On certain Properties of the Mean Motions and the Secular Accelerations of the principal Arguments used in the Lunar Theory. By ERNEST W. BROWN. Read November 12th, 1896. Received October 25th, 1896.

(i.) In an earlier communication to the Society* I have shown how Delaunay's lunar theory may be exhibited in a sufficiently simple manner by means of Jacobi's dynamical method. This plan of treatment, namely, the discovery by an indirect process of a solution of the partial differential equation satisfied by the principal function, I have also applied, in the preceding paper, to the general problem of three bodies, to prove the theorems obtained by Newcomb† with reference to the properties of a special series of canonical constants, and to add one or two results of minor importance.

In this paper I apply the same method: firstly, to show how we may pass directly from the elements of the ellipse to the final system of constants used by Delaunay after all his operations have been completed. Secondly, it is shown how the constant part of the parallax is connected with the motions of the perigee and the node: the equation found here is probably the most general form of this connexion. Certain previous results should be mentioned. In a "Note on a remarkable Property of the Analytical Expression for the Constant Term in the Reciprocal of the Moon's Radius Vector,"‡ Adams has proved that, if we neglect the square of the solar parallax, all the terms in the constant part of the lunar parallax which have the factors e^2 , γ^2 are identically zero, and further that the coefficients of e^4 , $e^2\gamma^2$, γ^4 are connected by two ratios to the four coefficients of e^2 , γ^2 in the expressions which give the mean motions of the perigee and the node. I have given a more definite form to these ratios, by obtaining them in the form of four relations instead of two;§ the new factors introduced being certain functions of the coefficients of the periodic terms. All these results will now be shown to be simple deductions from a single equation numbered (13) below.

Thirdly, these properties are used in connexion with the remarkable theorem obtained by Newcomb in the latter of his two papers

* *Proceedings* May, 1896, pp. 385-390.

† "General Integrals of Planetary Motion," *Smithsonian Contributions*, Vol. xxi.; "Action of the Planets on the Moon," *Amer. Eph. Papers*, Vol. v.

‡ *Monthly Notices R.A.S.*, Vol. xxxviii.

§ "Investigations in the Lunar Theory," *Amer. Jour. Math.*, Vol. xvii.

just referred to,* by which the secular accelerations are obtainable immediately from the final values of Delaunay's constants L, G, H . In particular, I shall show that the principal part of the acceleration of the mean motion (that independent of $e, \gamma, a/a'$) is obtainable directly from the corresponding portion of the constant term in the expression for the lunar parallax; and that, when this is found, the principal parts of the accelerations of the perigee and the node may be deduced from the expressions for their mean motions.

The basis which furnishes these results is a consideration of the constant parts of the various functions which naturally arise in the solution obtained by varying the arbitrary constants. In all problems of celestial mechanics there are certain arbitrary constants which must be determined from observation; and there are certain others which may also be accurately found by observation, but which depend on those previously found. The comparison of the two sets of values forms so important a test of the sufficiency of the theory that no means which will give tests of the accuracy of the theoretical calculations, or which will enable us to obtain the second class of constants without serious risk of error, will be unimportant from a practical point of view, apart from any theoretical interest which they may possess.

(ii.) The general canonical system of equations used by Delaunay is

$$\begin{aligned}\frac{dL}{dt} &= \frac{\partial R_1}{\partial l}, & \frac{dG}{dt} &= \frac{\partial R_1}{\partial g}, & \frac{dH}{dt} &= \frac{\partial R_1}{\partial h}, \\ \frac{dl}{dt} &= -\frac{\partial R_1}{\partial L}, & \frac{dg}{dt} &= -\frac{\partial R_1}{\partial G}, & \frac{dh}{dt} &= -\frac{\partial R_1}{\partial H},\end{aligned}$$

where

$$R_1 = \frac{(E+M)^2}{2L^3} + m' \left\{ \frac{1}{\sqrt{(x-x')^2 + (y-y')^2 + (z-z')^2}} - \frac{xx' + yy' + zz'}{r'^3} - \frac{1}{r'} \right\}.$$

Here x, y, z, x', y', z' are the coordinates of the Moon and Sun referred to fixed axes through the Earth; E, M, m' are the masses of the Earth, Moon, and Sun; L, G, H, l, g, h are the well-known functions of the elliptic elements given by

$$L = a^3 (E+M)^{\frac{1}{2}}, \quad G = L(1-e^2)^{\frac{1}{2}}, \quad H = G \cos i = G(1-2\gamma^2);$$

$$l = nt + \epsilon - \pi = \text{mean anomaly,}$$

$$g = 0 - \pi = \text{distance from node to perigee,}$$

$$h = \theta = \text{longitude of node.}$$

* Also proved in § (xi.) of the preceding paper.

