XLVIII. Some ellipsoidal potentials, Æolotropic and isotropic

R. Hargreaves M.A.

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- (2) Different equations of state, all of which are fair approximations to the behaviour of a real gas, indicate very different values for these inversion temperatures.
- (3) The sensitiveness of the positions of these points to change in the characteristic equation of the fluid makes a knowledge of their actual position, as determined experimentally, a very valuable means of discriminating between the relative validity of any proposed equations of state.

In conclusion attention may be drawn to the fact that all the results given in this paper are *exact* consequences of the equations of state to which they relate.

XLVIII. *Some Ellipsoidal Potentials, Æolotropic and Isotropic.* By R. HARGREAVES, M.A.*

IN a former paper on Æolotropic Potential, the potential functions corresponding to normal distribution on a conducting ellipsoid and to uniform volume-distribution were treated. The functions are $\int_{\lambda}^{\infty} \frac{d\lambda}{\sqrt{J}}$ and $\int_{\lambda}^{\infty} \frac{(1-u_a) d\lambda}{\sqrt{J}}$, where J is the cubic in λ which replaces the isotropic form $(a^2+\lambda)(b^2+\lambda)(c^2+\lambda)$, and $u_a=1$ represents the æolian quadric which replaces a confocal. If the additional factor $\frac{\partial u_a}{\partial x}$ occurs under the integral sign a vector type is given, which was also considered in the second, but not in the first case.

It is proposed now to deal with the more general types

$$\int_{\lambda}^{\infty} \frac{d\lambda}{\sqrt{J}} [u_a^s, \text{ or } u_a^s(1-u_a), \text{ or } (1-u_a)^s],$$

and with types in which the factor $\frac{\partial u_a}{\partial x}$ is attached to the several forms. The second and third forms belong to volume distributions with densities depending on powers of u_a ; the first has also a normal surface-density.

The energies attaching to these potentials take the simple form of numerical multiples of that of a conductor. They are obtained by the use of what (in the paper cited) were called wave-forms of the potential. Two theorems are required for the purpose: one to connect the wave-forms with

* Communicated by the Author.

those written above, and the other to deal with moments and products of inertia of any even order for an ellipsoid. The evaluation of energy turns on a similarity in the forms of the two theorems, and they prove to be closely related.

A potential is then considered which is a linear function of members of the second type, namely,

$$\int_{\lambda}^{\infty} (1-u_a)(k_1+k_2u_a+k_3u_a^2+\dots k_nu_a^{n-1})\frac{d\lambda}{\sqrt{J}}.$$

The energy is a quadric in the k 's, persymmetric in form; while the whole charge is a linear function of the k 's. For any value of n this potential may be determined so as to give a minimum of energy subject to constancy of total charge. The minima have simple values which decrease as n increases, ranging from that for uniform volume-distribution ($n=1$) to that for normal distribution on a conductor ($n=\infty$). Thus the potentials form an interesting series of links connecting the two potentials commonly considered with reference to an ellipsoid; and each potential, with a distribution depending on n constants, in some measure simulates that of a structure containing n parts brought into relation by a minimum condition.

To give greater generality the work is written in *æolotropic* form; but most of the energies evaluated vary as $\int_0^{\infty} \frac{d\lambda}{\sqrt{J}}$, and the only difference between isotropic and *æolotropic* forms consists in the relation of the constants in the cubic J to the axes of the ellipsoid.

§ 1. Of the three forms*

$$\begin{aligned}\psi_s &= \frac{\rho}{4} \int_{\lambda}^{\infty} \frac{u_a^s d\lambda}{\sqrt{J}}, & U_s &= \frac{\rho}{4} \int_{\lambda}^{\infty} \frac{u_a^s (1-u_a) d\lambda}{\sqrt{J}}, \\ V_s &= \frac{\rho}{4} \int_{\lambda}^{\infty} \frac{(1-u_a)^s d\lambda}{\sqrt{J}}, & . & . & . \quad (1)\end{aligned}$$

the first will be taken as fundamental, results for U_s and V_s being directly deducible from those for ψ_s . Since u_a may be written =1 when it is not under the sign of integration, we have

$$\frac{d\psi_s}{dx} = -\frac{\rho}{4\sqrt{J}} \frac{d\lambda}{dx} + \frac{s\rho}{4} \int_{\lambda}^{\infty} u_a^{s-1} \frac{\partial u_a}{\partial x} \frac{d\lambda}{\sqrt{J}}.$$

* The notation is that of the previous paper on '*Æolotropic Potential*' in *Phil. Mag.* April 1905, quoted as *Æ*.

