in the first sheet

$$u = u_0 = \frac{e^{-[r^2+r'^2-2rr'\cos(\theta-\theta')]/4\kappa t}}{t}, \quad \text{when } t = 0. \tag{48}$$

This vanishes, unless at the point (r', θ').

Hence we see that, for finite values of r, the integral is zero, for $t = 0$, unless at the point (r', θ'), when it takes the form

$$\mathrm{Lt}_{\substack{t=0\\R=0}} \left(\frac{e^{-(R^2/4\kappa t)}}{t}\right).$$

The term $e^{-(r^2/4\kappa t)}$ causes the integral to vanish at infinity on all the sheets.

Finally, we have the relation between the values of u at underlying points on the surface at any time. This is proved, just as before, from Fig. 10, and is expressed by the equation

$$u_1 + u_2 + \ldots + u_n = u_0.$$

To sum up, the function u just found has the following properties :—

(i.) *It is a solution of the equation*

$$\frac{\partial u}{\partial t} = \kappa \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right).$$

(ii.) *It is uniform on the n-sheeted Riemann's surface considered; in other words, it is of period* $2n\pi$ *in* θ.

(iii.) *On the first sheet of the surface, i.e., when* $|\theta - \theta'| < \pi$, $u = u_0$, *when* $t = 0$; *on the other sheets* $u = 0$. *At the point* (r', θ'), u *takes the form*

$$\left(\frac{e^{-(R^2/4\kappa t)}}{t}\right)_{\substack{R=0\\t=0}}.$$

(iv.) *It vanishes at infinity on all the sheets.*

(v.) *The values at the corresponding points on the different sheets satisfy the equation*

$$u_1 + u_2 + \ldots + u_n = u_0.$$

Evaluation of u for n = 2.

The case to which we desire to apply our multiform solution is that in which $n = 2$.

As before, $$u_1 + u_2 = u_0 = \frac{e^{-[r^2+r'^2-2rr'\cos(\theta-\theta')]/4\kappa t}}{t}.$$

Also we can deform the path (A), for u_2, into the two lines $a = \theta+\pi$, $a = \theta+3\pi$, taken in opposite directions, and we obtain

$$u_2 = \frac{1}{4\pi t} e^{-(r^2+r'^2)/4\kappa t} \int_{-\infty}^{\infty} e^{-(rr'/2\kappa t)\cosh b} \frac{1}{\cos\frac{1}{2}(\theta-\theta'-ib)} db. \tag{49}$$

Let us write C and c for

$$\frac{1}{\pi t} e^{-(r^2+r'^2)/4\kappa t} \text{ and } \frac{rr'}{2\kappa t}.$$

Then $u_2 = C\cos\frac{1}{2}(\theta-\theta') \int_0^{\infty} e^{-c\cosh b} \frac{\cosh\frac{1}{2}b}{\cos(\theta-\theta')+\cosh b} db$;

and

$$\frac{u_2}{u_0} = \frac{1}{\pi}\cos\frac{1}{2}(\theta-\theta') \int_0^{\infty} e^{-c[\cosh b+\cos(\theta-\theta')]} \frac{\cosh\frac{1}{2}b}{\cos(\theta-\theta')+\cosh b} db$$

$$= X, \text{ say};$$

therefore

$$\frac{\partial X}{\partial r} = -\frac{c}{r\pi}\cos\frac{1}{2}(\theta-\theta') e^{-2c\cos^2\frac{1}{2}(\theta-\theta')} \int_0^{\infty} e^{-2c\sinh^2\frac{1}{2}b}\cosh\frac{1}{2}b\, db,$$

i.e., $$\frac{\partial X}{\partial r} = -\frac{1}{2}\sqrt{\frac{r'}{\pi r\kappa t}}\cos\frac{1}{2}(\theta-\theta') e^{-(rr'/2\kappa t)\cos^2\frac{1}{2}(\theta-\theta')};$$

therefore $$\frac{\partial X}{\partial r} = -\frac{1}{\sqrt{\pi}}\frac{\partial}{\partial r}\int_0^{\sqrt{rr'/\kappa t}\cos\frac{1}{2}(\theta-\theta')} e^{-\lambda^2} d\lambda,$$

therefore $$X = -\frac{1}{\sqrt{\pi}}\int_0^{T} e^{-\lambda^2} d\lambda + X_0,$$

where $$T = \sqrt{\frac{rr'}{\kappa t}}\cos\frac{1}{2}(\theta-\theta'),$$

and X_0 is the value of X when $r = 0$.

