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XXXVII. *Note on the Resistance and Self-Induction of Branched Circuits.* By Prof. A. ANDERSON, M.A.*

ON page 134 of Fleming's 'Alternate Current Transformer,' vol. i., the values are given of the effective resistance and inductance of a system of conductors connected in parallel, and acted on by an impressed electromotive force varying according to a simple harmonic law. The conductors are supposed to have no mutual induction. The results are taken from a paper by Lord Rayleigh in the Philosophical Magazine of May 1886, to which the reader is referred.

The following method of obtaining these results, although it really does not differ much from that given in Lord Rayleigh's paper, will perhaps be more easily understood.

Let the resistances of the conductors be $R_1, R_2, R_3, \dots R_n$, the coefficients of self-induction $L_1, L_2, L_3, \dots L_n$, and suppose the impressed electromotive force to be $E \sin pt$. The currents in the branches may be denoted by $I_1 \sin (pt - \theta_1), I_2 \sin (pt - \theta_2), \dots I_n \sin (pt - \theta_n)$. Hence, if i denote the total current, we have

$$i = \sin pt \sum I \cos \theta - \cos pt \sum I \sin \theta.$$

Now, since

$$\begin{aligned} E \sin pt &= R_1 I_1 \sin (pt - \theta_1) + p L_1 I_1 \cos (pt - \theta_1) \\ &= R_2 I_2 \sin (pt - \theta_2) + p L_2 I_2 \cos (pt - \theta_2) \\ &= \dots, \end{aligned}$$

we have

$$\begin{aligned} E &= R_1 I_1 \cos \theta_1 + p L_1 I_1 \sin \theta_1, \\ 0 &= p L_1 I_1 \cos \theta_1 - R_1 I_1 \sin \theta_1, \end{aligned}$$

whence

$$I_1 \cos \theta_1 = \frac{E R_1}{R_1^2 + p^2 L_1^2}, \quad I_1 \sin \theta_1 = \frac{p E L_1}{R_1^2 + p^2 L_1^2};$$

and similar expressions hold for $I_2 \cos \theta_2, I_2 \sin \theta_2$, &c.

Hence

$$i = E \sin pt \sum \frac{R}{R^2 + p^2 L^2} - p E \cos pt \sum \frac{L}{R^2 + p^2 L^2},$$

or

$$i = A E \sin pt - p B E \cos pt, \text{ where } A \text{ is written for}$$

$$\sum \frac{R}{R^2 + p^2 L^2}, \text{ and } B \text{ for } \sum \frac{L}{R^2 + p^2 L^2}.$$

* Communicated by the Author.

Now, denoting the effective resistance of the system by R' and the inductance by L' , we have

$$E \sin pt = R'i + L' \frac{di}{dt},$$

or

$$\sin pt = (AR' + p^2 BL') \sin pt + p(L'A - R'B) \cos pt;$$

and, therefore,

$$L'A - R'B = 0,$$

$$AR' + p^2 BL' = 1,$$

so that

$$R' = \frac{A}{A^2 + p^2 B^2}, \quad L' = \frac{B}{A^2 + p^2 B^2},$$

which are the results quoted.

We may employ the same method to find the effective resistance and inductance in the case where the conductors have mutual induction. Let M_{rs} , or M_{sr} , denote the coefficient of mutual induction between the conductors r and s . It will now be convenient to discontinue the use of the symbol L , and to denote the coefficient of self-induction of the conductor r by M_{rr} . We have, in this case,

$$\begin{aligned} E \sin pt &= R_1 I_1 \sin (pt - \theta_1) + p[M_{11} I_1 \cos (pt - \theta_1) \\ &\quad + M_{12} I_2 \cos (pt - \theta_2) + \dots + M_{1n} I_n \cos (pt - \theta_n)] \\ &= R_2 I_2 \sin (pt - \theta_1) + p[M_{21} I_1 \cos (pt - \theta_1) \\ &\quad + M_{22} I_2 \cos (pt - \theta_2) + \dots + M_{2n} I_n \cos (pt - \theta_n)] \\ &= \dots \end{aligned}$$

Hence

$$\begin{aligned} E &= R_1 I_1 \cos \theta_1 + p[M_{11} I_1 \sin \theta_1 + M_{12} I_2 \sin \theta_2 + \dots + M_{1n} I_n \sin \theta_n], \\ 0 &= R_1 I_1 \sin \theta_1 - p[M_{11} I_1 \cos \theta_1 + M_{12} I_2 \cos \theta_2 + \dots + M_{1n} I_n \cos \theta_n]. \end{aligned}$$

