Problems relating to the Impact of Waves on a Spherical Obstacle in an Elastic Medium. By Prof. HORACE LAMB, F.R.S.
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The scattering of elastic waves by a spherical obstacle whose properties differ slightly from those of the surrounding medium was discussed long ago, and applied to the illustration of optical phenomena, by Lord Rayleigh.* The present paper takes up the same problem in a more general form; thus in § 5 the obstacle is supposed to be rigid and fixed; in § 6 it is rigid but movable; we next consider the case of a rigid shell containing a vibratory mechanism (§ 7) or a gyrostat (§ 8); in § 9 the effect of a spherical cavity in the medium is examined; finally, in § 10, we have the case of an elastic sphere whose density and rigidity may differ to any extent from those of the surrounding medium.

The investigation was undertaken originally with a view of obtaining a mechanical illustration of the selective absorption of light by a gas, on the hypothesis that this (apparent) absorption is due to a coincidence between the period of the incident waves and that of a free mode of vibration of the molecules. It was found, however, that in this field, as well as in others, the electromagnetic theory furnishes in some respects a much more complete analogy with optical conditions. It has also the advantage of greater mathematical simplicity. For these reasons the calculations of this paper were for a time suspended; they are now completed and published in the hope that they may still present some features of dynamical interest.

The point above referred to, having been in the meantime discussed at length in the papers cited below, is here treated more briefly in § 11. It appears that the law of dissipation of radiant energy by scattering (when this is a maximum) has the same form as in the acoustical analogy considered in a former paper,† viz., we find that the ratio (I) which the energy dissipated outwards per

* *Phil. Mag.*, June, 1871; *Scientific Papers*, Vol. I., p. 104.

† "A Problem in Resonance, illustrative of the Mechanical Theory of Selective Absorption of Light by a Gas," *Proc. Lond. Math. Soc.*, Vol. xxxii., p. 11.

unit time from the sphere bears to the energy-flux in the incident waves is given by

$$I = \frac{2n+1}{2\pi} \lambda^2, \quad (1)$$

where λ is the wave-length, and n is the index of the spherical harmonic belonging to the particular type of free vibration in question. It is remarkable that this ratio is independent of the size or constitution of the spherical obstacle, and involves only the wave-length. The same law (as might be expected from the nature of the analysis) is met with in the corresponding electro-magnetic problem,* and appears to have a very wide range of generality.

The notation for the various solutions of the differential equation

$$\frac{d^2 F}{d\xi^2} + \frac{2(n+1)}{\xi} \frac{dF}{d\xi} + F = 0 \quad (2)$$

which are required in investigations such as the present can hardly be said to be definitely fixed. In the author's *Hydrodynamics*,† it has been proposed to write

$$\begin{aligned} \psi_n(\xi) &= (-)^n \left(\frac{d}{\xi d\xi} \right)^n \frac{\sin \xi}{\xi} \\ &= \frac{1}{1.3 \dots (2n+1)} \left\{ 1 - \frac{\xi^2}{2(2n+3)} + \frac{\xi^4}{2.4(2n+3)(2n+5)} - \dots \right\}, \end{aligned} \quad (3)$$

$$\begin{aligned} \Psi_n(\xi) &= (-)^n \left(\frac{d}{\xi d\xi} \right)^n \frac{\cos \xi}{\xi} \\ &= \frac{1.3 \dots (2n-1)}{\xi^{2n+1}} \left\{ 1 - \frac{\xi^2}{2(1-2n)} + \frac{\xi^4}{2.4(1-2n)(3-2n)} - \dots \right\}, \end{aligned} \quad (4)$$

$$\text{and} \quad f_n(\xi) = (-)^n \left(\frac{d}{\xi d\xi} \right)^n \frac{e^{-\xi}}{\xi} = \Psi_n(\xi) - i\psi_n(\xi); \quad (5)$$

these notations will be adhered to in the present paper. The above

* *Camb. Trans.*, Vol. xviii., p. 348. [See also L. Lorenz, *Œuvres Scientifiques*, Copenhagen, 1898, t. i., p. 499, where the formula (1) is arrived at from a somewhat different basis. The investigation was published (in Danish) in 1890; the writer regrets that he has only recently become acquainted with it.—*Sept.*, 1900.]

† §§ 267, 309.

functions all satisfy equations of the types

$$\psi'_n(\xi) = -\xi\psi_{n+1}(\xi), \quad (6)$$

$$\xi\psi'_n(\xi) + (2n+1)\psi_n(\xi) = \psi_{n-1}(\xi), \quad (7)$$

which are frequently required in reductions. We shall also have occasion to appeal to the formula

$$x\phi_n = \frac{r^3}{2n+1} \left(\frac{d\phi_n}{dx} - r^{2n+1} \frac{d}{dx} \frac{\phi_n}{r^{2n+1}} \right) \quad (8)$$

and its companions, where ϕ_n is a solid harmonic of (positive or negative) degree n .

General Equations.

1. The component displacements (u, v, w) at any point of an incompressible isotropic medium satisfy the equations

$$\frac{d^2u}{dt^2} = c^2\nabla^2u + \frac{1}{\rho} \frac{dp}{dx}, \quad \frac{d^2v}{dt^2} = c^2\nabla^2v + \frac{1}{\rho} \frac{dp}{dy}, \quad \frac{d^2w}{dt^2} = c^2\nabla^2w + \frac{1}{\rho} \frac{dp}{dz}, \quad (9)$$

$$\frac{du}{dx} + \frac{dv}{dy} + \frac{dw}{dz} = 0, \quad (10)$$

where c is the wave-velocity, connected with the rigidity (μ) and the density (ρ) by the relation

$$c^2 = \mu/\rho, \quad (11)$$

and p denotes the mean normal traction about the point (x, y, z). From these we deduce

$$\nabla^2p = 0. \quad (12)$$

In the case of simple-harmonic motion we assume a time-factor e^{ikt} , so that $2\pi/k$ denotes the wave-length of plane waves of the same frequency. The equations (9) then take the forms

$$(\nabla^2 + k^2)u = -\frac{1}{\mu} \frac{dp}{dx}, \quad (\nabla^2 + k^2)v = -\frac{1}{\mu} \frac{dp}{dy}, \quad (\nabla^2 + k^2)w = -\frac{1}{\mu} \frac{dp}{dz}. \quad (13)$$

In the problems which we have in view the medium is limited internally by a spherical surface (of radius a) described about the origin, and the disturbance may be regarded as made up of two parts, viz., the extraneous disturbance due to given sources at a distance, and the waves scattered outwards by the sphere.

If the medium were uninterrupted, the displacements due to the extraneous disturbance would be finite at the origin. Hence for the incident wave-system we may write

$$\left. \begin{aligned} p &= -\mu k^3 \Sigma r^n S_n \\ u &= \Sigma u_n, \quad v = \Sigma v_n, \quad w = \Sigma w_n \end{aligned} \right\}, \quad (14)$$

where

$$\left. \begin{aligned} u_n &= \frac{d}{dx} r^n S_n + (n+1) \psi_{n-1}(kr) \frac{d}{dx} r^n T_n - nk^3 r^{2n+3} \psi_{n+1}(kr) \frac{d}{dx} \frac{T_n}{r^{n+1}} \\ &\quad + \psi_n(kr) \left(y \frac{d}{dz} - z \frac{d}{dy} \right) r^n U_n \\ v_n &= \frac{d}{dy} r^n S_n + (n+1) \psi_{n-1}(kr) \frac{d}{dy} r^n T_n - nk^3 r^{2n+3} \psi_{n+1}(kr) \frac{d}{dy} \frac{T_n}{r^{n+1}} \\ &\quad + \psi_n(kr) \left(z \frac{d}{dx} - x \frac{d}{dz} \right) r^n U_n \\ w_n &= \frac{d}{dz} r^n S_n + (n+1) \psi_{n-1}(kr) \frac{d}{dz} r^n T_n - nk^3 r^{2n+3} \psi_{n+1}(kr) \frac{d}{dz} \frac{T_n}{r^{n+1}} \\ &\quad + \psi_n(kr) \left(x \frac{d}{dy} - y \frac{d}{dx} \right) r^n U_n \end{aligned} \right\} \quad (15)$$

The function $\psi_n(kr)$ is defined by (3), and S_n, T_n, U_n are spherical surface-harmonics of order n . The sign Σ indicates summation with respect to n , which may have the values 1, 2, 3, These formulæ make

$$xu + yv + zw = \Sigma \{ nr^n S_n + n(n+1)(2n+1) \psi_n(kr) r^n T_n \}, \quad (16)$$

$$x \left(\frac{dw}{dy} - \frac{dv}{dz} \right) + y \left(\frac{du}{dz} - \frac{dw}{dx} \right) + z \left(\frac{dv}{dx} - \frac{du}{dy} \right) = -\Sigma n(n+1) \psi_n(kr) r^n U_n. \quad (17)$$

When the extraneous disturbance consists of a train of plane waves, the harmonics S_n, T_n, U_n are determined as follows. Assuming that the direction of propagation is that of x -negative, and that the vibration is parallel to y , we may write, symbolically,

$$u = 0, \quad v = e^{ikx}, \quad w = 0, \quad (18)$$

the time-factor e^{ikt} being omitted, here and elsewhere, for shortness. If this be resolved into the sum of a series of disturbances of the

type (15), we must have, by (16) and (17),

$$\Sigma \{ n r^n S_n + n(n+1)(2n+1) \psi_n(kr) r^n T_n \} = y e^{ikr}, \quad (19)$$

$$\Sigma n(n+1) \psi_n(kr) r^n U_n = -ikz e^{ikr}. \quad (20)$$

Now, if we put

$$x = r \cos \theta, \quad y = r \sin \theta \cos \omega, \quad z = r \sin \theta \sin \omega, \quad (21)$$

$$\text{we have } iky e^{ikr} = \Sigma (2n+1)(ikr)^n \psi_n(kr) \sin \theta \cos \omega P'_n(\cos \theta), \quad (22)$$

where $P_n(\cos \theta)$ is the ordinary zonal harmonic. We infer, by comparison with (19), that $S_n = 0$, and

$$T'_n = -\frac{(ik)^{n-1}}{n(n+1)} \sin \theta \cos \omega P'_n(\cos \theta). \quad (23)$$

Similarly, we find

$$U_n = -\frac{2n+1}{n(n+1)} (ik)^n \sin \theta \sin \omega P'_n(\cos \theta). \quad (24)$$

In particular,

$$T_1 = \frac{1}{2} \sin \theta \cos \omega = \frac{1}{2} \frac{y}{r}, \quad U_1 = -\frac{3}{2} ik \sin \theta \sin \omega = -\frac{3}{2} ik \frac{z}{r}, \quad (25)$$

$$T_2 = \frac{1}{2} ik \sin \theta \cos \theta \cos \omega = \frac{1}{2} ik \frac{xy}{r^2}, \quad U_2 = \frac{5}{2} k^2 \sin \theta \cos \theta \sin \omega = \frac{5}{2} k^2 \frac{xz}{r^2}. \quad (26)$$

The component tractions p_{rx} , p_{ry} , p_{rz} on a spherical surface of radius r described about the origin are given by

$$\left. \begin{aligned} r p_{rx} &= x p + \mu \left(r \frac{d}{dr} - 1 \right) u + \mu \frac{d}{dx} (xu + yv + zw) \\ r p_{ry} &= y p + \mu \left(r \frac{d}{dr} - 1 \right) v + \mu \frac{d}{dy} (xu + yv + zw) \\ r p_{rz} &= z p + \mu \left(r \frac{d}{dr} - 1 \right) w + \mu \frac{d}{dz} (xu + yv + zw) \end{aligned} \right\} \quad (27)$$

The parts of p_{rx} , p_{ry} , p_{rz} which arise from (u_n, v_n, w_n) may be conveniently denoted by F_n, G_n, H_n . Their values are written down here for reference, although they are not really required until § 6.