It is easy to show that $X_0 = \frac{1}{2}$. ...

Hence
$$u_2 = u_0\left(\tfrac{1}{2} - \frac{1}{\sqrt{\pi}}\int_0^T e^{-\lambda^2}\,d\lambda\right)$$
$$= \frac{1}{\sqrt{\pi}}\,u_0\int_{-\infty}^{-T} e^{-\lambda^2}\,d\lambda; \tag{50}$$

and
$$u_1 = u_2 - u_0$$
$$= \frac{1}{\sqrt{\pi}}\,u_0\int_{-\infty}^{T} e^{-\lambda^2}\,d\lambda. \tag{51}$$

Remembering that this expression for u_2 is that for u at the point $(r,\ \theta+2\pi)$, when $|\ \theta-\theta'\ | < \pi$, we obtain for u on the second sheet at the point $(r,\ \theta)$ the same form
$$\frac{1}{\sqrt{\pi}}\,u_0\int_{-\infty}^{T} e^{-\lambda^2}\,d\lambda$$
as for u on the first sheet.

We have thus found a function
$$u = \frac{1}{\sqrt{\pi}}\,\frac{e^{-[r^2+r'^2-2rr'\cos(\theta-\theta')]/4\kappa t}}{t}\int_{-\infty}^{\sqrt{rr'/\kappa t}\cos\frac{1}{2}(\theta-\theta')} e^{-\lambda^2}\,d\lambda, \tag{52}$$
with the following properties :—

(i.) *It is a solution of the equation*
$$\frac{\partial u}{\partial t} = \kappa\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right).$$

(ii.) *On the Riemann's surface considered it is uniform ; or, in other words, it is periodic in θ, and of period 4π.*

(iii.) *On this surface it has only the one pole, and that at the point $(r',\ \theta')$, at which point u takes the value u_0, while at all other points u vanishes for $t = 0$.*

(iv.) *When $r = \infty$, u vanishes.*

Since the function is periodic and of period 4π, there is no reason why we should retain the range
$$-(\pi-\theta') < \theta < (3\pi+\theta').$$
We may take any more suitable one with 4π as its magnitude, and the simplest is
$$-2\pi < \theta < 2\pi.$$

In this region our function u would have but the one pole, and would satisfy the conditions above, care being taken to discriminate between the sheets of the surface; in other words, to choose the proper value of u for the point considered.

9. *Application to the Theory of the Conduction of Heat.—The Problem of an Instantaneous Line Source in an Infinite Body of Uniform Conductivity* κ *in which there is a Semi-infinite Plane bounded by a Straight Edge: the Plane either* (i.) *kept always at Zero Temperature, or* (ii.) *coated in such a way that no Transference of Heat is possible across it.*

Taking a plane normal to the line as the plane $z = 0$, our problem is one in two dimensions. With the origin at the edge, the given plane as $\theta = 0$, and the line source passing through (r', θ'), we are able at once to apply the solution of the last section. We consider the physical space as defined by $0 < \theta < 2\pi$, and we introduce the imaginary space $-2\pi < \theta < 0$.

Then
$$\bar{u} = u(\theta') \mp u(-\theta') \quad (0 < \theta' < 2\pi) \tag{53}$$
are the solutions corresponding to the two cases.

In the space $0 < \theta < 2\pi$, $\bar{u}$ is zero at time $t = 0$ for all values of r, except at the point (r', θ'), where it takes the form
$$\left(\frac{e^{-(R^2/4\kappa t)}}{t}\right)_{\substack{t=0 \\ R=0}}.$$

Further, at infinity $\bar{u} = 0$.

The symmetry of the expression shows us that the boundary conditions are satisfied at $\theta = 0$ and $\theta = 2\pi$.