The change necessary in order that x', y', z' may be the coordinates of the Sun referred to the centre of mass of the Earth and Moon is known to be sufficiently accounted for if we replace a/a' by $(E-M)a/(E+M)a'$ in the final results.

It will be advantageous to introduce a change of variables. Let

$$\begin{aligned} p_1 &= L, & p_2 &= G-L, & p_3 &= H-G; \\ q_1 &= l+g+h, & q_2 &= g+h, & q_3 &= h. \end{aligned}$$

The equations satisfied by the new variables will still be canonical, and, if we put

$$R = -R_1,$$

they will become

$$\frac{dp_i}{dt} = -\frac{\partial R}{\partial q_i}, \quad \frac{dq_i}{dt} = \frac{\partial R}{\partial p_i}, \quad (i = 1, 2, 3). \quad (1)$$

The new variables, while possessing all the special properties of the old ones, have this further advantage, namely, that p_2, p_3 are small quantities of the orders e^2, γ^2 , respectively; also, q_1, q_2, q_3 are the longitudes of the Moon, its perigee, and node, the non-periodic parts of them being the mean values of these angles.

(iii.) The solution of the equations (1) will be carried out in a manner similar to that which has been used in the previous paper; but, as the time now occurs explicitly in R through its presence in the elliptic values of x', y', z' , one or two modifications are necessary. We suppose, with Delaunay, that $z' = 0$, and that x', y' have their elliptic values.

The equations satisfied by the principal function are

$$p_i = \frac{\partial S}{\partial q_i}, \quad \frac{\partial S}{\partial t} = -R, \quad (2)$$

and it may be shown as before that p_1, p_2, p_3 are expressible by series of the form

$$p_i = c_i + \sum s j_i \cos N', \quad (3)$$

c_1, c_2, c_3 being arbitrary constants, the coefficients s depending only on c_1, c_2, c_3, n', e' , and N' being an angle of the form

$$N' = j_1 q_1 + j_2 q_2 + j_3 q_3 + j' n' t + \alpha,$$

where α is an absolute constant and j_1, j_2, j_3, j' are positive or negative integers or zero.

As

$$\frac{\partial p_i}{\partial t} = \frac{\partial^2 S}{\partial q_i \partial t} = -\frac{\partial R}{\partial q_i},$$

all the quantities being expressed in terms of q_i, c_i, t , we have

$$R = -B - \Sigma s j' n' \cos N', \quad (4)$$

where B is a constant of the same nature as s . A solution of the partial differential equation for S , involving the q_i , the c_i , and the time, will therefore be

$$S = c_1 q_1 + c_2 q_2 + c_3 q_3 + Bt + \Sigma s \sin N'. \quad (5)$$

The remaining integrals are given by

$$l_i = \frac{\partial S}{\partial c_i} = q_i + t \frac{\partial B}{\partial c_i} + \Sigma \frac{\partial s}{\partial c_i} \sin N', \quad (6)$$

l_1, l_2, l_3 being three new arbitrary constants.

(iv.) The relation of the foregoing solution to that of Delaunay is easily seen. The three expressions

$$l_i - t \frac{\partial B}{\partial c_i}$$

are the non-periodic parts of the variables q_1, q_2, q_3 , and therefore

$$-\frac{\partial B}{\partial c_1}, \quad -\frac{\partial B}{\partial c_2}, \quad -\frac{\partial B}{\partial c_3}$$

are the mean motions of the Moon, its perigee, and node respectively. The symbol B is the same as that used by Delaunay,* and it represents the constant remainder of his disturbing function after all the periodic terms have been eliminated. We therefore see that $c_1, c_1 + c_3, c_1 + c_2 + c_3, B$ are the same as Delaunay's L, G, H, R † after the completion of the operations. We have therefore shown that the same quantities are the constant parts of Delaunay's initial variables L, G, H , and of R , when the latter are expressed in terms of the c_i , the q_i , the solar constants, and the time.