Denoting

$$I_1 \cos \theta_1, I_2 \cos \theta_2, \dots \text{ by } x_1, x_2, \dots,$$

and

$$I_1 \sin \theta_1, I_2 \sin \theta_2, \dots \text{ by } y_1, y_2, \dots,$$

we have the equations

$$\begin{aligned} E &= R_1 x_1 + p \sum_{r=1}^{r=n} M_{1r} y_r = R_2 x_2 + p \sum_{r=1}^{r=n} M_{2r} y_r = \dots = \dots, \\ 0 &= R_1 y_1 - p \sum_{r=1}^{r=n} M_{1r} x_r = R_2 y_2 - p \sum_{r=1}^{r=n} M_{2r} x_r = \dots = \dots \end{aligned}$$

Substituting in the former equations the values of $y_1, y_2, \dots$ found from the latter, we get the system

$$\begin{aligned} E &= R_1 x_1 + p^2 \sum_{r=1}^{r=n} \sum_{s=1}^{s=n} \frac{M_{1r} M_{rs}}{R_r} x_s \\ &= R_2 x_2 + p^2 \sum_{r=1}^{r=n} \sum_{s=1}^{s=n} \frac{M_{2r} M_{rs}}{R_r} x_s \\ &= \dots \end{aligned}$$

If we denote

$$\sum_{r=1}^{r=n} \frac{M_{\mu r} M_{\nu r}}{R_r} \text{ by } P_{\mu\nu},$$

we get

$$\begin{aligned} \frac{E}{p^2} &= \frac{R_1 x_1}{p^2} + \sum_{r=1}^{r=n} P_{1r} x_r \\ &= \frac{R_2 x_2}{p^2} + \sum_{r=1}^{r=n} P_{2r} x_r \\ &= \dots \end{aligned}$$

Hence

$$x_1 = \frac{EQ_1}{p^2 \Delta}, \quad x_2 = \frac{EQ_2}{p^2 \Delta}, \quad \dots \quad x_r = \frac{EQ_r}{p^2 \Delta},$$

where

$$\Delta = \begin{vmatrix} P_{11} + \frac{R_1}{p^2}, & P_{12}, & P_{13}, & \dots & P_{1n} \\ P_{21}, & P_{22} + \frac{R_2}{p^2}, & P_{23}, & \dots & P_{2n} \\ P_{31}, & P_{32}, & P_{33} + \frac{R_3}{p^2}, & \dots & P_{3n} \\ \cdot & \cdot & \cdot & \dots & \cdot \\ \cdot & \cdot & \cdot & \dots & \cdot \\ \cdot & \cdot & \cdot & \dots & \cdot \\ P_{n1}, & P_{n2}, & P_{n3}, & \dots & P_{nn} + \frac{R_n}{p^2} \end{vmatrix},$$

and Q_r is the result of equating to unity the terms in the r th row or r th column of this determinant.

It follows, then, immediately, that

$$y_1 = \frac{EU_1}{p\Delta}, \quad y_2 = \frac{EU_2}{p\Delta}, \quad \dots \quad y = \frac{EU_r}{p\Delta}, \quad \dots,$$

where

$$U_r = \frac{1}{R_r} \sum_{s=1}^{s=n} M_{rs} Q_s.$$

Now, as before, we have

$$i = \sin pt \Sigma x - \cos pt \Sigma y,$$

and, therefore,

$$E \sin pt = R'(\sin pt \Sigma x - \cos pt \Sigma y) + pL'(\cos pt \Sigma x + \sin pt \Sigma y),$$

whence

$$E = R' \Sigma x + pL' \Sigma y,$$

$$0 = pL' \Sigma x - R' \Sigma y,$$

and

$$R' = \frac{E \Sigma x}{(\Sigma x)^2 + (\Sigma y)^2}, \quad L' = \frac{E \Sigma y}{(\Sigma x)^2 + (\Sigma y)^2},$$

or

$$R' = \frac{p^2 \Delta \Sigma Q}{(\Sigma Q)^2 + p^2 (\Sigma U)^2}, \quad L' = \frac{p^2 \Delta \Sigma U}{(\Sigma Q)^2 + p^2 (\Sigma U)^2}.$$

In the simple case of two branches of resistances R and S , coefficients of self-induction L and N , and coefficient of mutual induction M , we easily find

$$\Delta = \frac{R^2 S^2 + p^2 (2M^2 RS + L^2 S^2 + N^2 R^2) + p^4 (LN - M^2)^2}{p^4 RS},$$

$$\Sigma Q = \frac{RS(R + S) + p^2 S(L - M)^2 + p^2 R(N - M)^2}{p^2 RS},$$

and

$$\Sigma U = \frac{2MRS + LS^2 + NR^2 + p^2 (LN - M^2)(L + N - 2M)}{p^2 RS},$$

The values of L' and R' are obtained by substitution. If $p=0$, we see by inspection that

$$R' = \frac{RS}{R + S} \quad \text{and} \quad L' = \frac{2MRS + LS^2 + NR^2}{(R + S)^2},$$

and if $p=\infty$, that

$$R' = \frac{S(L - M)^2 + R(N - M)^2}{(L + N - 2M)^2}, \quad L' = \frac{LN - M^2}{L + N - 2M}.$$