This is clear when we note that
$$\bar{u} = \frac{1}{\sqrt{\pi}\, t}\left(e^{-(R^2/4\kappa t)} \int_{-\infty}^{\sqrt{rr'/\kappa t}\cos\frac{1}{2}(\theta-\theta')} e^{-\lambda^2}\, d\lambda \mp e^{-(R'^2/4\kappa t)} \int_{-\infty}^{\sqrt{rr'/\kappa t}\cos\frac{1}{2}(\theta+\theta')} e^{-\lambda^2}\, d\lambda\right), \tag{54}$$
where
$$R^2 = (x-x')^2 + (y-y')^2,$$
$$R'^2 = (x-x')^2 + (y+y')^2.$$

In the first case $\bar{u} = 0$, when $\theta = 0$ and $\theta = 2\pi$.

In the second $\dfrac{\partial \bar{u}}{\partial \theta} = 0$, when $\theta = 0$ and $\theta = 2\pi$.

The pole in the space $-2\pi < \theta < 0$, does not affect the validity of the result, as we may fix upon one complete revolution about the axis of z as defining absolutely and covering wholly the range of points entering into the space or body considered.

In the paper "On Conduction of Heat" in *Math. Ann.*, Bd. XLV., this result is quoted by Sommerfeld, and he states that it had been obtained by him after a somewhat laborious calculation from the suitable expression in Bessel's functions

$$u = \cos \tfrac{1}{2}n(\theta-\theta') \int_{-\infty}^{\infty} e^{-\kappa\lambda^2 t} \sum_{-\infty}^{\infty} J_{\frac{1}{2}n}(\lambda r)\, J_{\frac{1}{2}n}(\lambda r')\, \lambda\, d\lambda. \tag{55}$$

The importance of the method here developed is that, as will be shown immediately, there is no difficulty in at once extending the results to the three-dimensional case. Also it places these problems on the same level with those in sound, light, &c., and the extensions to cases in which the physical conditions are different will find their application at once in the conduction of heat.

10. *Multiform Solution of the Equation* $\dfrac{\partial u}{\partial t} = \kappa \nabla^2 u$.

The work here follows the same lines as in the two-dimensional case.

We start from the particular solution

$$u_0 = \frac{1}{t^{\frac{3}{2}}}\, e^{-[(x-x')^2+(y-y')^2+(z-z')^2]/4\kappa t}; \tag{56}$$

or, in cylindrical coordinates,

$$u_0 = \frac{1}{t^{\frac{3}{2}}}\, e^{-[r^2+r'^2+(z-z')^2-2rr'\cos(\theta-\theta')]/4\kappa t}.$$

Then we obtain the identical transformation

$$u_0 = \frac{1}{2\pi}\, \frac{e^{-[r^2+r'^2+(z-z')^2]/4\kappa t}}{t^{\frac{3}{2}}} \int e^{(rr'/2\kappa t)\cos(a-\theta)}\, \frac{e^{ia}}{e^{ia}-e^{i\theta'}}\, da \tag{57}$$

over a circuit in the a-plane, enclosing $a = \theta'$ and no other singularity of the integrand.

This reduces to the integral over the path (A) of the former section.

To obtain the Multiform Solution, it is only necessary to consider the integral

$$u = \frac{1}{2n\pi}\, \frac{e^{-[r^2+r'^2+(z-z')^2]/4\kappa t}}{t^{\frac{3}{2}}} \int e^{(rr'/2\kappa t)\cos(a-\theta)}\, \frac{e^{ia/n}}{e^{ia/n}-e^{i\theta'/n}}\, da, \tag{58}$$

taken over the path (A) corresponding to the value of θ. This is the multiform solution with a pole at (r', θ', z') in the range $-(\pi-\theta') < \theta < (2n-1)\pi+\theta'$.

Thus we see that the sole difference in our results for the three-dimensional case consists in the introduction of the factors

$$e^{-(z-z')^2/4\kappa t} \quad \text{and} \quad \frac{1}{t^{\frac{3}{2}}}.$$

In the particular case when $n = 2$,

$$u = \frac{1}{\sqrt{\pi}} \frac{e^{-[r^2+r'^2+(z-z')^2-2rr'\cos(\theta-\theta')]/4\kappa t}}{t^{\frac{3}{2}}} \int_{-\infty}^{\sqrt{rr'/\kappa t}\cos\frac{1}{2}(\theta-\theta')} e^{-\lambda^2}\, d\lambda. \qquad (59)$$

11. These physical applications of the multiform solutions found in this paper have been given because of their simplicity and the possibility of testing their agreement with the facts of nature.