(v.) The solution contained in (iii.) gives a method of performing any one of Delaunay's operations which may possibly be useful in certain cases. Assume the values (3) of the p_i : to each periodic term there is one unknown coefficient s . We substitute these values of the p_i in the portion of R under consideration, and equate the result to the expression of R in (4). The equations of condition thus furnished will give B and all the coefficients s in terms of c_1, c_2, c_3, n', e'^2 .

* *Théorie de la Lune*, Vol. II., p. 234.

† Delaunay's R is the R_1 of this paper.

With these results we enter equations (6), and a reversion of series will give the values of q_1, q_2, q_3 in terms of the constants and the time. Substituting in (3), we then obtain p_1, p_2, p_3 in terms of the time. The six new canonical variables for the next operation are the c_i and $l_i - t \partial B / \partial c_i$, and the passage to it is made in the manner shown in my paper on Delaunay's method already referred to.

It is not necessary to consider only one periodic term of the disturbing function: we may take two or more at any operation, if any advantage is to be gained thereby. The method might possibly be of some advantage, if the determination of the mean motions of the perigee and node was under consideration, since no reversion of series is necessary in the latter operations. If we decide to take one periodic term at a time, the calculation of such functions as $p_1^{-2}, p_1^4, (p_2/p_1)^4$, with indeterminate values of the coefficients of the periodic series, might be done once for all, so that the coefficients s could be determined by mere numerical substitutions. The reversion of the series, necessary in order to obtain the q_i in terms of the time, could be treated in a similar way. A somewhat extended series of calculations would be necessary to see if any real advantage over Delaunay's method of procedure is gained. It may be mentioned that the introduction of e, γ at each stage will not be necessary, since c_2, c_3 are respectively of the orders e^2, γ^2 .

(vi.) I now proceed to prove the property connecting the mean motions with the constant term of the parallax.

Representing by b_1, b_2, b_3 the coefficients of t in the non-periodic parts of q_1, q_2, q_3 , that is, putting

$$b_i = -\frac{\partial B}{\partial c_i}, \quad (i = 1, 2, 3), \quad (7)$$

it will be shown that

$$b_1 c_1 + b_2 c_2 + b_3 c_3 + B = \frac{3}{2} \left(\frac{E + M}{r} \right)_0,$$

where in all cases $(Q)_0$ represents the non-periodic part of any function Q expressed as a sum of sines or cosines in terms of the time and the constants.

First, since S is supposed to be a function of q_1, q_2, q_3, t , we have

$$\begin{aligned} \frac{dS}{dt} &= \sum_i \dot{q}_i \frac{\partial S}{\partial q_i} + \frac{\partial S}{\partial t} \\ &= \sum_i p_i \dot{q}_i - R \quad \text{by (2).} \end{aligned}$$

Hence, remembering the notation just explained,

$$\left(\frac{dS}{dt}\right)_0 = (\Sigma_i p_i \dot{q}_i)_0 - (R)_0.$$

The left-hand member of this equation is the coefficient of t in the non-periodic part of S , when the latter has been expressed in terms of the time and the arbitraries. If we look at equation (5), and suppose the values of q_1, q_2, q_3 in terms of the time to be substituted therein, it is evident that the periodic terms $\Sigma s \sin N'$ will only produce periodic terms. The non-periodic part of q_i is $l_i + b_i t$, and therefore the non-periodic part of the coefficient of t in S is

$$b_1 c_1 + b_2 c_2 + b_3 c_3 + B.$$

$$\text{Hence} \quad (\Sigma_i p_i \dot{q}_i)_0 = b_1 c_1 + b_2 c_2 + b_3 c_3 + B + (R)_0. \quad (8)$$

Secondly, denote the ordinary disturbing function by Ω , that is, let

$$\Omega = m' \left\{ \frac{1}{\sqrt{(x-x')^2 + (y-y')^2 + (z-z')^2}} - \frac{xx' + yy' + zz'}{r^3} - \frac{1}{r'} \right\};$$

so that, if $E + M = \kappa$,

$$R = -\Omega - \frac{\kappa^2}{2p_1^2}; \quad (9)$$

and the equations of motion are

$$\ddot{x} = \frac{\partial}{\partial x} \left(\frac{\kappa}{r} + \Omega \right), \quad \ddot{y} = \frac{\partial}{\partial y} \left(\frac{\kappa}{r} + \Omega \right), \quad \ddot{z} = \frac{\partial}{\partial z} \left(\frac{\kappa}{r} + \Omega \right).$$