We have

$$\left. \begin{aligned} rF_n/\mu &= A_n(kr) \frac{d}{dx} r^n S_n + B_n(kr) r^{2n+1} \frac{d}{dx} \frac{S_n}{r^{n+1}} + C_n(kr) \frac{d}{dx} r^n T_n \\ &\quad + D_n(kr) r^{2n+1} \frac{d}{dx} \frac{T_n}{r^{n+1}} + E_n(kr) \left(y \frac{d}{dz} - z \frac{d}{dy} \right) r^n U_n \\ rG_n/\mu &= A_n(kr) \frac{d}{dy} r^n S_n + B_n(kr) r^{2n+1} \frac{d}{dy} \frac{S_n}{r^{n+1}} + C_n(kr) \frac{d}{dy} r^n T_n \\ &\quad + D_n(kr) r^{2n+1} \frac{d}{dy} \frac{T_n}{r^{n+1}} + E_n(kr) \left(z \frac{d}{dx} - x \frac{d}{dz} \right) r^n U_n \\ rH_n/\mu &= A_n(kr) \frac{d}{dz} r^n S_n + B_n(kr) r^{2n+1} \frac{d}{dz} \frac{S_n}{r^{n+1}} + C_n(kr) \frac{d}{dz} r^n T_n \\ &\quad + D_n(kr) r^{2n+1} \frac{d}{dz} \frac{T_n}{r^{n+1}} + E_n(kr) \left(x \frac{d}{dy} - y \frac{d}{dx} \right) r^n U_n \end{aligned} \right\}, \quad (28)$$

$$\text{where } \left. \begin{aligned} A_n(kr) &= -\frac{k^2 r^2}{2n+1} + 2(n-1) \\ B_n(kr) &= \frac{k^2 r^3}{2n+1} \end{aligned} \right\}, \quad (29)$$

$$\left. \begin{aligned} C_n(kr) &= (n+1) \{ kr \psi'_{n-1}(kr) + 2(n-1) \psi_{n-1}(kr) \} \\ D_n(kr) &= -nk^2 r^3 \{ kr \psi'_{n+1}(kr) - \psi_{n+1}(kr) \} \end{aligned} \right\}, \quad (30)$$

$$E_n(kr) = kr \psi'_n(kr) + (n-1) \psi_n(kr).^* \quad (31)$$

In the case of plane waves the terms in S_n , as we have seen, are absent.

Divergent Waves.

2. We have next to consider the expressions for the divergent waves. For these we have

$$\left. \begin{aligned} p &= -\mu k^3 \Sigma \frac{S'_n}{r^{n+1}} \\ u &= \Sigma u_n, \quad v = \Sigma v_n, \quad w = \Sigma w_n \end{aligned} \right\}, \quad (32)$$

* The steps of the calculation are indicated in *Hydrodynamics*, § 306.

where

$$\left. \begin{aligned} u_n' &= \frac{d}{dx} \frac{S_n'}{r^{n+1}} + (n+1)f_{n-1}(kr) \frac{d}{dx} r^n T_n' - nk^2 r^{2n+3} f_{n+1}(kr) \frac{d}{dx} \frac{T_n'}{r^{n+1}} \\ &\quad + f_n(kr) \left(y \frac{d}{dz} - z \frac{d}{dy} \right) r^n U_n' \\ v_n' &= \frac{d}{dy} \frac{S_n'}{r^{n+1}} + (n+1)f_{n-1}(kr) \frac{d}{dy} r^n T_n' - nk^2 r^{2n+3} f_{n+1}(kr) \frac{d}{dy} \frac{T_n'}{r^{n+1}} \\ &\quad + f_n(kr) \left(z \frac{d}{dx} - x \frac{d}{dz} \right) r^n U_n' \\ w_n' &= \frac{d}{dz} \frac{S_n'}{r^{n+1}} + (n+1)f_{n-1}(kr) \frac{d}{dz} r^n T_n' - nk^2 r^{2n+3} f_{n+1}(kr) \frac{d}{dz} \frac{T_n'}{r^{n+1}} \\ &\quad + f_n(kr) \left(x \frac{d}{dy} - y \frac{d}{dx} \right) r^n U_n' \end{aligned} \right\} \quad (33)$$

$f_n(kr)$ being the function defined by (5). The surface-harmonics S_n' , T_n' , U_n' are to be found in terms of S_n , T_n , U_n by means of the conditions which hold at the surface $r = a$. The admissible values of n are 1, 2, 3, ..., as before, the case $n = 0$ being excluded, since the volume enclosed by the spherical boundary will be assumed to be constant. These formulæ make

$$xu + yv + zw = \Sigma \left\{ -(n+1) \frac{S_n'}{r^{n+1}} + n(n+1)(2n+1)f_n(kr) r^n T_n' \right\}, \quad (34)$$

$$x \left(\frac{dw}{dy} - \frac{dv}{dz} \right) + y \left(\frac{du}{dz} - \frac{dw}{dx} \right) + z \left(\frac{dv}{dx} - \frac{du}{dy} \right) = -\Sigma n(n+1)f_n(kr) T_n'. \quad (35)$$

Again, if we denote the parts of p_{rx} , p_{ry} , p_{rz} which are due to (u_n', v_n', w_n') by F_n' , G_n' , H_n' , we find

$$\left. \begin{aligned} rF_n'/\mu &= A_n'(kr) \frac{1}{r^{2n+1}} \frac{d}{dx} r^n S_n' + B_n'(kr) \frac{d}{dx} \frac{S_n'}{r^{n+1}} + C_n'(kr) \frac{d}{dx} r^n T_n' \\ &\quad + D_n'(kr) r^{2n+1} \frac{d}{dx} \frac{T_n'}{r^{n+1}} + E_n'(kr) \left(y \frac{d}{dz} - z \frac{d}{dy} \right) r^n U_n' \\ rG_n'/\mu &= A_n'(kr) \frac{1}{r^{2n+1}} \frac{d}{dy} r^n S_n' + B_n'(kr) \frac{d}{dy} \frac{S_n'}{r^{n+1}} + C_n'(kr) \frac{d}{dy} r^n T_n' \\ &\quad + D_n'(kr) r^{2n+1} \frac{d}{dy} \frac{T_n'}{r^{n+1}} + E_n'(kr) \left(z \frac{d}{dx} - x \frac{d}{dz} \right) r^n U_n' \\ rH_n'/\mu &= A_n'(kr) \frac{1}{r^{2n+1}} \frac{d}{dz} r^n S_n' + B_n'(kr) \frac{d}{dz} \frac{S_n'}{r^{n+1}} + C_n'(kr) \frac{d}{dz} r^n T_n' \\ &\quad + D_n'(kr) r^{2n+1} \frac{d}{dz} \frac{T_n'}{r^{n+1}} + E_n'(kr) \left(x \frac{d}{dy} - y \frac{d}{dx} \right) r^n U_n' \end{aligned} \right\}, \quad (36)$$

$$\text{where } \left. \begin{aligned} A_n(kr) &= -\frac{k^2 r^2}{2n+1} \\ B_n(kr) &= \frac{k^2 r^2}{2n+1} - 2(n+2) \end{aligned} \right\}, \quad (37)$$

$$\left. \begin{aligned} C_n(kr) &= (n+1) \{kr f'_{n-1}(kr) + 2(n-1)f_{n-1}(kr)\} \\ D_n(kr) &= -nk^2 r^2 \{kr f'_{n+1}(kr) - f_{n+1}(kr)\} \end{aligned} \right\}, \quad (38)$$

$$E_n(kr) = kr f'_n(kr) + (n-1)f_n(kr). \quad (39)$$

3. We have to examine the properties of the scattered waves at a great distance from the origin. When kr is large, we have

$$f_n(kr) = \frac{i^n}{(kr)^{n+1}} e^{-ikr}, \quad (40)$$

approximately, and therefore, keeping only the most important terms,

$$\left. \begin{aligned} u_n &= (2n+1) \frac{i^{n-1}}{(kr)^n} e^{-ikr} \left(\frac{d}{dx} r^n T_n^n - nxr^{n-2} T_n^n \right) \\ &\quad + \frac{i^n}{(kr)^{n+1}} e^{-ikr} \left(y \frac{d}{dz} - z \frac{d}{dy} \right) r^n U_n^n \\ v_n &= (2n+1) \frac{i^{n-1}}{(kr)^n} e^{-ikr} \left(\frac{d}{dy} r^n T_n^n - ny r^{n-2} T_n^n \right) \\ &\quad + \frac{i^n}{(kr)^{n+1}} e^{-ikr} \left(z \frac{d}{dx} - x \frac{d}{dz} \right) r^n U_n^n \\ w_n &= (2n+1) \frac{i^{n-1}}{(kr)^n} e^{-ikr} \left(\frac{d}{dz} r^n T_n^n - nz r^{n-2} T_n^n \right) \\ &\quad + \frac{i^n}{(kr)^{n+1}} e^{-ikr} \left(x \frac{d}{dy} - y \frac{d}{dx} \right) r^n U_n^n \end{aligned} \right\}, \quad (41)$$

where a slight transformation has been effected by means of the formula (8). We notice that u_n, v_n, w_n are ultimately of the order $1/r$, whilst the radial displacement $(xu_n + yv_n + zw_n)/r$ is zero to the present degree of approximation. It is really of the order $1/r^2$, as is easily seen from (34).