The cases in which the planes meet at an angle $\frac{n\pi}{m}$ (n, m positive integers) may be discussed by the same method. Here we should require the n-fold Riemann's surface, or space; or, in other words, our physical space would be defined by one complete revolution round the axis of z, and we should bring to our aid $(n-1)$ imaginary spaces, built up by the successive $(n-1)$ revolutions of the radius vector in the cylindrical coordinate system.

No attempt has been made here to prove the uniqueness of the solutions in the particular cases. This was done for the problems in potential in the often-quoted paper in our *Proceedings*. The physical applications prove that they are unique. An analytical proof I hope to give later.

The next advance in this method ought to be the solution of the problems where the obstacle consists of an infinite plane in which there is a slit with parallel edges; or an infinite plane with parallel edges. The system of bipolar coordinates

$$\rho = \log\left(\frac{r_1}{r_2}\right),$$

$$\phi = \theta_1 - \theta_2$$

gives us a suitable transformation for this case. We have to deal with the integration of our partial differential equations on a Riemann's surface, or space, which has $\phi = 0$ for branch-section, or membrane, and two branch-points, or lines, at the points $\rho = \pm\infty$.

It is obvious that this amounts to defining our physical space by the range

$$0 < \phi < 2\pi,$$

and putting the image in the space defined by

$$-2\pi < \phi < 0.$$

The problem—for the equation of potential—was discussed in Sommerfeld's paper on that equation. [See note by Dr. Sommerfeld, below.] The solutions of the corresponding problems for the equations with which this paper deals at present occupy my attention.

It only requires the discovery of a proper coordinate system to advance our knowledge to the cases examined by the method of series and in approximation by Prof. Lamb, and such a discovery ought to give us, not only exact solutions, but solutions also applicable to three-dimensional work.

The question of the solution of these partial differential equations on other Riemann's surfaces should be a fruitful one also for the pure mathematician, and all these questions which, in the theory of functions, have circled round the potential would enter here for discussion.*

Note by Dr. Sommerfeld to Mr. Carslaw's paper.

Dr. Sommerfeld takes this opportunity of calling attention to an error in his discussion of the problem in potential, where a point charge is placed in the region bounded by an infinite conducting plane, in which there is a slit with parallel edges:—

In den folgenden Zeilen bitte ich ein Versehen berichtigen zu dürfen, welches sich in § 5 meiner in Vol. XXVIII. der *Proc. Lond. Math. Soc.* abgedruckten Arbeit eingeschlichen hat. Ich thue dieses um so lieber, als Herr H. S. Carslaw auf den vorangehenden Seiten zu meiner Freude und auf meine Anregung hin gezeigt hat, dass sich die Methode jener Arbeit in der am Schluss (p. 429) angedeuteten Weise auf andere physikalische Differentialgleichungen ausdehnen lässt.

Der Fehler besteht darin, dass bei Benutzung des p. 421 angegebenen Wertes von R'^2 die Function u aus Gleichung (5), p. 422, zwar allen übrigen Bedingungen des Problems, aber nicht der Differential-

* *Cf.* Pockel's, *Über die partielle Differential-Gleichung* $\nabla^2 u + \kappa^2 u = 0$, pp. 225, 238, 339.

gleichung des Potentials genügt. Um Letzteres zu erreichen, muss man vielmehr nach dem p. 405 genannten Principe den Winkel ϕ' in dem Ausdrucke von R^2 *durchweg* durch die Integrationsvariable α ersetzen, und, dementsprechend, R'^2 folgendermassen definiren:

$$R'^2 = 2\frac{\cos i(\rho-\rho')-\cos(\phi-\alpha)}{(\cos i\rho-\cos\phi)(\cos i\rho'-\cos\alpha)}+(z-z')^2.$$

Gleichzeitig wird es nötig, die Wahl der Function $f(\alpha)$ etwas abzuändern, damit $f(\alpha)/R'$ für $\alpha=\infty$ verschwindet. Man nehme zu dem Zwecke statt der p. 422 angegebenen beiden Werte

$$f(\alpha)=\frac{ie^{i\alpha}}{e^{i\alpha}-e^{i\phi'}}\sqrt{\frac{\cos i\rho'-\cos\phi'}{\cos i\phi'-\cos\alpha}},$$

bez.
$$f(\alpha)=\frac{i}{n}\,\frac{e^{i\alpha/n}}{e^{i\alpha/n}-e^{i\phi'/n}}\sqrt{\frac{\cos i\rho'-\cos\phi'}{\cos i\rho'-\cos\alpha}}.$$