We shall now neglect the ratio of the parallax of the Moon to that of the Sun. As the results obtained below refer only to the constant parts of the various functions, and as this ratio is well known to occur in these constant parts to an order not less than $m^2 a^2 / a'^2$, the error committed will be almost negligible. It amounts to about 2" in the annual motion of the perigee (that is to say, a quantity of the same order as the difference between the observed value and Hansen's theoretical value), and is less than 0''001 in the secular accelerations per century. The accuracy of the results obtained for the latter quantities will therefore not be impaired by this limitation.

We have, on expanding Ω and neglecting the ratio of the parallaxes

$$\Omega = \frac{m'}{r^3} \left\{ \frac{1}{2} \frac{(xx' + yy' + zz')^2}{r^2} - \frac{1}{2} r^2 \right\},$$

and therefore

$$x \frac{\partial \Omega}{\partial x} + y \frac{\partial \Omega}{\partial y} + z \frac{\partial \Omega}{\partial z} = 2\Omega.$$

Multiply the equations of motion by x, y, z , respectively, and add. Using the last equation, the result may be written

$$\frac{1}{2} \frac{d^2}{dt^2} (r^2) - 2T = -\frac{\kappa}{r} + 2\Omega,$$

where

$$2T = \dot{x}^2 + \dot{y}^2 + \dot{z}^2.$$

Since the first term cannot give rise to a constant portion, this equation gives

$$2(T)_0 = \left(\frac{\kappa}{r}\right)_0 - 2(\Omega)_0. \quad (9')$$

Again, from the theory of elliptic motion,

$$T = \frac{\kappa}{r} - \frac{\kappa^2}{2p_1^2}$$

and therefore, by (9),

$$T - R = \frac{\kappa}{r} + \Omega.$$

Taking the constant parts of this equation, and eliminating $(T)_0$ by means of the previous value obtained for it, we get

$$(R)_0 + 2(\Omega)_0 = -\frac{1}{2} \left(\frac{\kappa}{r}\right)_0 \quad (10).$$

Thirdly, the p_i are of the order $[\text{time}]^{-1} \times [\text{length}]^3$, that is, of the order $[\text{mass}]^{\frac{1}{2}} [\text{length}]^{\frac{1}{2}}$. As the coefficients of the elliptic values of x, y, z are expressible by means of p_1, p_2, p_3 only, and are homogeneous with respect to these quantities, x, y, z must be of the order $[p]^{\frac{1}{2}}$, abstraction being made of the masses. Hence Ω is of the order $[p]^4$. Euler's theorem of homogeneous functions then gives

$$p_1 \frac{\partial \Omega}{\partial p_1} + p_2 \frac{\partial \Omega}{\partial p_2} + p_3 \frac{\partial \Omega}{\partial p_3} = 4\Omega.$$

Using this result with (9), we find

$$\begin{aligned} \Sigma_i p_i \dot{q}_i &= \Sigma_i p_i \frac{\partial R}{\partial p_i} = \frac{\kappa^2}{p_1^2} - 4\Omega \\ &= -6\Omega - 2R. \end{aligned}$$

Taking the non-periodic terms in this equation and combining with (10), we obtain

$$(\Sigma_i p_i \dot{q}_i)_0 = \frac{3}{2} \left(\frac{\kappa}{r}\right)_0 + (R)_0,$$

whence, by (8), $b_1 c_1 + b_2 c_2 + b_3 c_3 + B = \frac{3}{2} \left(\frac{\kappa}{r}\right)_0$, (11)
the required result.

(vii.) Before proceeding to the special applications of this equation, one or two general properties may be noted.