The geometrical interpretation of the two types of terms which appear in (41) is very simple. The terms in T_n^n express a state of motion in which the particles on a sphere of large radius r are

vibrating in directions orthogonal to the contour-lines of the harmonic T_n , with an amplitude inversely proportional to the distance between consecutive contours, if we imagine these to be drawn for equal infinitesimal increments of T_n . The terms in U_n , on the other hand, express a vibration *along* the contour lines of U_n , with a similar law of amplitude.*

For example, if (in accordance with a later notation) we write

$$rT_1 = \frac{1}{2}B_1y, \quad rU_1 = 0, \quad (42)$$

$$\text{we have} \quad (u_1, v_1, w_1) = \frac{3B_1}{2k} \left(-\frac{xy}{r^3}, \frac{x^2+z^2}{r^3}, -\frac{yz}{r^3} \right) \frac{e^{-ikr}}{r}. \quad (43)$$

The particles on a sphere of large radius r are here vibrating in the directions of the meridians in which the sphere is cut by planes through the axis of y , with an amplitude proportional to the distance from this axis.

On the other hand, if

$$rT_1 = 0, \quad rU_1 = -\frac{3}{2}ikC_1z, \quad (44)$$

$$\text{we find} \quad (u_1, v_1, w_1) = \frac{3C_1}{2k} \left(\frac{y}{r}, -\frac{x}{r}, 0 \right) \frac{e^{-ikr}}{r}. \quad (45)$$

The lines of motion are circles about the axis of z , and the amplitude, for points at the same distance r from the origin, varies as the distance from this axis.

$$\text{Again, if} \quad r^2T_2 = \frac{1}{2}ikB_2xy, \quad U_2 = 0, \quad (46)$$

we find

$$(u_2, v_2, w_2) = -\frac{5B_2}{2k} \left(y \frac{(y^2+z^2-x^2)}{r^3}, \frac{x(z^2+x^2-y^2)}{r^3}, \frac{2xyz}{r^3} \right) \frac{e^{-ikr}}{r}. \quad (47)$$

The lines of motion on a sphere of radius r are the orthogonal trajectories of the curves in which the sphere is cut by the hyperbolic cylinders $xy = \text{const.}$ It is easily proved that these trajectories are the intersections of the sphere with the cones $z^2 = \lambda(x^2 - y^2)$, where λ is a variable parameter. They all pass through the four points

$$\left(\pm \frac{r}{\sqrt{2}}, \pm \frac{r}{\sqrt{2}}, 0 \right).$$

The values of F_n, G_n, H_n at a great distance from the origin are

* Cf. *Proc. Lond. Math. Soc.*, Vol. XIII., p. 194.

most easily obtained from (27) and (41). Having regard to the order of magnitude of u_n , v_n , w_n and $(xu_n + yv_n + zw_n)/r$, we readily find

$$F_n = -ik\mu u_n, \quad G_n = -ik\mu v_n, \quad H_n = -ik\mu w_n. \quad (48)$$

It appears that the radial stress is relatively very small, as might have been anticipated from the fact that the waves tend to become ultimately plane.

Propagation of Energy Outwards.

4. The formulæ (41) and (48) enable us to calculate a very useful expression for the rate at which energy is carried off by the system of scattered waves represented by (32), (33). If, after restoring the time-factor, we take only the real parts of our expressions, the rate in question is equal to the mean value (with respect to the time) of the integral

$$-\iint \left(p_{rx} \frac{du}{dt} + p_{ry} \frac{dv}{dt} + p_{rz} \frac{dw}{dt} \right) dS, \quad (49)$$

taken over the surface of a sphere, of *any* radius r , described about the origin. For simplicity of calculation we may suppose r to be infinitely great.

It appears on examination that, if m and n be distinct,

$$\iint \left(F_m \frac{du_n}{dt} + G_m \frac{dv_n}{dt} + H_m \frac{dw_n}{dt} \right) dS = 0, \quad (50)$$

in virtue of the conjugate property of spherical harmonics of different orders, so that the expression (49) reduces to the form

$$-\Sigma \iint \left(F_n \frac{du_n}{dt} + G_n \frac{dv_n}{dt} + H_n \frac{dw_n}{dt} \right) dS. \quad (51)$$

Moreover, it is easily seen that the terms in this expression which involve combinations of T_n with U_n will disappear.*

Let us now write in (41)

$$r^n T_n = \Phi_n + i\phi_n. \quad (52)$$

Taking real parts, we have, so far as the terms in T_n are concerned,

$$\frac{du_n}{dt} = kc \frac{2n+1}{(kr)^n} \left\{ \left(\frac{d\Phi_n}{dv} - n \frac{x}{r^2} \Phi_n \right) \cos k(ct-r+\epsilon) - \left(\frac{d\phi_n}{dv} - n \frac{x}{r^2} \phi_n \right) \sin k(ct-r+\epsilon) \right\}, \quad (53)$$

* See *Camb. Trans.*, Vol. XVIII., p. 357.

where ϵ may be 0 or π or $\pm\frac{1}{2}\pi$. Also

$$-F_n = \mu k \frac{2n+1}{(kr)^n} \{ \dots \}, \quad (54)$$

where the terms in $\{ \}$ are the same as in (53). Hence, so far as it depends on T_n , the mean value of the expression (51) will be

$$\begin{aligned} & \frac{1}{2} \mu k^2 c \frac{(2n+1)^2}{(kr)^{2n}} \iint \left\{ \left(\frac{d\Phi_n}{dx} - n \frac{x}{r^3} \Phi_n \right)^2 + \left(\frac{d\Phi_n}{dy} - n \frac{y}{r^3} \Phi_n \right)^2 + \left(\frac{d\Phi_n}{dz} - n \frac{z}{r^3} \Phi_n \right)^2 \right. \\ & \quad \left. + \left(\frac{d\phi_n}{dx} - n \frac{x}{r^3} \phi_n \right)^2 + \left(\frac{d\phi_n}{dy} - n \frac{y}{r^3} \phi_n \right)^2 + \left(\frac{d\phi_n}{dz} - n \frac{z}{r^3} \phi_n \right)^2 \right\} dS \\ &= \frac{1}{2} \mu k^2 c \frac{(2n+1)^2}{(kr)^{2n}} \iint \left\{ \left(\frac{d\Phi_n}{dx} \right)^2 + \left(\frac{d\Phi_n}{dy} \right)^2 + \left(\frac{d\Phi_n}{dz} \right)^2 - \frac{n^2}{r^3} \Phi_n^2 \right. \\ & \quad \left. + \left(\frac{d\phi_n}{dx} \right)^2 + \left(\frac{d\phi_n}{dy} \right)^2 + \left(\frac{d\phi_n}{dz} \right)^2 - \frac{n^2}{r^3} \phi_n^2 \right\} dS. \end{aligned} \quad (55)$$

In the same way, if

$$r^n U_n = X_n + iX_n, \quad (56)$$

we find that the mean value of (51), so far as it depends on U_n , will be

$$\begin{aligned} & \frac{1}{2} \frac{\mu k^2 c}{(kr)^{2n+2}} \iint \left\{ \left(y \frac{dX_n}{dz} - z \frac{dX_n}{dy} \right)^2 + \left(z \frac{dX_n}{dx} - x \frac{dX_n}{dz} \right)^2 + \left(x \frac{dX_n}{dy} - y \frac{dX_n}{dx} \right)^2 \right. \\ & \quad \left. + \left(y \frac{dX_n}{dz} - z \frac{dX_n}{dy} \right)^2 + \left(z \frac{dX_n}{dx} - x \frac{dX_n}{dz} \right)^2 + \left(x \frac{dX_n}{dy} - y \frac{dX_n}{dx} \right)^2 \right\} dS \\ &= \frac{1}{2} \frac{\mu c}{(kr)^{2n}} \iint \left\{ \left(\frac{dX_n}{dx} \right)^2 + \left(\frac{dX_n}{dy} \right)^2 + \left(\frac{dX_n}{dz} \right)^2 - \frac{n^2}{r^3} X_n^2 \right. \\ & \quad \left. + \left(\frac{dX_n}{dx} \right)^2 + \left(\frac{dX_n}{dy} \right)^2 + \left(\frac{dX_n}{dz} \right)^2 - \frac{n^2}{r^3} X_n^2 \right\} dS. \end{aligned} \quad (57)$$

The expressions under the integral signs in (55) and (57) are the sums of the squares of the tangential components of the vectors $(d\Phi_n/dx, d\Phi_n/dy, d\Phi_n/dz)$, &c., as might have been foreseen from the geometrical interpretation indicated in § 3. Now, turning to polar coordinates, if Ω_n be a surface-harmonic of order n , we have

$$\int_0^{2\pi} \int_0^\pi \left\{ \left(\frac{d\Omega_n}{d\theta} \right)^2 + \left(\frac{d\Omega_n}{\sin \theta d\omega} \right)^2 \right\} \sin \theta d\theta d\omega = n(n+1) \int_0^{2\pi} \int_0^\pi \Omega_n^2 \sin \theta d\theta d\omega. \quad (58)$$

Hence, summing the expressions (55) and (57), we obtain

$$\frac{1}{2}\mu c \Sigma \frac{n(n+1)}{k^{2n}} \int_0^{2\pi} \int_0^\pi \{ (2n+1)^2 k^2 |T_n^\lambda|^2 + |U_n^\lambda|^2 \} \sin \theta \, d\theta \, d\omega. \quad (59)$$

In the case of incident plane waves, the formulæ (23) and (24) suggest that we should write

$$T_n^\lambda = B_n^\lambda T_n = B_n^\lambda \frac{(ik)^{n-1}}{n(n+1)} \sin \theta \cos \omega P_n'(\cos \theta), \quad (60)$$

$$U_n^\lambda = C_n^\lambda U_n = -C_n^\lambda \frac{2n+1}{n(n+1)} (ik)^n \sin \theta \sin \omega P_n'(\cos \theta). \quad (61)$$

$$\text{Since } \int_0^{2\pi} \int_0^\pi \{ \sin \theta \cos \omega P_n'(\cos \theta) \}^2 \sin \theta \, d\theta \, d\omega = \frac{2\pi n(n+1)}{2n+1}, \quad (62)$$

the expression (59) now takes the form

$$\pi \mu c \Sigma (2n+1) \{ |B_n^\lambda|^2 + |C_n^\lambda|^2 \}. \quad (63)$$

The proper standard of comparison here is the energy propagated per unit time across unit area (of the wave-front) in the primary waves represented by (18). This is equal to $\frac{1}{2}\mu k^2 c$. Hence, denoting by I the ratio which the energy dissipated by the scattered waves bears to the energy-flux in the primary waves, we have

$$I = \frac{2\pi}{k^2} \Sigma (2n+1) \{ |B_n^\lambda|^2 + |C_n^\lambda|^2 \}. \quad (64)$$

This has exactly the same form as in the corresponding electromagnetic problem.*

Fixed Rigid Obstacle.