$f(\alpha)$ besitzt dann immer noch die Eigenschaft, für $\alpha=\phi'$ von der ersten Ordnung mit dem Residuum 1 unendlich zu werden. Als Verzweigungspunkte des Integranden kommen ausser $\alpha=\infty$ nur diejenigen Stellen der α-Ebene in Betracht, für welche $R'^2=0$, d. h.,

$$\cos(\phi-\alpha)-\cos i(\rho-\rho')=\tfrac{1}{2}(z-z')^2(\cos i\rho-\cos\phi)(\cos i\rho'-\cos\alpha)$$

wird. Sie sind sehr leicht zu bestimmen, wenn $z-z'=0$; dann haben wir nämlich einfach

$$\alpha=\phi+2k\pi\pm i(\rho-\rho').$$

Im anderen Falle muss man die Gleichung für α auflösen, und erhält

$$\alpha=a+2k\pi\pm ib,$$

wo die Grössen a und b reelle Zahlen bedeuten, die von ϕ, ρ, ρ' und $z-z'$ abhängen.

Die Deformation des Integrationsweges lässt sich darauf gerade so ausführen wie p. 422 angegeben. Der mit W bezeichnete Weg führt, vom Unendlichen ausgehend und dahin zurückkehrend, in einer Schleife um die Verzweigungspunkte $\alpha=a+ib$ und $\alpha=a-ib$ herum.

Die Schlussformel (5) ist hiernach folgendermassen abzuändern:

$$\text{(V)}\quad u=\frac{1}{2\pi n}\int\frac{1}{R'}\sqrt{\frac{\cos i\rho'-\cos\phi'}{\cos i\rho-\cos\alpha}}\,\frac{e^{i\alpha/n}\,d\alpha}{e^{i\alpha/n}-e^{i\phi'/n}}.$$

Die folgenden Bemerkungen über Näherungsformeln in der Nähe der Verzweigungslinien und über die Ausführung der Integration im Falle $n = 2$ sind in der pp. 423 und 424 gegebenen Form unmittelbar aufrecht zu halten, wenn man sich auf Punkte der Ebene $z = z'$ beschränkt; im diesem Falle stimmt nämlich die berichtigte Form (V) mit der früher angegebenen (5) genau überein. An der p. 425 beschriebenen Figur, welche sich gerade auf diese Ebene $z = z'$ bezieht, ist daher nichts zu corrigiren.

Ein geringfügiger Schreibfehler, auf den mich Herr Carslaw aufmerksam machte, findet sich ausserdem p. 417. Die Gleichung (3) muss nämlich lauten:

$$v = \frac{2}{\pi R} \text{ arc tan} \sqrt{\frac{\sigma + \tau}{\sigma - \tau}} - \frac{2}{\pi R'} \text{ arc tan} \sqrt{\frac{\sigma + \tau'}{\sigma - \tau'}},$$

wobei zur Abkürzung

$$R^2 = r^2 + r'^2 - 2rr' \cos(\phi - \phi') + (z - z')^2,$$

$$R'^2 = r^2 + r'^2 - 2rr' \cos(\phi + \phi') + (z - z')^2$$

gesetzt ist, und wobei σ, τ, τ' die pp. 413 und 417 angegebene Bedeutung haben.

Thursday, January 12th, 1899.

Prof. ELLIOTT, F.R.S., Vice-President; and subsequently Lt.-Col. CUNNINGHAM, R.E., Vice-President, and Dr. HOBSON, F.R.S., in the Chair.

Fourteen members, and a visitor, present.

Prof. Elliott referred, in feeling terms, to the recent death of the Rev. B. Price, F.R.S., who was elected a member of the Society June 26th, 1866.

Dr. Morrice read a paper on "Linear Transformation by Inversions."

Mr. H. M. Macdonald spoke on "The Zeroes of the Bessel Functions" (in continuation of his previous paper on the subject).

Lt.-Col. Cunningham communicated a paper by Mr. D. Biddle, entitled "A Simple Method of Factorizing large Composite Numbers of any unknown Form."