(a) Let arbitrary variations δc_i be given to the c_i . We have

$$\Sigma_i (b_i \delta c_i + c_i \delta b_i) + \delta B = \frac{3}{2} \delta \left(\frac{\kappa}{r} \right)_0.$$

But $b_i = -\partial B / \partial c_i$, and therefore, if we suppose $(\kappa/r)_0$ to be expressed in terms of b_1, b_2, b_3 , and the solar constants,

$$c_i = \frac{3}{2} \frac{\partial}{\partial b_i} \left(\frac{\kappa}{r} \right)_0. \quad (12)$$

[The result corresponding to this in the general problem of three bodies is given by the equations*

$$c_i = -3 \frac{\partial H}{\partial b_i} = \frac{3}{2} \frac{\partial}{\partial b_i} (F)_0.]$$

(\beta) The equations (12), (7) show that

$$\frac{\partial c_i}{\partial b_r} = \frac{\partial c_r}{\partial b_i}, \quad \frac{\partial b_i}{\partial c_r} = \frac{\partial b_r}{\partial c_i}.$$

(\gamma) A formula for obtaining the c_i from the expressions for the rectangular coordinates when the latter are known can be deduced from this result.† We have

$$\begin{aligned} c_i &= \frac{3}{2} \frac{\partial}{\partial b_i} \left(\frac{\kappa}{r} \right)_0 \\ &= \frac{\partial}{\partial b_i} \left(T + \Omega + \frac{\kappa}{r} \right)_0, \quad \text{by (9'),} \\ &= \text{non-periodic part of } \frac{\partial T}{\partial b_i} + \left(\frac{\partial x}{\partial b_i} \frac{\partial}{\partial x} + \frac{\partial y}{\partial b_i} \frac{\partial}{\partial y} + \frac{\partial z}{\partial b_i} \frac{\partial}{\partial z} \right) \left(\frac{\kappa}{r} + \Omega \right) \\ &= \quad \quad \quad \left(\dot{x} \frac{\partial \dot{x}}{\partial b_i} + \dots + \dots + \dot{x} \frac{\partial x}{\partial b_i} + \dots + \dots \right) \\ &= \quad \quad \quad \frac{d}{dt} \left(\dot{x} \frac{\partial x}{\partial b_i} + \dot{y} \frac{\partial y}{\partial b_i} + \dot{z} \frac{\partial z}{\partial b_i} \right). \end{aligned}$$

Assuming

$$x = \Sigma k \cos N, \quad y = \Sigma k \sin N, \quad z = \Sigma k' \sin N,$$

where N is defined as before, we obtain from this result

$$c_i = \Sigma k^2 (j_1 b_1 + j_2 b_2 + j_3 b_3 + j' n') j_i + \frac{1}{2} \Sigma k'^2 (j_1 b_1 + j_2 b_2 + j_3 b_3 + j' n') j_i,$$

* See §(vi.)(c) of my paper "On the Application of Jacobi's Dynamical Method to the Problem of Three Bodies."

† *Ibid.*, §(viii.).

a result similar to that found by Newcomb (*Action of the Planets on Moon*, § 8) in the general problem.

(δ) We can also show,* as in the general problem, that, if p_i, q_i be expressed in terms of the time by series of the forms

$$\begin{aligned} p_i &= (P)_i + \Sigma P_i \cos N, \\ q_i &= l_i + b_i t + \Sigma Q_i \sin N, \end{aligned}$$

the c_i may be found, if necessary, from the equations

$$c_i = (P)_i + \Sigma P_i Q_i j_i.$$

We suppose the l_i to receive variations δl_i , and we then have

$$\Sigma p_i \delta q_i = \delta S.$$

The constant part of the coefficient of δl_i in δS is easily seen, as before, to be c_i , and the result follows immediately.

(viii.) *Connexion between $(\kappa/r)_0$ and the Mean Motion of the Perigee and the Node. Adams' Theorems.*—Replace b_1, b_2, b_3 , which are respectively the mean motions of the Moon, its perigee, and its node, by the letters which usually represent these quantities, namely, n, π_1, θ_1 . The equation (11) is therefore written

$$nc_1 + \pi_1 c_2 + \theta_1 c_3 + B = \frac{3}{2} \left(\frac{\kappa}{r} \right)_0; \quad (13)$$

and the equations (7) become

$$n = -\frac{\partial B}{\partial c_1}, \quad \pi_1 = -\frac{\partial B}{\partial c_2}, \quad \theta_1 = -\frac{\partial B}{\partial c_3}. \quad (14)$$

We now consider all the quantities in these equations to be expressed as functions of $n, e^2, \gamma^2, n', e'^2, \kappa$, where e, γ are the eccentricity and the sine of half the inclination, defined in Delaunay's manner. These definitions are used for the sake of clearness, although it will be evident that any of the various possible modes of defining them are available. The definitions of n, π_1, θ_1 are fixed. It may be remarked, however, that the expression $b_1 c_1 + b_2 c_2 + b_3 c_3$ is an invariant for any linear tangential transformation of the variables p_i, q_i .