5. We may first suppose that the internal boundary ($r = a$) is absolutely fixed. The surface-conditions then are

$$\Sigma (u_n + u_n^\lambda) = 0, \quad \Sigma (v_n + v_n^\lambda) = 0, \quad \Sigma (w_n + w_n^\lambda) = 0. \quad (65)$$

If we substitute the values of $u_n, v_n, w_n, u_n^\lambda, v_n^\lambda, w_n^\lambda$ from (15) and (33), we deduce†

$$S_n + (n+1) \psi_{n-1}(ka) T_n + (n+1) f_{n-1}(ka) T_n^\lambda = 0, \quad (66)$$

$$-nk^2 a^2 \psi_{n+1}(ka) T_n + \frac{S_n^\lambda}{a^{2n+1}} - nk^2 a^2 f_{n+1}(ka) T_n^\lambda = 0, \quad (67)$$

$$\psi_n(ka) U_n + f_n(ka) U_n^\lambda = 0. \quad (68)$$

* *Camb. Trans.*, Vol. XVIII., p. 358. [See also Lorenz, *loc. cit.*, p. 492.]

† The proof of these and similar inferences in the sequel is indicated in *Hydrodynamics*, § 307.

These determine the harmonics S_n^1 , T_n^1 , U_n^1 in terms of S_n , T_n , U_n , which are regarded as given.

When there is no extraneous disturbance, we have $S_n = 0$, $T_n = 0$, $U_n = 0$. The free vibrations involving T_n^1 have values of k determined by

$$f_{n-1}(ka) = 0; \quad (69)$$

thus, for $n = 2$, we find $ika = -1$. Since the time-factor is e^{ikt} , this gives an "aperiodic" vibration $e^{-ct/a}$. The free vibrations involving U_n^1 have values of k determined by

$$f_n(ka) = 0. \quad (70)$$

The characteristic of all these free vibrations, whether oscillatory or not, is that the "modulus of decay" is comparable with the time a plane wave would take to traverse a distance equal to the radius of the sphere.

In the case of plane incident waves, we have $S_n = 0$; and, if we further suppose that the wave-length is large compared with the dimensions of the spherical obstacle, ka will be a small quantity. Hence, keeping only the most important terms, we find

$$T_n^1 = -\frac{\psi_{n-1}(ka)}{f_{n-1}(ka)} T_n = -\frac{(ka)^{2n-1}}{\{1.3 \dots (2n-3)\}^2 (2n-1)} T_n, \quad (71)$$

$$U_n^1 = -\frac{\psi_n(ka)}{f_n(ka)} U_n = -\frac{(ka)^{2n+1}}{\{1.3 \dots (2n-1)\}^2 (2n+1)} U_n. \quad (72)$$

The most important part of the scattered waves is that represented by T_1^1 . Since, from (71) and (25),

$$T_1^1 = -kaT_1 = -\frac{1}{2}ka \frac{y}{r}, \quad (73)$$

the corresponding vibration (u_1^1 , v_1^1 , w_1^1) at a distance from the origin is found by making $B_1^1 = -ka$ in (43). This type of vibration is familiar in problems such as we are engaged on, but the amplitude is now much greater than usual, viz., in a ratio of the order $(ka)^{-2}$. This is accounted for by the abnormal degree of constraint imposed on our medium at a rigid and immovable boundary. In the corresponding acoustical problem,* where plane waves of sound impinge on a fixed sphere, there is complete freedom of lateral motion at the surface.

* Lord Rayleigh, *Theory of Sound*, § 334.

The formula (64) gives for the dissipation-ratio in the present case

$$I = 6\pi a^2, \quad (74)$$

approximately.

Rigid Movable Sphere.

6. Next suppose that a perfectly rigid but movable sphere is embedded in the medium. The tractions on the surface will reduce to a force (X, Y, Z) and a couple (L, M, N). When the incident wave-system is that given by (18), the components Y of force and N of couple will (in the absence of gyrostatic influence) be the only ones which do not vanish, and the motion of the sphere will consist of a linear oscillation in the axis of y and an angular oscillation about the axis of z . It is convenient, however, in the first instance, to treat the problem more generally. In any case it appears that the only parts of our general formulæ which will contribute anything to the values of X, Y, Z, L, M, N are those for which $n = 1$; moreover, that the linear and angular motions of the sphere are independent, the former depending on the harmonics S_1, T_1 , and the latter on U_1 . For $n > 1$, the determination of S'_n, T'_n, U'_n in terms of S_n, T_n, U_n will be exactly the same as when the sphere was fixed (§ 5).

$$\text{We find } \left. \begin{aligned} X &= \iint (F_1 + F'_1) dS \\ Y &= \iint (G_1 + G'_1) dS \\ Z &= \iint (H_1 + H'_1) dS \end{aligned} \right\}, \quad (75)$$

$$\left. \begin{aligned} L &= \iint \{y(H_1 + H'_1) - z(G_1 + G'_1)\} dS \\ M &= \iint \{z(F_1 + F'_1) - x(H_1 + H'_1)\} dS \\ N &= \iint \{x(G_1 + G'_1) - y(F_1 + F'_1)\} dS \end{aligned} \right\}, \quad (76)$$

where the integrations extend over the surface $r = a$. Hence, substituting from (28) and (36), we find

$$\begin{aligned} X &= -\frac{4}{3}\pi\mu k^2 a^3 \left\{ \frac{d}{dx} rS_1 + \frac{1}{a^3} \frac{d}{dx} rS'_1 \right\} \\ &\quad + 8\pi\mu ka^2 \left\{ \psi'_0(ka) \frac{d}{dx} rT_1 + f'_0(ka) \frac{d}{dx} rT'_1 \right\}, \end{aligned} \quad (77)$$

$$L = -\frac{8}{3}\pi\mu ka^4 \left\{ \psi'_1(ka) \frac{d}{dx} rU_1 + f'_1(ka) \frac{d}{dx} rU'_1 \right\}, \quad (78)$$

with similar equations. We may therefore write

$$-\frac{4}{3}\pi\mu k^2 a^4 \left(S_1 + \frac{S_1'}{a^3} \right) + 8\pi\mu k a^3 \{ \psi_0'(ka) T_1' + f_0'(ka) T_1'' \} = Xx + Yy + Zz, \quad (79)$$

$$-\frac{8}{3}\pi\mu k a^5 \{ \psi_1'(ka) U_1 + f_1'(ka) U_1' \} = Lx + My + Nz. \quad (80)$$

Again, if (α, β, γ) and (ξ, η, ζ) be the linear and angular displacements of the sphere, the kinematical relations to be satisfied at the surface $r = a$ are

$$u_1 + u_1' = \alpha + \eta z - \zeta y, \quad v_1 + v_1' = \beta + \zeta x - \xi z, \quad w_1 + w_1' = \gamma + \xi y - \eta x. \quad (81)$$

Hence, by (16), (17), and (34), (35),

$$a \left(S_1 - 2 \frac{S_1'}{a^3} \right) + 6a \{ \psi_1(ka) T_1' + f_1(ka) T_1'' \} = \alpha x + \beta y + \gamma z, \quad (82)$$

$$-a \{ \psi_1(ka) U_1 + f_1(ka) U_1' \} = \xi x + \eta y + \zeta z. \quad (83)$$

It follows also from (15) and (33) that we must have

$$\frac{S_1'}{a^3} - k^3 a^3 \{ \psi_2(ka) T_1' + f_2(ka) T_1'' \} = 0, \quad (84)$$

since the surface-harmonics of order 2, in the values of $u_1 + u_1'$, $v_1 + v_1'$, $w_1 + w_1'$ must cancel.

We have, further, the dynamical equations

$$m \frac{d^2 \alpha}{dt^2} = X, \quad m \frac{d^2 \beta}{dt^2} = Y, \quad m \frac{d^2 \gamma}{dt^2} = Z, \quad (85)$$

$$K \frac{d^2 \xi}{dt^2} = L, \quad K \frac{d^2 \eta}{dt^2} = M, \quad K \frac{d^2 \zeta}{dt^2} = N, \quad (86)$$

where m denotes the mass of the sphere, and K its moment of inertia. There is an obvious convenience in writing

$$m = \kappa \cdot \frac{4}{3}\pi\rho a^3, \quad K = \nu \cdot \frac{4}{3}\pi\rho a^5 \cdot \frac{2}{5}a^2. \quad (87)$$

Remembering that the time-factor is e^{ikrt} , and that $c = \mu/\rho$, the equations (85), (86) give

$$Xx + Yy + Zz = -\kappa \cdot \frac{4}{3}\pi\mu k^3 a^3 (\alpha x + \beta y + \gamma z), \quad (88)$$

$$Lx + My + Nz = -\nu \cdot \frac{8}{15}\pi\mu k^3 a^5 (\xi x + \eta y + \zeta z). \quad (89)$$

Hence, from (79) and (82),

$$(\kappa - 1) S_1 - (2\kappa + 1) \frac{S_1'}{a^3} + 6(\kappa - 1) \{ \psi_1(ka) T_1' + f_1(ka) T_1'' \} = 0. \quad (90)$$

Combined with (84) this gives

$$(\kappa-1)S_1 + \{(2\kappa+1)\psi_0(ka) - 9\psi_1(ka)\}T_1 \\ + \{(2\kappa+1)f_0(ka) - 9f_1(ka)\}T_1' = 0, \quad (91)$$

where a simplification has been made by means of (6) and (7).

Similarly, from (80), (83), and (89), we deduce

$$\{\tfrac{1}{2}\nu ka \psi_1(ka) + \psi_1'(ka)\}U_1 + \{\tfrac{1}{2}\nu ka f_1(ka) + f_1'(ka)\}U_1' = 0. \quad (92)$$

The free *rectilinear* vibrations which the sphere can execute in the absence of extraneous disturbance are found by putting $S_1 = 0$, $T_1 = 0$ in (91), whence

$$(2\kappa+1)f_0(ka) - 9f_1(ka) = 0, \quad (93)$$

$$\text{or} \quad (2\kappa+1)(ika)^2 + 9(ika) + 9 = 0. \quad (94)$$

If $\kappa > \frac{5}{8}$, the roots of this quadratic in ika are complex, with a real negative part, viz.,

$$ika = -\frac{9}{4\kappa+2} \pm i \frac{3\sqrt{(8\kappa-5)}}{4\kappa+2} \quad (95)$$

This indicates a periodic oscillation with a gradually diminishing amplitude. When the ratio (κ) which the mass of the sphere bears to that of an equal volume of the medium is a large number, the wave-length corresponding to the period will be large compared with the radius of the sphere, and a considerable number of oscillations will take place in the interval required for the amplitude to diminish in the ratio $1/e$. When $\kappa < \frac{5}{8}$, the values of ika are real and negative, and the vibration is aperiodic.