Thus the surface-discontinuity or density is the same for all the functions ψ and is that of ψ_0 ; while potentials U_s and V_s containing differences of powers of u_a have only a volume-density. A second differentiation gives

$$\frac{d^2\psi^s}{dx\,dy} = -\frac{\rho}{4\sqrt{J}}\left(\frac{d^2\lambda}{dx\,dy} - \frac{J}{2J}\frac{d\lambda}{dx}\frac{d\lambda}{dy}\right) - \frac{s\rho}{4\sqrt{J}}\frac{\partial u_a}{\partial x}\frac{d\lambda}{dy} \\ + \frac{s\rho}{4}\int_{\lambda}^{\infty}\left\{2\gamma'u_a^{s-1} + \overline{s-1}u_a^{s-2}\frac{\partial u_a}{\partial x}\frac{\partial u_a}{\partial y}\right\}\frac{d\lambda}{\sqrt{J}}.$$

When $\nabla_e^2\psi_s$ is formed, the sum of terms such as those in the first bracket vanishes, and the total outside the sign of integration is $-s\rho/\sqrt{J}$. Under the sign of integration the bracket gives

$$2u_a^{s-1}\Sigma(p\alpha + 2p'\alpha') - 4(s-1)u_a^{s-2}i_a,$$

or

$$2u_a^{s-1}J/J - 4(s-1)u_a^{s-2}i_a \text{ quoting } \mathcal{A}E. (76) \text{ and } (86).$$

Hence the integral

$$= -s\rho\int_{\lambda}^{\infty}\frac{\partial}{\partial\lambda}\left(\frac{u_a^{s-1}}{\sqrt{J}}\right)d\lambda = s\rho u_a^{s-1}/\sqrt{J}.$$

Externally the lower limit is variable, and the two terms are cancelled in virtue of $u_a=1$; thus $\nabla_e^2\psi_s=0$. Internally the lower limit is constant and $=0$, and there are no terms outside the integral sign; since $\lambda=0$ makes $J=1$, $u_a=u_a$, the integral is $s\rho u_a^{s-1}$, that is $\nabla_e^2\psi_s=s\rho u_a^{s-1}$ and the volume-density is $-s\rho u_a^{s-1}$. The surface-density σ

$$= -\left[l\left(p\frac{d\psi_0}{dx} + r'\frac{d\psi_0}{dy} + q'\frac{d\psi_0}{dz}\right) + m\left(r'\frac{d\psi_0}{dx} + q'\frac{d\psi_0}{dy} + p'\frac{d\psi_0}{dz}\right) \right. \\ \left. + n\left(q'\frac{d\psi_0}{dx} + p'\frac{d\psi_0}{dy} + r'\frac{d\psi_0}{dz}\right)\right] \\ = \frac{\rho}{4}\left[l\left(p\frac{d\lambda}{dx} + r'\frac{d\lambda}{dy} + q'\frac{d\lambda}{dz}\right) + \dots\right] \\ = \frac{\rho\varpi}{8}\left[\frac{\partial u_a}{\partial x}\left(p\frac{d\lambda}{dx} + r'\frac{d\lambda}{dy} + q'\frac{d\lambda}{dz}\right) + \dots\right] = \frac{\rho\varpi}{2},$$

ϖ being a perpendicular from the centre on a tangent plane and $l\,m\,n$ direction-cosines of a normal.