Differentiate (13) partially with regard to n, e^2, γ^2 , successively. Since

$$\begin{aligned} \frac{\partial B}{\partial n} &= \frac{\partial B}{\partial c_1} \frac{\partial c_1}{\partial n} + \frac{\partial B}{\partial c_2} \frac{\partial c_2}{\partial n} + \frac{\partial B}{\partial c_3} \frac{\partial c_3}{\partial n} \\ &= -n \frac{\partial c_1}{\partial n} - \pi_1 \frac{\partial c_2}{\partial n} - \theta_1 \frac{\partial c_3}{\partial n}, \quad \&c., \end{aligned}$$

* *Ibid.*, § (ix.).

we obtain

$$\left. \begin{aligned} c_1 + c_3 \frac{\partial \pi_1}{\partial n} + c_3 \frac{\partial \theta_1}{\partial n} &= \frac{3}{2} \frac{\partial}{\partial n} \left(\frac{\kappa}{r} \right)_0, \\ c_3 \frac{\partial \pi_1}{\partial e^3} + c_3 \frac{\partial \theta_1}{\partial e^3} &= \frac{3}{2} \frac{\partial}{\partial e^3} \left(\frac{\kappa}{r} \right)_0, \\ c_3 \frac{\partial \pi_1}{\partial \gamma^3} + c_3 \frac{\partial \theta_1}{\partial \gamma^3} &= \frac{3}{2} \frac{\partial}{\partial \gamma^3} \left(\frac{\kappa}{r} \right)_0 \end{aligned} \right\}, \quad (15)$$

where $\partial \pi_1 / \partial n$, $\partial \pi_1 / \partial e^3$, &c., evidently denote the partial differentials of π_1 , θ_1 with respect to n , e^3 , γ^3 , when expressed in terms of n , e^3 , γ^3 , n' , e'^3 . It will be noticed that these three equations have the advantage of not containing B .

By means of these forms of (13) we can deduce Adams' theorems [see reference given in § (i.)].

Define a by means of the equation $\kappa = n^2 a^3$, and write

$$\begin{aligned} \left(\frac{a}{r} \right)_0 &= A + B e^3 + C \gamma^3 + E e^4 + 2 F e^2 \gamma^3 + G \gamma^4 + \dots, \\ \pi_1 &= n (P + H e^2 + K \gamma^3 + \dots), \\ \theta_1 &= n (T + M e^2 + N \gamma^3 + \dots), \end{aligned}$$

where $A, B, \dots P, H, \dots T, M, \dots$ are functions of m, e^2 only. Substitute these results in the second and third of equations (15). These being identities, we can equate to zero the coefficients of the various powers of e^3, γ^3 .

Equate the coefficients of e^3, γ^3 to zero. Remembering that c_2, c_3 respectively contain the factors e^3, γ^3 in all their terms, we find

$$B = 0, \quad C = 0.$$

This is Adams' first theorem.

Equate the coefficients of $e^4, e^2 \gamma^3, \gamma^4$ to zero. We find

$$\left. \begin{aligned} c_2 H &= \frac{3}{2} n a^3 E e^3, & c_3 M &= \frac{3}{2} n a^3 F \gamma^3, \\ c_3 K &= \frac{3}{2} n a^3 F e^3, & c_3 N &= \frac{3}{2} n a^3 G \gamma^3 \end{aligned} \right\}, \quad (16)$$

$c_2/e^3, c_3/\gamma^3$ being limited to terms of the form $n a^2 f(m, e^2)$. Hence, eliminating c_2, c_3 ,

$$\frac{E}{F} = \frac{H}{K}, \quad \frac{F}{G} = \frac{M}{N},$$

the second theorem of Adams. The further results which he obtains for the higher powers of e^3, γ^3 follow in like manner.