Again, the free *rotational* vibrations of the sphere are found by putting $U_1 = 0$ in (92); thus

$$\tfrac{1}{2}\nu ka f_1(ka) + f_1'(ka) = 0, \quad (96)$$

$$\text{or} \quad \nu(ika)^3 + (\nu+5)(ika)^2 + 15ika + 15 = 0. \quad (97)$$

When ν is large one root of this cubic in ika is $-1 - \frac{1}{5}\nu$, nearly, corresponding to an aperiodic motion. The others are given by

$$ika = -\frac{75}{2\nu^2} \pm i \sqrt{\left(\frac{15}{\nu}\right)}, \quad (98)$$

approximately; and it is to be observed that the modulus of decay is, relatively to the period, very much longer than in the case of the rectilinear vibrations.

In the case of the plane incident system (18) we have $S_1 = 0$, and the values of T'_1 and U'_1 in terms of T_1 and U_1 , respectively, are given by (91) and (92). When κ and ν have only moderate values, simple approximate expressions are easily found for the case where ka is small; thus

$$T'_1 = \frac{2}{3}(\kappa - 1)k^3a^3T_1, \quad U'_1 = \frac{1}{45}(\nu - 1)k^5a^5U_1, \quad (99)$$

showing that the *angular* motion of the sphere is of secondary importance. We must remember that T'_2 is now of equal rank with T'_1 , being given by (71). The most important parts of the disturbance at a distance are therefore found by putting

$$B'_1 = \frac{2}{3}(\kappa - 1)k^3a^3, \quad B'_2 = -\frac{1}{3}k^3a^3, \quad (100)$$

in (43) and (47) respectively. The dissipation-ratio is

$$I = \left\{ \frac{8}{27}(\kappa - 1)^2 + \frac{40}{81} \right\} \pi a^2 (ka)^4, \quad (101)$$

approximately, showing the usual law of dependence upon wavelength.

The expressions (99) vanish, to our present degree of approximation, when $\kappa = 1$, $\nu = 1$. For $\kappa = 0$, $\nu = 0$ they coincide with the results obtained below in § 9 for the case of a spherical cavity in the medium.

When, on the other hand, κ and ν are so large as to be comparable with $(ka)^{-2}$, the above approximation is insufficient. It is clear however that, except in the neighbourhood of the critical wavelengths referred to in § 11, the results must tend to coincide with those obtained in § 5 for a fixed sphere, and in particular the dissipation-ratio must tend to the value (74).

Rigid Shell containing a Vibratory Mechanism.

7. The investigation of the preceding section may be readily extended in various ways, by imagining that the sphere is hollow and contains a dynamical system capable of storing potential and kinetic energy. Various arrangements of this kind have been suggested by Lord Kelvin. As a sufficient example of a system devoid of gyrostatic influence, we may take the case of a rigid spherical shell containing a central spherical nucleus attached to it by springs.

To calculate first the linear vibration parallel to y , such as will be called into play by waves of the type (18), let m denote the mass of the shell, m_0 that of the nucleus, and let β, β_0 be the displacements of

their centres. If $2\pi/\sigma_0$ be the period of vibration of the nucleus when the shell is held fixed, we have

$$\frac{d^2\beta_0}{dt^2} = -\sigma_0^2(\beta_0 - \beta) \quad (102)$$

and
$$m \frac{d^2\beta}{dt^2} + m_0 \frac{d^2\beta_0}{dt^2} = Y, \quad (103)$$

where Y denotes, as before, the effect of the stresses on the outer surface. Writing $-k^2c^2$ for d^2/dt^2 , and eliminating β_0 , we find

$$- \frac{(m + m_0) \sigma_0^2 - m k^2 c^2}{\sigma_0^2 - k^2 c^2} k^2 c^2 \beta = Y. \quad (104)$$

If we put
$$m = \kappa \cdot \frac{4}{3} \pi \rho a^3, \quad m_0 = \kappa_0 \cdot \frac{4}{3} \pi \rho a^3, \quad (105)$$

it appears that to adapt the various results of § 6 to the present circumstances we have merely to replace κ by

$$\kappa' = \frac{(\kappa + \kappa_0) \sigma_0^2 - \kappa k^2 c^2}{\sigma_0^2 - k^2 c^2} \quad (106)$$

The limiting form of the result when the inertia of the nucleus is very great is obtained by putting $\kappa_0 = \infty$, $\sigma_0 = 0$, whilst $\kappa_0 \sigma_0^2$ is finite. The shell then behaves as if (in addition to the force Y) it were urged to a fixed position by a force varying as the distance.

Similarly, if the springs resist rotation of the nucleus relatively to the containing shell, we have to replace ν by

$$\nu' = \frac{(\nu + \nu_0) \sigma_0^2 - \nu k^2 c^2}{\sigma_0^2 - k^2 c^2}, \quad (107)$$

where the constants ν , ν_0 , σ_0 have analogous meanings, as regards rotation, to those adopted for κ , κ_0 , σ_0 in the case of translation.

It is evident in each case that the approximations we have made on the assumption that ka is small will require revision when ka is nearly equal to $\sigma_0 a/c$. The further consideration of such questions is, however, deferred to § 11.

Rigid Shell containing a Gyrostat.

8. To examine the kind of modification that will be introduced into our results by gyrostatic influence, we may imagine that the sphere of § 6 is replaced by a spherical shell containing a gyrostat in rapid rotation. The first part of the investigation proceeds as before,

until we come to the dynamical equations of rotation (86). These now take the forms

$$\left. \begin{aligned} K \frac{d^2 \xi}{dt^2} + R \frac{d\eta}{dt} - Q \frac{d\zeta}{dt} &= L, \\ K \frac{d^2 \eta}{dt^2} + P \frac{d\zeta}{dt} - R \frac{d\xi}{dt} &= M, \\ K \frac{d^2 \zeta}{dt^2} + Q \frac{d\xi}{dt} - P \frac{d\eta}{dt} &= N \end{aligned} \right\}. \quad (108)$$

The symbols P, Q, R here denote the components of angular momentum of the gyrostator when the containing shell is at rest in its equilibrium position.

Special interest attaches to the two cases where the axis of the gyrostator is parallel and perpendicular, respectively, to the direction of propagation of the plane waves.

In the former case we have $Q = R = 0$, and the equations (108) reduce to

$$K \frac{d^2 \xi}{dt^2} = L, \quad K \frac{d^2 \eta}{dt^2} + P \frac{d\zeta}{dt} = M, \quad K \frac{d^2 \zeta}{dt^2} - P \frac{d\eta}{dt} = N. \quad (109)$$

The last two are equivalent to

$$\left. \begin{aligned} K \frac{d^2}{dt^2} (\eta + i\zeta) - iP \frac{d}{dt} (\eta + i\zeta) &= M + iN \\ K \frac{d^2}{dt^2} (\eta - i\zeta) + iP \frac{d}{dt} (\eta - i\zeta) &= M - iN \end{aligned} \right\}. \quad (110)$$

The rotational oscillations about the axis of x call for no special treatment. In the remaining types we have $\xi = 0, L = 0$, and *either*

$$\left. \begin{aligned} \eta &= i\zeta, \quad M = iN \\ K \frac{d^2 \eta}{dt^2} - iP \frac{d\eta}{dt} &= M \end{aligned} \right\}, \quad (111)$$

with

$$\left. \begin{aligned} \eta &= -i\zeta, \quad M = -iN \\ K \frac{d^2 \eta}{dt^2} + iP \frac{d\eta}{dt} &= M \end{aligned} \right\}. \quad (112)$$

with

In the case of (111) we may write

$$rU_1 = B(y - iz), \quad rU_1' = B'(y - iz). \quad (113)$$

Hence, from (83) and (80),

$$\eta = - \{ B\psi_1(ka) + B'f_1(ka) \}, \quad (114)$$

$$M = - \frac{8}{3}\pi\mu ka^4 \{ B\psi_1'(ka) + B'f_1'(ka) \}. \quad (115)$$

If we write $K = \nu \cdot \frac{4}{3}\pi\rho a^3 \cdot \frac{2}{5}a^2$, $P = K\varpi$, (116)

ϖ being a quantity of the nature of an angular velocity, and if we assume the time-factor e^{ikt} in (111), we find

$$\left\{ \frac{1}{5}\nu \left(ka - \frac{\varpi a}{c} \right) \psi_1(ka) + \psi_1'(ka) \right\} B \\ + \left\{ \frac{1}{5}\nu \left(ka - \frac{\varpi a}{c} \right) f_1(ka) + f_1'(ka) \right\} B' = 0, \quad (117)$$

which is the altered form of (92).

In the *free* oscillations of the type (111) we have $B = 0$, and

$$\frac{1}{5}\nu \left(ka - \frac{\varpi a}{c} \right) f_1(ka) + f_1'(ka) = 0, \quad (118)$$

or $\frac{1}{5}\nu ika(ika+1) \left(ika - i\frac{\varpi a}{c} \right) + (ika)^2 + 3ika + 3 = 0. \quad (119)$

If ν is a large number, one root of this is $ika = -1$, nearly; this indicates a rapidly decaying vibration. If, further, $\varpi a/c$ be small compared with $\sqrt{(15/\nu)}$, the remaining roots will be

$$ika = ik_1a, \quad -ik_2a, \quad (120)$$

where $k_1a = \sqrt{\frac{15}{\nu} + \frac{1}{2}\frac{\varpi a}{c}}$, $k_2a = \sqrt{\frac{15}{\nu} - \frac{1}{2}\frac{\varpi a}{c}}$, (121)

nearly. It may be shown that in the motion corresponding to the positive value of k the axis of the gyrostat has a positive precessional motion about the axis of x , whilst the negative root gives a negative precessional motion. The period ($2\pi/k_1c$ or $2\pi/k_2c$) is shorter when the direction of the precessional motion agrees with that of the rotation of the gyrostat than when it is opposed to it.