Since $\frac{d\psi_s}{dx} = \frac{d\psi_0}{dx} + \frac{\partial\psi_s}{\partial x}$, and $\frac{d\psi_s}{dx}$ and $\frac{d\psi_0}{dx}$ are external

potentials, so also is $\frac{\partial \psi_s}{\partial x}$. It is more convenient to use an independent notation for the vector type, and we take

$$\chi_s = \int_{\lambda}^{\infty} \frac{u_a^s(\alpha x + \gamma' y + \beta' z) d\lambda}{\sqrt{J}}. \quad (2)$$

Since
$$\frac{d\chi_s}{dx} = -\frac{\rho}{4\sqrt{J}}(\alpha x + \gamma' y + \beta' z) \frac{d\lambda}{dx} + \frac{\rho}{4} \int_{\lambda}^{\infty} \left[\alpha u_a^s + s u_a^{s-1} (\alpha x + \gamma' y + \beta' z) \frac{\partial u_a}{\partial x} \right] \frac{d\lambda}{\sqrt{J}},$$

the surface-discontinuity is the same as for χ_0 , and the surface-density is $\frac{\rho \omega}{2} (\alpha x + c' y + b' z)$. The volume-density is $-s\rho(\alpha x + c' y + b' z)u_a^{s-1}$ as appears from the connexion with ψ_{s+1} ; or by work following closely the lines of *Æ. p.* 442 we show that

$$\begin{aligned} \nabla_e^2 \chi_s &= -s\rho(\alpha x + \gamma' y + \beta' z) / \sqrt{J} \\ &\quad - s\rho \int_{\lambda}^{\infty} \frac{\partial}{\partial \lambda} \left\{ \frac{u_a^{s-1} (\alpha x + \gamma' y + \beta' z)}{\sqrt{J}} \right\} d\lambda \\ &= -s\rho(\alpha x + \gamma' y + \beta' z) / \sqrt{J} + s\rho u_a^{s-1} (\alpha x + \gamma' y + \beta' z) / \sqrt{J}, \end{aligned}$$

from which

$$\nabla_e^2 \chi_s = 0 \text{ externally and } = s\rho (\alpha x + c' y + b' z) u_a^{s-1} \text{ internally.}$$

§ 2. We proceed to the formulæ by which the above are transferred to wave-forms. In the latter it is more convenient to work with $\mu = \lambda \Delta_a$ and $\Delta(\mu) = \Delta_a^2 J(\lambda)$, identity in final results following from $d\mu / \sqrt{\Delta(\mu)} = d\lambda / \sqrt{J}$. The fundamental formula for transference is

$$\frac{\Delta_a^s}{2\pi} \int \frac{(lx + my + nz)^{2s} d\omega}{u_p(u_A + \mu u_p)^{s+1/2}} = \int_{\mu}^{\infty} \frac{u_a^s d\mu}{\sqrt{\Delta(\mu)}} = \int_{\lambda}^{\infty} \frac{u_a^s d\lambda}{\sqrt{J}} = \psi_s, \quad (3)$$

and a particular case is

$$\frac{\Delta_a^s}{2\pi} \int \frac{(lx + my + nz)^{2s} d\omega}{u_p u_A^{s+1/2}} = \int_0^{\infty} \frac{u_a^s d\lambda}{\sqrt{J}}; \quad (3b)$$

the general case applies to external potentials, the particular to internal potentials, the latter specially required for energy. Now (3) is an immediate integral, with regard to μ , of

$$\begin{aligned} \frac{1}{4\pi} \int \frac{(lx + my + nz)^{2s} d\omega}{(u_A + \mu u_p)^{s+3/2}} &= \frac{1}{2s+1} \left(\frac{u}{\Delta_a} \right)^s \frac{1}{\sqrt{\Delta(\mu)}} \\ &= \frac{u_A^{s(\mu)}}{(2s+1) \{ \Delta(\mu) \}^{s+1/2}}, \quad (4) \end{aligned}$$

the particular case with $\mu=0$ being

$$\frac{1}{4\pi} \int \frac{(lx + my + nz)^{2s} d\omega}{u_A^{s+3/2}} = \frac{u_a^s}{(2s+1)\Delta_a^{s+1}}, \quad (4b)$$