It is of some interest to compare these results with those obtained

by the use of rectangular coordinates.* In the paper referred to, I found†

$$HT_e = 6E, \quad MT_e = 6F,$$

$$KT_e = 6F, \quad NT_e = 6G,$$

where T_e, T , were certain functions of m, e^2 , obtained from the coefficients of the periodic terms in the expressions for the moving rectangular coordinates there used. A comparison of the two sets of results gives

$$c_2 = \frac{1}{4}na^2e^2T_e, \quad c_3 = \frac{1}{4}na^2\gamma^2T_e.$$

On comparing the values obtained for T_e, T , in the memoir referred to, and those found for c_2, c_3 in § (vii.) above, the correctness of these two identities may be easily verified.

One or two further results of the equations (15) may be mentioned.

If all powers of e^2, γ^2 be neglected, the first of them gives

$$c_1 = \frac{3}{2} \frac{\partial}{\partial n} \left(\frac{\kappa}{r} \right)_0. \quad (17)$$

If we consider the first powers of e^2, γ^2 only, we find from the same equation, combined with Adams' first theorem,

$$\text{coefficient of } e^2 \text{ in } c_1 = -c_3 \frac{\partial \pi_1}{\partial n},$$

$$,, \quad \gamma^2 ,, \quad c_2 = -c_3 \frac{\partial \theta_1}{\partial n}.$$

When $c_1, c_2, c_3, \pi_1, \theta_1$ are known in terms of $n, e^2, \gamma^2, n', e^2$, the value of the constant term of the lunar parallax can be obtained to the same degree of accuracy as c_1 , so far as its part independent of e, γ is concerned, and to two orders higher than c_2, c_3 , so far as the parts depending on $e^4, e^2\gamma^2, \gamma^4$ are concerned. The integration of equation (17) presents no difficulty, since the form of the expression for the parallax is known, and it contains no term independent of n .

By using the value of the constant term of the parallax obtained by Adams‡ in a different manner, we have a useful verification of Delaunay's results.

* "Investigations in the Lunar Theory," *Amer. Jour. Math.*, Vol. xvii.

† The meaning given to E, F, G is slightly different from that above.

‡ Given in his paper referred to in § (i.). The values of c_1, c_2, c_3 in terms of $n, e^2, \gamma^2, n', e^2$ are given by Newcomb in § 11 of his memoir on "The Action of the Planets on the Moon"; the values of π_1, θ_1 are found in § 12 of the same paper. The latter were calculated by Delaunay (*Comptes Rendus*, Vol. lxxiv., p. 19), and by Cayley (*Monthly Notices R.A.S.*, Vol. xxxii., pp. 8-16, 74; *Coll. Works*, Vol. vii., pp. 532-534).

(ix.) *The Secular Accelerations.*—Professor Newcomb has shown [see references at the end of § (i.)] that the secular accelerations may be obtained by means of the equations

$$\delta c_1 = 0, \quad \delta c_2 = 0, \quad \delta c_3 = 0.$$

Here c_1, c_2, c_3 are supposed to have been expressed in terms of $n, e^2, \gamma^2, n', e'^2, \kappa$. A variation δe^2 , due to planetary action, induces variations $\delta n, \delta e^2, \delta \gamma^2$, which may be found by means of the above equations. The secular accelerations are then given by

$$\int \delta n dt, \quad \int \delta \pi_1 dt, \quad \int \delta \theta_1 dt.$$

Applying the variation δ to the three equations (15), we find, since $\delta c_1, \delta c_2, \delta c_3$ are all zero,

$$\left. \begin{aligned} c_3 \delta \frac{\partial \pi_1}{\partial} + c_3 \delta \frac{\partial \theta_1}{\partial n} &= \frac{3}{2} \delta \frac{\partial}{\partial n} \left(\frac{\kappa}{r} \right)_0, \\ c_2 \delta \frac{\partial \pi_1}{\partial e^2} + c_3 \delta \frac{\partial \theta_1}{\partial e^2} &= \frac{3}{2} \delta \frac{\partial}{\partial e^2} \left(\frac{\kappa}{r} \right)_0, \\ c_3 \delta \frac{\partial \pi_1}{\partial \gamma^2} + c_3 \delta \frac{\partial \theta_1}{\partial \gamma^2} &= \frac{3}{2} \delta \frac{\partial}{\partial \gamma^2} \left(\frac{\kappa}{r} \right)_0 \end{aligned} \right\}. \quad (18)$$

We shall use $\delta e^2, \delta \gamma^2, \delta e'^2$ to denote the variations of e^2, γ^2, e'^2 , respectively.