If we substitute from (113) in (41), we find that at a distance from the sphere

$$(u_1, v_1, w_1) = \frac{iB'}{k^3r} \left(-\frac{iy+z}{r}, \frac{ix}{r}, \frac{x}{r} \right) \begin{cases} e^{ik_1(ct-r)} \\ e^{-ik_2(ct-r)} \end{cases}. \quad (122)$$

At points on the axis of x the vibrations are circular; in the wave corresponding to k_1 they have the same sense as ϖ , and in the wave corresponding to k_2 they have the sense opposite to that of ϖ . At points of the plane yz the vibrations are rectilinear, and parallel to x .

The consideration of the system (112) leads to similar conclusions.

To examine the effect of the obstacle on a system of plane waves it is necessary to analyse these into right-handed and left-handed circular vibrations. For a train of right-handed circular vibrations, say

$$u = 0, \quad v = e^{ikx}, \quad w = -ie^{ikx}, \quad (123)$$

the formulæ (113) will apply, provided $B = \frac{3}{2}k$. The value of B is then given by (117), and the scattered waves by

$$(u_1, v_1, w_1) = \frac{iB}{k^3 r} \left(-\frac{iy+z}{r}, \frac{ix}{r}, \frac{x}{r} \right) e^{ik(ct-r)}, \quad (124)$$

which has just been interpreted. The case of a train of left-handed vibrations could be treated in a similar manner, but the result will evidently be equivalent to reversing the sign of ϖ in (117).

To investigate the case where the axis of the gyrostat is perpendicular to the direction of propagation of the plane waves, it will be sufficient to take the case where the vibrations in these are rectilinear, as expressed by (18), and the axis of the gyrostat is parallel to the direction of vibration. If we put $P = R = 0$, the equations (108) reduce to

$$K \frac{d^3 \xi}{dt^3} - Q \frac{d\xi}{dt} = L, \quad K \frac{d^2 \eta}{dt^2} = M, \quad K \frac{d^3 \zeta}{dt^3} + Q \frac{d\xi}{dt} = N. \quad (125)$$

So far as the effect of the plane waves is concerned we may suppose that $\eta = 0$, $M = 0$. Also, assuming

$$rU_1 = Cz, \quad rU_1 = A'x + C'z, \quad (126)$$

$$\text{we find} \quad \xi = -A'f_1(ka), \quad \zeta = -\{C\psi_1(ka) + C'f_1(ka)\}, \quad (127)$$

$$L = -\frac{2}{3}\pi\mu ka^4 A'f_1'(ka), \quad N = -\frac{2}{3}\pi\mu ka^4 \{C\psi_1'(ka) + C'f_1'(ka)\}. \quad (128)$$

$$\text{Writing} \quad Q = K\varpi \quad (129)$$

and substituting in (125), we find

$$\left\{ \frac{1}{5} \nu k a f_1(ka) + f_1'(ka) \right\} A' + \frac{1}{5} i \nu \frac{\varpi a}{c} f_1(ka) C' = - \frac{1}{5} i \nu \frac{\varpi a}{c} \psi_1(ka) C, \quad (130)$$

$$\begin{aligned} - \frac{1}{5} i \nu \frac{\varpi a}{c} f_1(ka) A' + \left\{ \frac{1}{5} \nu k a f_1(ka) + f_1'(ka) \right\} C' \\ = - \left\{ \frac{1}{5} \nu k a \psi_1(ka) + \psi_1'(ka) \right\} C, \quad (131) \end{aligned}$$

whence

$$\begin{aligned} \left[\left\{ \frac{1}{5} \nu k a f_1(ka) + f_1'(ka) \right\}^2 - \left\{ \frac{1}{5} \nu \frac{\varpi a}{c} f_1(ka) \right\}^2 \right] \frac{A'}{C} \\ = - i \nu \frac{\varpi a}{c} \{ \psi_1(ka) f_1'(ka) - f_1(ka) \psi_1'(ka) \} \\ = i \nu \frac{\varpi a}{c} \frac{1}{k^4 a^4}, \quad (132) \end{aligned}$$

and

$$\begin{aligned} \left[\left\{ \frac{1}{5} \nu k a f_1(ka) + f_1'(ka) \right\}^2 - \left\{ \frac{1}{5} \nu \frac{\varpi a}{c} f_1(ka) \right\}^2 \right] \frac{C'}{C} \\ = - \left[\frac{1}{5} \nu k a \psi_1(ka) + \psi_1'(ka) \right] \left\{ \frac{1}{5} \nu k a f_1(ka) + f_1'(ka) \right\} \\ - \nu^2 \frac{\varpi^2 a^2}{c^2} \psi_1(ka) f_1(ka) \Big]. \quad (133) \end{aligned}$$

If we substitute from (126) in (41), the component displacements at a distance from the origin are given by

$$(u_1, v_1, w_1) = \frac{i}{k^2 r} \left(\frac{C' y}{r}, \frac{A' z - C' x}{r}, - \frac{A' y}{r} \right) e^{ik(st-r)}. \quad (134)$$

For sufficiently small values of ϖ the value of C' will be sensibly the same as in the problem of § 6, whilst A' will be relatively small. The chief effect of the terms in A' will be to introduce a small vibration parallel to y at points on the axis of z , whilst at points on the axis of y the previously rectilinear vibration will become elliptic, the elliptic orbits being very elongated, with their major axes parallel to the axis of x .

Spherical Cavity.

9. By way of contrast with the problem of § 5, we may next consider the case where the inner boundary $r = a$ is free from stress; in other words, there is a spherical cavity in the medium. It

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appears from (28) and (36) that the surface-conditions

$$\Sigma (F_n + F_n^*) = 0, \quad \Sigma (G_n + G_n^*) = 0, \quad \Sigma (H_n + H_n^*) = 0 \quad (135)$$

will be satisfied, provided

$$A_n(ka) S_n + C_n(ka) T_n + A_n^*(ka) \frac{S_n^*}{a^{2n+1}} + C_n^*(ka) T_n = 0, \quad (136)$$

$$B_n(ka) S_n + D_n(ka) T_n + B_n^*(ka) \frac{S_n^*}{a^{2n+1}} + D_n^*(ka) T_n = 0, \quad (137)$$

$$E_n(ka) U_n + E_n^*(ka) U_n^* = 0. \quad (138)$$

Leaving aside the "free" vibrations, we have, when ka is small,

$$A_n(ka) = 2(n-1), \quad B_n(ka) = \frac{k^2 a^3}{2n+1}, \quad (139)$$

$$\left. \begin{aligned} C_n(ka) &= \frac{2(n^2-1)}{1.3 \dots (2n-1)}, & D_n(ka) &= \frac{nk^2 a^3}{1.3 \dots (2n+3)} \\ E_n^*(ka) &= \frac{n-1}{1.3 \dots (2n+1)} \end{aligned} \right\}, \quad (140)$$

$$A_n^*(ka) = -\frac{k^2 a^3}{2n+1}, \quad B_n^*(ka) = -2(n+2), \quad (141)$$

$$\left. \begin{aligned} C_n^*(ka) &= -\frac{1.3 \dots (2n-3)(n+1)}{(ka)^{2n-1}} \\ D_n^*(ka) &= \frac{1.3 \dots (2n+1) 2n(n+2)}{(ka)^{2n+1}} \\ E_n^*(ka) &= -\frac{1.3 \dots (2n-1)(n+2)}{(ka)^{2n+1}} \end{aligned} \right\}, \quad (142)$$

approximately. Substituting in (136), (137), (138), we find that when $S_n = 0$, as in the case of incident plane waves,

$$T_n^* = \frac{2(n^2-1)(ka)^{2n-1}}{\{1.3 \dots (2n-3)\}^2 (2n-1)(2n^2+1)} T_n \quad (143)$$

$$U_n^* = \frac{(n-1)(ka)^{2n+1}}{\{1.3 \dots (2n-1)\}^2 (2n+1)(n+2)} U_n \quad (144)$$

The case $n = 1$ is exceptional, and demands a closer approximation. We find

$$A_1(ka) = -\frac{1}{3}k^2a^3, \quad B_1(ka) = \frac{1}{3}k^2a^3, \quad (145)$$

$$C_1(ka) = -\frac{2}{3}k^3a^3, \quad D_1(ka) = \frac{1}{15}k^3a^3, \quad E_1(ka) = -\frac{1}{15}k^3a^3, \quad (146)$$

$$A_1'(ka) = -\frac{1}{3}k^2a^2, \quad B_1'(ka) = -6, \quad (147)$$

$$C_1'(ka) = -\frac{2}{ka}, \quad D_1'(ka) = \frac{18}{k^3a^3}, \quad E_1'(ka) = -\frac{3}{k^3a^3}, \quad (148)$$

and thence $T_1' = -\frac{2}{3}k^3a^3T_1, \quad U_1' = -\frac{1}{45}k^5a^5U_1. \quad (149)$

Since, from (143), $T_2' = \frac{2}{3}k^3a^3T_2, \quad (150)$

it appears that the most important terms in the expressions for the scattered wave are those which arise from T_1' and T_2' . In the case of the wave-system represented by (18), we have

$$T_1' = -\frac{1}{3}k^3a^3\frac{y}{r}, \quad T_2' = \frac{1}{3}ik^4a^3\frac{2y}{r^2}; \quad (151)$$

the terms in question are therefore obtained by writing $B_1' = -\frac{2}{3}k^3a^3$, $B_2' = \frac{2}{3}k^3a^3$, in (43) and (47). And from (64) the dissipation-ratio is found to be

$$I = \frac{8}{27}\pi k^4a^6 + \frac{4}{81}\pi k^4a^6 = \frac{64}{81}\pi a^2(ka)^4, \quad (152)$$

approximately. This gives the usual law of intensity varying as the inverse fourth power of the wave-length.

Elastic Spherical Obstacle.

