A particular case of this is the well-known hyper-elliptic integral which can be reduced to elliptic elements, viz.,

$$\frac{dx}{\sqrt{1-x^2 \,.\, 1-k^2x^2 \,.\, 1-k'^2x^2 \,.\, 1-k^2k'^2x^2}}.$$

It being obvious that $(1-x^2)(1-k^2k'^2x^2)$ and $(1-k^2x^2)(1-k'^2x^2)$ are quartics derivable from the same fundamental quartic. In short, they have the same sextic covariants.

I hope to be able to continue the discussion of the elliptic transformations on the covariant basis in a future paper. I think, at any rate, it is very interesting to see how the transformations of the different orders are to be derived from the covariants in their most general forms.

Note on the Arc of a Sphero-Conic. *By* H. F. W. Burstall, St. John's College, Cambridge.

[*Read November* 11*th*, 1886.]

Let the sphere be of unit radius; the polar equation of a sphero-conic

is $$\sin^2\rho \left\{ \sin^2\theta \left(1+\frac{1}{a^2}\right) + \cos^2\theta \left(1+\frac{1}{b^2}\right) \right\} = 1.$$

If P is the pole of a great circle touching the sphero-conic, the locus of P is called the polar reciprocal of the first conic, and its equation is well known to be

$$\sin^2\rho \left\{ \sin^2\theta \,(1+a^2) + \cos^2\theta \,(1+b^2) \right\} = 1.$$

Let AQP be the arc of a sphero-conic; A being the end of the major axis, TA and TP the tangents at A and P, AP the chord of contact, and $A'P'$ the arc of the reciprocal.

We have A' the pole of the tangent at A, and as the point of contact moves round to P the pole moves from A' to P', and the chord $A'P'$ is the locus of the poles of all great circles passing through T. Thus the triangle formed by the two tangents and the arc of the conic is reciprocal to the triangle formed by the arc $A'P'$ and the chord $A'P'$.

Let the area $OA'P'Q' = \Delta_1$ and the area $OA'P' = \Delta$, let $TP = t_1$, $TA = t_2$, arc $AP = s$, and $\angle A'OP' = \theta'$.

Then (Williamson's " Integral Calculus," Art. 188)

$$\Delta_1 - \Delta + s + t_1 + t_2 = 2\pi,$$

and, remembering that the sphere is of unit radius,

$$\Delta_1 = \int_0^{\theta'} (1-\cos\rho)\, d\theta'$$

$$= \theta' - \int_0^{\theta'} \left\{1 - \frac{1}{\cos\theta'(1+a^2)+\sin\theta'(1+b^2)}\right\}^{\frac{1}{2}} d\theta',$$

therefore

$$s + t_1 + t_2 + \theta' - \Delta = 2\pi + \int_0^{\theta'} \left\{\frac{a^2\cos^2\theta' + b^2\sin^2\theta'}{1+a^2\cos^2\theta' + b^2\sin^2\theta'}\right\}^{\frac{1}{2}} d\theta'.$$

[In the integral

$$\int_0^{\psi} \left\{\frac{a^2\cos^2\psi + b^2\sin^2\psi}{\cos^2\psi(1+a^2)+\sin^2\psi(1+b^2)}\right\}^{\frac{1}{2}} d\psi, \qquad a > b,$$

put $$\frac{b}{a}\tan\psi = \tan\phi,$$

and it reduces to

$$\int_0^{\phi} \frac{a^2b^2}{\cos\phi\sqrt{b^2(1+a^2)+(1+b^2)a^2\tan^2\phi}} \; \frac{d\phi}{b^2\cos^2\phi + a^2\sin^2\phi}$$

$$= \int_0^{\phi} \frac{a^2b^2\, d\phi}{\{a^2-(a^2-b^2)\cos^2\phi\}\sqrt{a^2(1+b^2)-\cos^2\phi\,\{a^2(1+b^2)-b^2(1+a^2)\}}}$$

$$= \frac{a^2b^2}{a^3(1+b^2)^{\frac{1}{2}}}\int_0^{\phi} \frac{d\phi}{\left(1-\dfrac{a^2-b^2}{a^2}\cos^2\phi\right)\sqrt{1-\dfrac{a^2-b^2}{a^2(1+b^2)}\cos^2\phi}},$$

which is an elliptic integral of the third kind. (See Chasles on " Spherical Conics," translated by Graves, Appendix, p. 111.]

In any spherical curve, let ON be the perpendicular from the pole on the tangent, and let $NP = t$, $ON = p$, $\angle NOA = \omega$.

We have $$s_1 - t = \int_0^{\omega} \sin p\, d\omega;$$

where $s_1 = \text{arc } AP$. To find $\sin p$ in terms of ω; the tangent plane at

P is
$$\frac{x_1 x}{a^2}+\frac{y_1 y}{b^2}=z_1 z.$$

The equation of a plane perpendicular to this, and containing the axis of z, is
$$\frac{x_1}{a^2}y-\frac{y_1}{b^2}x=0,$$

therefore
$$\tan\omega=\frac{y}{x}=\frac{y_1}{x_1}\cdot\frac{a^2}{b^2}.$$

Now, O is the point (0, 0, 1), therefore the perpendicular from O on the tangent plane is
$$\frac{z_1}{\sqrt{\frac{x_1^2}{a^4}+\frac{y_1^2}{b^4}+z_1^2}};$$

but
$$z_1^2=\frac{x_1^2}{a^2}+\frac{y_1^2}{b^2},$$

therefore
$$(\text{perpendicular})^2=\frac{\frac{x_1^2}{a^2}+\frac{y_1^2}{b^2}}{\frac{x_1^2}{a^4}+\frac{y_1^2}{b^4}+\frac{x_1^2}{a^2}+\frac{y_1^2}{b^2}}$$
$$=\frac{a^2\cos^2\omega+b^2\sin^2\omega}{1+a^2\cos^2\omega+b^2\sin^2\omega}.$$

Now,
$$p=\sin p,$$

therefore
$$s_1-t=\int_0^{\omega}\sqrt{\frac{a^2\cos^2\omega+b^2\sin^2\omega}{1+a^2\cos^2\omega+b^2\sin^2\omega}}\,d\omega.$$

Making $\omega=\theta'$, we have
$$s+t_1+t_2+\theta'-\Delta=2\pi+s_1-t,$$

therefore
$$s_1-s=t+t_1+t_2+\theta'-\Delta-2\pi.$$

Hence we have the analogue of Fagnani's theorem for a sphero-conic.