Messrs. Lawrence, Larmor, Hobson, and Western spoke upon one or more of the above papers.

The following papers were communicated, in abstract, by Dr. Hobson, viz.:—

On a Determinant each of whose Elements is the Product of k Factors: Prof. Metzler.

Properties of Hyperspace, in relation to Systems of Forces, the Kinematics of Rigid Bodies, and Clifford's Parallels: Mr. A. N. Whitehead.

On the Reduction of a Linear Substitution to its Canonical Form: Prof. W. Burnside.

The following presents were made to the Library:—

Koenigsberger, L.—"The Investigations of Hermann von Helmholtz on the Fundamental Principles of Mathematics and Mechanics," 8vo; Washington, 1898 (from "Smithsonian Report," 1896, pp. 93–124).

Oltramare, G.—"Calcul de Généralisation," 8vo; Paris, 1899. Two copies: one presented by the Author and the other by the Publisher.

"Educational Times," January, 1899.

"Indian Engineering," Vol. xxiv., Nos. 21–25, Nov. 19–Dec. 17, 1898.

"Reciprocal Polygons," by Jamshedji Edalji, B.A., B.Sc.; Ahmedabad, 1898. From the Author.

The following is the list of exchanges received:—

"Proceedings of the Royal Society," Vol. LXIV., No. 405.

"Beiblätter zu den Annalen der Physik und Chemie," Bd. XXII., St. 11; Leipzig, 1898.

"Memoirs and Proceedings of the Manchester Literary and Philosophical Society," Vol. XLII., Pt. 5, 1897–98.

"Berichte über die Verhandlungen der Königl. Sächs. Gesellschaft der Wissenschaften zu Leipzig," Bd. L., Pt. 5, 1898.

"Proceedings of the Physical Society of London," Vol. XVI., Pt. 3; November, 1898.

"Proceedings of the Canadian Institute," Vol. I., Pt. 6; November, 1898.

"Proceedings of the Royal Irish Academy," Vol. v., No. 1; December, 1898.

"Bulletin of the American Mathematical Society," Series 2, Vol. v., No. 3; December, 1898.

"Rendiconto dell' Accademia delle Scienze Fisiche e Matematiche," Vol. IV., Fasc. 8–11; Napoli, 1898.

"Rendiconti del Circolo Matematico di Palermo," Tomo XII., Fasc. 6; November and December, 1898.

"Bulletin des Sciences Mathématiques," Tome xxii., Dec., 1898; Paris.

"Acta Mathematica," xxii., 3; Stockholm, 1898.

"Annali di Matematica," Serie 3, Tomo ii., Fasc. 1; Milano, November, 1898.

"Atti della Reale Accademia dei Lincei—Rendiconti," Sem. 2, Vol. vii., Fasc. 10, 11; Roma, 1898.

Zeroes of the Bessel Functions. By H. M. Macdonald.

Read January 12th, 1899. Received February 1st, 1899.

In a previous paper, the zeroes of $J_n(z)/z^n$, where n is any real quantity, have been considered. There is no difficulty in extending the results there obtained to the case where n is any quantity. When the real part of n is greater than -1, all the zeroes are associated with the essential singularity at infinity, and are obtainable from Stokes' formula. When the real part of n lies between $-m-1$ and $-m$, m being an integer, there are, in addition to the zeroes associated with the essential singularity at infinity, $2m$ zeroes which can be derived from a formula similar to that given in § 11 of the previous paper. The method used for discussing the zeroes of $J_n(z)/z^n$ depended on the fact that it is a holomorphic function of z at all points of the z-plane not at an infinite distance from the origin. A solution of Bessel's equation is not, in general, expressible as a holomorphic function multiplied by a power of z; the only case where it is so is that of n half an odd integer. The most important solutions of Bessel's equation other than J_n are Hankel's second solution Y_n, and that solution of the equation which vanishes at the positive imaginary infinity, usually denoted by K_n. In what follows, the function K_n will be principally discussed.

In §§ 1, 2, the elementary properties of the function $K_n(z)$ are investigated, and the function is defined to be that solution of the differential equation

$$\frac{d^2y}{dz^2}+\frac{1}{z}\frac{dy}{dz}-\left(1+\frac{n^2}{z^2}\right)y=0$$

which vanishes at the real positive infinity, the plane being bounded