The presence of the factor e^2 in $c_2, \delta e^2$ and of γ^2 in $c_3, \delta \gamma^2$ enables us to calculate the principal parts of $\delta n, \delta \pi_1, \delta \theta_1$ without the intervention of c_1, c_2, c_3 , when $\pi_1, \theta_1, (\kappa/r)_0$ are known.

(a) *Neglect all powers of e^2, γ^2 .* The first of equations (18) gives

$$\delta \frac{\partial}{\partial n} \left(\frac{\kappa}{r} \right)_0 = 0,$$

or
$$\delta n = \left\{ \frac{\partial^2}{\partial n \partial e^2} \left(\frac{\kappa}{r} \right)_0 \right\} \delta e^2 + \left\{ \frac{\partial^2}{\partial n \partial \gamma^2} \left(\frac{\kappa}{r} \right)_0 \right\} \delta \gamma^2,$$

giving the part of δn , independent of e^2, γ^2 .

When δn has been thus found, the corresponding portions of e^2, γ^2 are given by

$$\delta \pi_1 = \frac{\partial \pi_1}{\partial n} \delta n + \frac{\partial \pi_1}{\partial e^2} \delta e^2,$$

$$\delta \theta_1 = \frac{\partial \theta_1}{\partial n} \delta n + \frac{\partial \theta_1}{\partial e^2} \delta e^2.$$

(β) Neglect all powers of e^3 , γ^3 above the first. Denote by $(\delta n)_e$, $(\delta n)_\gamma$ the parts of δn which contain the factors e^i , γ^i respectively. Using Adams' first theorem, we find from (18)

$$c_2 \left\{ \frac{\partial^2 \pi_1}{\partial n \partial n} (\delta n)_e + \frac{\partial^2 \pi_1}{\partial n \partial e^2} \delta e^2 \right\} = \frac{3}{2} (\delta n)_e \frac{\partial^2}{\partial n \partial n} \left(\frac{\kappa}{r} \right)_0,$$

$$c_3 \left\{ \frac{\partial^2 \theta_1}{\partial n \partial n} (\delta n)_e + \frac{\partial^2 \theta_1}{\partial n \partial e^2} \delta e^2 \right\} = \frac{3}{2} (\delta n)_\gamma \frac{\partial^2}{\partial n \partial n} \left(\frac{\kappa}{r} \right)_0,$$

which give the terms in δn containing the factors e^2 , γ^2 .

To calculate the parts of $\delta \pi_1$, $\delta \theta_1$ which contain the factors e^3 , γ^3 it will probably be simplest and most accurate to find δe^3 , $\delta \gamma^3$ from the equations

$$\delta c_2 = 0, \quad \delta c_3 = 0,$$

using the known value of δn . We then substitute in the equations

$$\delta \pi_1 = \frac{\partial \pi_1}{\partial e^2} \delta e^2 + \frac{\partial \pi_1}{\partial \gamma^2} \delta \gamma^2 + \frac{\partial \pi_1}{\partial n} \delta n + \frac{\partial \pi_1}{\partial e^2} \delta e^2,$$

$$\delta \theta_1 = \frac{\partial \theta_1}{\partial e^2} \delta e^2 + \frac{\partial \theta_1}{\partial \gamma^2} \delta \gamma^2 + \frac{\partial \theta_1}{\partial n} \delta n + \frac{\partial \theta_1}{\partial e^2} \delta e^2.$$

The degree of accuracy obtained will, in general, correspond to that with which $\partial \pi_1 / \partial e^2$ is known. The terms of higher orders are almost insensible.

I hope shortly to publish a verification by this method of some of Delaunay's results for the secular accelerations;* a preliminary examination of some of the terms exhibits errors which are not unimportant.

* [Note added March 29th, 1897.—See *Monthly Notices, R.A.S.*, March, 1897, where complete numerical results are given.]