10. In the case of an elastic incompressible sphere of density ρ_0 and rigidity μ_0 embedded in an incompressible medium of density ρ and rigidity μ , we may write

$$c_0^2 = \mu_0/\rho_0, \quad (153)$$

and, since the time-factor (e^{ikt}) is everywhere the same,

$$k_0c_0 = kc, \quad (154)$$

whence

$$\mu_0k_0^2 : \mu k^2 = \rho_0 : \rho. \quad (155)$$

The formulæ appropriate to the interior of the sphere will be of the same character as those of § 1, with k_0 written for k , and μ_0 for μ . The typical expressions for the displacements inside the sphere may

therefore be taken to be

$$\left. \begin{aligned} u_n'' &= \frac{d}{dx} r^n S_n'' + (n+1) \psi_{n-1}(k_0 r) \frac{d}{dx} r^n T_n'' - n k_0^2 r^{2n+3} \psi_{n+1}(k_0 r) \frac{d}{dx} \frac{T_n''}{r^{n+1}} \\ &\quad + \psi_n(k_0 r) \left(y \frac{d}{dz} - z \frac{d}{dy} \right) r^n U_n'' \\ v_n'' &= \frac{d}{dy} r^n S_n'' + (n+1) \psi_{n-1}(k_0 r) \frac{d}{dy} r^n T_n'' - n k_0^2 r^{2n+3} \psi_{n+1}(k_0 r) \frac{d}{dy} \frac{T_n''}{r^{n+1}} \\ &\quad + \psi_n(k_0 r) \left(z \frac{d}{dx} - x \frac{d}{dz} \right) r^n U_n'' \\ w_n'' &= \frac{d}{dz} r^n S_n'' + (n+1) \psi_{n-1}(k_0 r) \frac{d}{dz} r^n T_n'' - n k_0^2 r^{2n+3} \psi_{n+1}(k_0 r) \frac{d}{dz} \frac{T_n''}{r^{n+1}} \\ &\quad + \psi_n(k_0 r) \left(x \frac{d}{dy} - y \frac{d}{dx} \right) r^n U_n'' \end{aligned} \right\}, \quad (156)$$

The corresponding stresses F_n'' , G_n'' , H_n'' on a sphere of radius r will be as in (28), with μ_0 written for μ , k_0 for k , and S_n'' , T_n'' , U_n'' for S_n , T_n , U_n , respectively.

The continuity of displacement and stress at the surface $r = a$ requires

$$u_n + u_n' = u_n'', \quad v_n + v_n' = v_n'', \quad w_n + w_n' = w_n'', \quad (157)$$

$$F_n + F_n' = F_n'', \quad G_n + G_n' = G_n'', \quad H_n + H_n' = H_n''. \quad (158)$$

These lead in the usual manner to the formulæ

$$S_n + (n+1) \psi_{n-1}(ka) T_n + (n+1) f_{n-1}(ka) T_n' = S_n'' + (n+1) \psi_{n-1}(k_0 a) T_n'', \quad (159)$$

$$-n k^2 a^3 \psi_{n+1}(ka) T_n + \frac{S_n'}{a^{2n+1}} - n k^2 a^3 f_{n+1}(ka) T_n' = -n k_0^2 a^3 \psi_{n+1}(k_0 a) T_n'', \quad (160)$$

$$\psi_n(ka) U_n + f_n(ka) U_n' = \psi_n(k_0 a) U_n'', \quad (161)$$

and

$$\begin{aligned} \mu A_n(ka) S_n + \mu C_n(ka) T_n + \mu A_n'(ka) \frac{S_n'}{a^{2n+1}} + \mu C_n'(ka) T_n' \\ = \mu_0 A_n(k_0 a) S_n'' + \mu_0 C_n(k_0 a) T_n'', \end{aligned} \quad (162)$$

$$\begin{aligned} \mu B_n(ka) S_n + \mu D_n(ka) T_n + \mu B_n'(ka) \frac{S_n'}{a^{2n+1}} + \mu D_n'(ka) T_n' \\ = \mu_0 B_n(k_0 a) S_n'' + \mu_0 D_n(k_0 a) T_n'', \end{aligned} \quad (163)$$

$$\mu E_n(ka) U_n + \mu E_n'(ka) U_n' = \mu_0 E_n(k_0 a) U_n'', \quad (164)$$

where the functions $A_n(ka)$, $A_n'(ka)$, &c., have the meanings assigned in (29), (30), (31), and (37), (38), (39).

When the extraneous disturbance consists of plane waves, we have $S_n = 0$. If, further, ka and k_0a be both small, the values of T_n'' and U_n'' in terms of T_n and U_n respectively may be approximated to without much difficulty. Thus, for $n = 1$, we find, making use of the formulæ (145) ... (148),

$$2T_1 + \frac{2}{ka} T_1' = S_1'' + 2T_1'', \quad (165)$$

$$-\frac{1}{15}k^2a^2T_1 + \frac{S_1'}{a^3} - \frac{3}{k^3a^3} T_1' = -\frac{1}{15}k_0^2a^2T_1'', \quad (166)$$

$$-\frac{2}{3}\mu k^2a^2T_1 - \frac{1}{3}\mu k^2a^3 \frac{S_1'}{a^3} - \frac{2\mu}{ka} T_1' = -\frac{1}{3}\mu_0k_0^2a^3S_1'' - \frac{2}{3}\mu_0k_0^2a^2T_1'', \quad (167)$$

$$\frac{1}{15}\mu k^2a^3T_1 - 6\mu \frac{S_1'}{a^3} + \frac{18\mu}{k^3a^3} T_1' = \frac{1}{3}\mu_0k_0^2a^3S_1'' + \frac{1}{15}\mu_0k_0^2a^2T_1'', \quad (168)$$

where only the most important part of each coefficient has been retained.

The solution of these is facilitated by the consideration that T_1' must be of the order k^3a^3 compared with T_1 , as is seen from the results of § 9, which deals with a particular case of the present problem. Moreover T_1'' must be of the same order as T_1 . We thence find, from (165) and (166),

$$S_1'' + 2T_1'' = 2T_1, \quad (169)$$

$$S_1'/a^3 = 3/k^3a^3 \cdot T_1'. \quad (170)$$

Then (167) leads to the result

$$T_1' = \frac{2}{3}ka \frac{\mu_0k_0^2a^2 - \mu k^2a^2}{\mu} T_1 = \frac{2}{3}k^3a^3 \left(\frac{\rho_0}{\rho} - 1 \right) T_1, \quad (171)$$

by (155). This agrees with (149) if we put $\rho_0 = 0$.*

Similarly, the approximate forms of (161) and (164) are

$$\frac{1}{3}U_1 + \frac{1}{k^3a^3} U_1' = \frac{1}{3}U_1'', \quad (172)$$

$$-\frac{1}{15}\mu k^3a^2U_1 - \frac{3\mu}{k^3a^3} U_1' = -\frac{1}{15}\mu_0k_0^2a^2U_1'', \quad (173)$$

* It is to be noticed that the equations (167) and (168) cannot be assumed to be sufficiently accurate for the determination of S_1'' and T_1'' , owing to the approximate cancelling of the second and third terms in (168). The main interest lies, however, in the determination of T_1' .

leading to
$$U_1 = \frac{1}{45} k^3 a^3 \left(\frac{\rho_0}{\rho} - 1 \right) U_1, \quad (174)$$

which also may be compared with (149).

It appears, on reference to (99), that the values of T_1' and U_1' are, to our present degree of approximation, the same as if the sphere were absolutely rigid. This might have been foreseen, for the terms in T_1'' represent, when $k_0 a$ is small, a mere translation of the sphere as a whole, whilst those in U_1'' represent a pure rotation.

For $n > 1$, I find

$$T_n'' = - \frac{2(n^2-1)(\mu_0-\mu)}{2(n^2-1)\mu_0 + (2n^2+1)\mu} \cdot \frac{(ka)^{2n-1}}{\{1.3 \dots (2n-3)\}^2 (2n-1)} T_n, \quad (175)$$

$$U_n'' = - \frac{(n-1)(\mu_0-\mu)}{(n-1)\mu_0 + (n+2)\mu} \cdot \frac{(ka)^{2n+1}}{\{1.3 \dots (2n-1)\}^2 (2n+1)} U_n. \quad (176)$$

These coincide with (143), (144), if we put $\mu_0 = 0$. In particular

$$T_2' = - \frac{2(\mu_0-\mu)}{6\mu_0+9\mu} k^3 a^3 \cdot T_2, \quad U_2' = - \frac{\mu_0-\mu}{45(\mu_0+4\mu)} k^3 a^3 U_2. \quad (177)$$

The most important part of the disturbance at a distance in the scattered waves is therefore obtained by writing

$$B_1' = \frac{2}{3} \left(\frac{\rho_0}{\rho} - 1 \right) k^3 a^3, \quad B_2' = - \frac{2(\mu_0-\mu)}{6\mu_0+9\mu} k^3 a^3, \quad (178)$$

in the formulæ (43) and (47), respectively. The approximate value of the dissipation-ratio is

$$I = \left\{ \frac{8}{27} \left(\frac{\rho_0}{\rho} - 1 \right)^2 + \frac{40(\mu_0-\mu)^2}{(6\mu_0+9\mu)^2} \right\} \pi a^2 (ka)^4. \quad (179)$$

Effect of Synchronism.

11. In the various problems above considered, general formulæ have been obtained for the disturbance produced in a train of plane waves by a spherical obstacle constituted in various ways; but no

* When $(\rho_0-\rho)/\rho_0$ and $(\mu_0-\mu)/\mu$ are small, we have

$$T_1' = \frac{1}{6} k^3 a^3 \frac{\Delta \rho}{\rho} T_1, \quad T_2' = - \frac{1}{15} k^3 a^3 \frac{\Delta \mu}{\mu} T_2.$$

If we substitute the values of T_1 and T_2 from (25) and (26), and work out the expressions for the component *rotations* of the medium, in the scattered wave, we reproduce results given by Lord Rayleigh in a paper already cited (p. 120).

special examination has been made of cases where there is coincidence, or approximate coincidence, between the period of the waves and a natural period of vibration of the sphere. In this respect interest attaches chiefly to cases where the free period in question corresponds to a wave-length large in comparison with the circumference of the sphere, but our results will not be altogether dependent on this supposition.

The first instance where a free period of this character was met with was in the rectilinear vibrations of a rigid but movable sphere (§ 6) whose inertia bears a large ratio (κ) to that of an equal volume of the surrounding medium. If in (91) we put $S_1 = 0$, the accurate formula for the scattered waves of type T_1^i takes the form

$$\frac{T_1^i}{T_1} = - \frac{(2\kappa+1)\psi_0(ka) - 9\psi_1(ka)}{(2\kappa+1)f_0(ka) - 9f_1(ka)} = - \frac{g(ka)}{G(ka) - ig(ka)}, \quad (180)$$

where

$$g(ka) = (2\kappa+1)\psi_0(ka) - 9\psi_1(ka), \quad G(ka) = (2\kappa+1)\Psi_0(ka) - 9\Psi_1(ka). \quad (181)$$

The modulus of the expression in the last member of (180) is never greater than unity; but it attains the value unity, and the amplitude of the scattered waves is therefore a maximum, when

$$G(ka) = 0. \quad (182)$$

If we substitute the values of $\Psi_0(ka)$, $\Psi_1(ka)$ from (4), this equation reduces to

$$\tan ka = \frac{2\kappa+1}{9}ka - \frac{1}{ka}. \quad (183)$$

The situation of the roots is easily found by a graphical construction, and it appears that when κ is large there is a small root given (to a first approximation) by

$$ka = \frac{3}{\sqrt{(2\kappa-8)}}; \quad (184)$$

a result which may be compared with (95). When (182) is satisfied exactly, we have

$$T_1^i = -iT_1. \quad (185)$$

Hence, putting $B_1^i = -i$ in (64), we have, for the dissipation-ratio,

$$I = \frac{6\pi}{k^3} = \frac{3}{2\pi}\lambda^3, \quad (186)$$

where λ is the wave-length.

A similar argument applies to the rotational vibrations. We find, from (92),

$$\frac{U_1}{U_1} = - \frac{g(ka)}{G(ka) - ig(ka)}, \quad (187)$$

where

$$g(ka) = \frac{1}{2} \nu ka \psi_1(ka) + \psi_1'(ka), \quad G(ka) = \frac{1}{2} \nu ka \Psi_1(ka) + \Psi_1'(ka). \quad (188)$$

The wave-length of maximum dissipation is determined, as before, by $G(ka) = 0$, or

$$\tan ka = - \frac{1}{ka} - \frac{5ka}{\nu k^2 a^2 - 15}. \quad (189)$$

The dissipation-ratio, when this is exactly satisfied, is given again by (186).

It is evident that the form of (180) or (187) is quite independent of the special hypotheses here made, or of the value of n . The general value of the dissipation-ratio I , when this is a maximum, is

$$I = \frac{2n+1}{2\pi} \lambda^2, \quad (190)$$

where n is the order of the spherical harmonic involved.

The wave-lengths of maximum scattering are in all cases sharply defined. It would be easy to institute calculations showing the effect of a slight deviation from a critical period; but this point has perhaps been sufficiently illustrated in the former paper.*

12. The particular example, just referred to, of a rigid sphere embedded in the medium is in one respect unfitted for the illustration of optical phenomena. In order to obtain a free period large compared with a/c , it was necessary to ascribe to the sphere a great relative inertia. It appears from § 5 that this implies a very considerable *general* scattering of incident waves, independently of any approximation to the critical wave-length. The dissipation, as given by (74), is, in fact, of the order $(\lambda/a)^4$ compared with what we know to be consistent with the actual degree of transparency of the atmosphere.†

The difficulty may be met, though perhaps rather artificially, if we have recourse to the more general suppositions of § 7. Let us

* *Proc. Lond. Math. Soc.*, Vol. xxxii., p. 18.

† Lord Rayleigh, *Phil. Mag.*, April, 1899.

suppose, for simplicity, that we have a rigid massless shell enclosing a nucleus attached to it by springs. Putting $\kappa = 0$ in (106), we have

$$\kappa' = \frac{\kappa_0 \sigma_0^2}{\sigma_0^2 - k^2 c^2}. \quad (191).$$

If we substitute this for κ in (94), we obtain the equation to determine the free periods, viz.,

$$\left(1 + \frac{2\kappa_0 \sigma_0^2}{\sigma_0^2 - k^2 c^2}\right) k^2 a^3 - 9ika - 9 = 0. \quad (192)$$

If the stiffness of the springs and the mass of the nucleus be so adjusted that $\sigma_0 a/c$ is small, this will be satisfied approximately by

$$k^2 a^2 = \frac{\sigma_0^2 a^2}{c^2} \div \left(1 + \frac{2}{9} \frac{\kappa_0 \sigma_0^2 a^2}{c^2}\right). * \quad (193)$$

The wave-length of this free period is large compared with a , independently of the value of κ_0 .

To determine the forced oscillations we find from (180), with κ' written for κ , and from (193),

$$\frac{T_1}{T_1'} = - \frac{2(\kappa' - 1)}{(2\kappa' + 1)k^2 a^2 - 9ika - 9} k^2 a^3, \quad (194)$$

provided the value of ka does not approximate to that given by (193). The dissipation-ratio will then be of the usual form

$$I = Ca^2 (ka)^4, \quad (195)$$

where C is a moderate numerical factor.

[Again, we might employ the hypothesis of an elastic sphere, considered in § 10. To obtain free periods with wave-lengths large compared with a , we must suppose the constants so adjusted that c_0 , defined by (153), is small compared with c ; in other words, we must have μ_0/ρ_0 small compared with μ/ρ . Since we are precluded, for the reason above given, from assigning any great relative inertia to the sphere, the only assumption left to us is to imagine that μ_0 is small compared with μ .

With molecules adjusted in this manner we might build up a model of a gaseous assemblage freely transparent to radiation

* This indicates that the frequency of vibration of the nucleus is less than if the containing shell were held fixed, in the ratio $\left(1 + \frac{2}{9} \frac{\kappa_0 \sigma_0^2 a^2}{c^2}\right)^{-1}$

generally, but with strong absorption for definite wave-lengths. It is to be noticed, however, that in reconciling these requirements we have made it impossible to get rid of that constituent of the general scattered light which is dependent on the harmonic T_2' and is represented by the formulæ (47).*. If we survey the various problems of this paper, we find that this constituent is always important (with an amplitude proportional to $k^3 a^3/r$) except in the case of § 10, where it can be annulled (approximately) by making $\mu_0 = \mu$. This is, however, inadmissible for the reason just explained.]†

Thursday, April 5th, 1900.

Lt.-Col. A. J. C. CUNNINGHAM, R.E., Vice-President,
in the Chair.

Nineteen members present.

Mr. F. W. B. Frankland, B.A., Fellow of Clare College, Cambridge, was elected a member.

The following persons were admitted into the Society :—Miss B. M. Cave Browne Cave, Miss F. E. Cave Browne Cave, and Mr. R. W. H. T. Hudson.

Mr. Tucker read a letter from Mrs. J. J. Walker, thanking the Society for its vote of condolence with her on the loss of her husband.

Mr. Macdonald read a note "On the Addition-Theorem for the Bessel Functions," and Mr. Basset made a communication on "The Orthoptic Loci of Curves of a given class."

Prof. Love gave a proof of "The Uniform Convergence of Fourier's Series."

Major MacMahon spoke on "An Extension of Orthogonal and Boolean Covariants."

* For the argument that this constituent is absent from the light scattered by finely divided particles, see Lord Rayleigh, *Scientific Papers*, Vol. I., p. 107.

† Added Sept., 1900.

generally, but with strong absorption for definite wave-lengths. It is to be noticed, however, that in reconciling these requirements we have made it impossible to get rid of that constituent of the general scattered light which is dependent on the harmonic T_2' and is represented by the formulæ (47).* If we survey the various problems of this paper, we find that this constituent is always important (with an amplitude proportional to $k^3 a^3/r$) except in the case of § 10, where it can be annulled (approximately) by making $\mu_0 = \mu$. This is, however, inadmissible for the reason just explained.]†

Thursday, April 5th, 1900.

Lt.-Col. A. J. C. CUNNINGHAM, R.E., Vice-President,
in the Chair.

Nineteen members present.

Mr. F. W. B. Frankland, B.A., Fellow of Clare College, Cambridge, was elected a member.

The following persons were admitted into the Society :—Miss B. M. Cave Browne Cave, Miss F. E. Cave Browne Cave, and Mr. R. W. H. T. Hudson.

Mr. Tucker read a letter from Mrs. J. J. Walker, thanking the Society for its vote of condolence with her on the loss of her husband.

Mr. Macdonald read a note "On the Addition-Theorem for the Bessel Functions," and Mr. Basset made a communication on "The Orthoptic Loci of Curves of a given class."

Prof. Love gave a proof of "The Uniform Convergence of Fourier's Series."

Major MacMahon spoke on "An Extension of Orthogonal and Boolean Covariants."

* For the argument that this constituent is absent from the light scattered by finely divided particles, see Lord Rayleigh, *Scientific Papers*, Vol. I., p. 107.

† Added Sept., 1900.

Messrs. Macaulay, R. W. H. T. Hudson, Whittaker, and the Chairman spoke to one or more of the papers.

A paper by Mr. Bromwich, entitled "Note on Weierstrass's Reduction of a Family of Bilinear Forms," was received.

The Chairman announced that the May meeting would be made "special," as announced by letter to the members of the Society.

The following presents were made to the Library:—

"Educational Times," April, 1900.

"Indian Engineering," Vol. xxvii., Nos. 7-10, Feb. 17-Mar. 10, 1900.

"Nautical Almanac for 1903," 8vo; Edinburgh, 1900. From Nautical Almanac Office.

"Mathematical Gazette," Vol. i., No. 20; 1900.

"Periodico di Matematica," Serie 2, Vol. ii., Fasc. 5; Livorno, 1900.

"Supplemento al Periodico di Matematica," Anno iii., Fasc. 5; Livorno, 1900.

Darboux, G.—"Sur la Déformation des Surfaces du second degré," 4to; Paris, 1900.

Ostwald, W.—"Periodische Erscheinungen bei der Auflösung des Chroms in Säuren," 8vo; Leipzig, 1900.

Rice, H.—"The Theory and Practice of Interpolation," 8vo; Lynn, Mass., 1900. From the author.

"Transactions of the American Mathematical Society," Vol. i., No. 1; 1900.

"Memoirs of the National Academy of Sciences," Vol. viii., Memoir 4; Washington, 1899.

"Grundlage der Geometrie" (3 pp.), and "Euklidische Grundlage der Geometrie" (7 pp.), from the author, Dr. W. Veltmann.

"Report on the Progress of the Solution of the Problem of Three Bodies," by E. T. Whittaker (reprint from "British Association Report," 1899).

The following exchanges were received:—

"Proceedings of the Royal Society," Vol. lxxvi., Nos. 426, 427; 1900.

"Beiblätter zu den Annalen der Physik und Chemie," Bd. xxiv., St. 2, 3; Leipzig, 1900.

"Annales de la Faculté des Sciences de Toulouse," Série 2, Tome i., 3^{me} Fascicule; 1899.

"Bulletin of the American Mathematical Society," Series 2, Vol. vi., Nos. 5, 6, Feb.-Mar., 1900; New York.

"Bulletin des Sciences Mathématiques," Tome xxiv., Jan., 1900; Paris.

"Rendiconto dell' Accademia delle Scienze Fisiche e Matematiche," Serie 3, Vol. vi., Nos. 1, 2; Napoli, 1900.

"Atti della Reale Accademia dei Lincei—Rendiconti," Sem. 1, Vol. ix., Fasc. 4, 5; Roma, 1900.

"Nyt Tidsskrift for Matematik," B. Aargang xi., Nr. 1; Copenhagen, 1900.

"Sitzungsberichte der Königl. Preuss. Akademie der Wissenschaften zu Berlin," Nos. 46, 47; Nov., 1897.