u_a and u_a on the right hand have (xyz) for variables, u_A and u_p on the left have (lmn) . We shall deduce (4) from

$$\frac{1}{4\pi} \int \frac{d\omega}{(u_A + \mu u_p)^{3/2}} = \frac{1}{\sqrt{\Delta(\mu)}}, \quad (5)$$

an integral obtained indirectly in the previous paper, and first give an independent proof of (5). The volume integral

$$\iiint dx dy dz \left| \sqrt{\frac{x^2}{\alpha^2} + \frac{y^2}{\beta^2} + \frac{z^2}{\gamma^2}} \right| = 2\pi\alpha\beta\gamma,$$

where the range of integration is the volume contained by the ellipsoid $\frac{x^2}{\alpha^2} + \frac{y^2}{\beta^2} + \frac{z^2}{\gamma^2} = 1$, is well known. Put

$(x, y, z) = r(l, m, n)$, and use a polar element of volume $r^2 dr d\omega$, where $d\omega$ is an element of area on a unit sphere; then integrate with regard to r , the boundary value being $1 = R^2 \left(\frac{l^2}{\alpha^2} + \frac{m^2}{\beta^2} + \frac{n^2}{\gamma^2} \right)$. The result is

$$\frac{1}{4\pi} \int d\omega \left/ \left(\frac{l^2}{\alpha^2} + \frac{m^2}{\beta^2} + \frac{n^2}{\gamma^2} \right)^{3/2} \right. = \alpha\beta\gamma,$$

or

$$\frac{1}{4\pi} \int d\omega / (al^2 + bm^2 + cn^2)^{3/2} = \frac{1}{\sqrt{abc}}.$$

If the ellipsoid is referred to general axes, then $u_a(l, m, n)$ takes the place of $al^2 + bm^2 + cn^2$, and abc must be replaced by Δ_a . If again for a we write $A + \mu p$, for a' , $A' + \mu p'$, then Δ_a is replaced by $\Delta(\mu)$, a determinant with the elements $A + \mu p, \dots$ and this is (5).

The step to (4) is made by using an operator

$$\Sigma \left(x^2 \frac{d}{dA} + yz \frac{d}{dA'} \right) \text{ say } O.$$

Since

$$\frac{d}{dA} u_A(l, m, n) = l^2, \quad \frac{d}{dA'} u_A = 2mn, \dots$$

we have

$$O \cdot u_A(l, m, n) = (lx + my + nz)^2.$$

And since

$$\frac{d}{dA} \Delta(\mu) = A(\mu) = \frac{\alpha \Delta(\mu)}{\Delta_a}, \text{ and } \frac{d}{dA'} \Delta(\mu) = 2A'(\mu) = \frac{2\alpha' \Delta(\mu)}{\Delta_a}, \dots$$

we have

$$O \cdot \Delta(\mu) = u_{A(\mu)}(x, y, z) = \frac{u_a \Delta(\mu)}{\Delta_a}.$$

Also $u_{A(\mu)}$ written at length is

$$\begin{aligned} & \Sigma x^2 \{ (BC - A'^2) + \mu(qC + rB - 2p'A') + \mu^2 P \} \\ & + 2\Sigma yz \{ B'C' - AA' + \mu(q'C' + r'B' - pA' - p'A) + \mu^2 P' \}, \end{aligned}$$

and therefore

$$\left. \begin{aligned} \frac{d}{dA} u_{A(\mu)} &= (C + \mu r)y^2 + (B + \mu q)z^2 - 2(A' + \mu p')yz, \\ \frac{1}{2} \frac{d}{dA'} u_{A(\mu)} &= (C' + \mu r')zx + (B' + \mu q')yx - (A + \mu p)yz - (A' + \mu p')x^2. \end{aligned} \right\} \quad (6)$$

When the whole operator is used all terms are cancelled, and we have the three results

$$\left. \begin{aligned} O \cdot u_A(l, m, n) &= (lx + my + nz)^2, \quad O \cdot \Delta(\mu) = u_{A(\mu)}(x, y, z) = \frac{u_a \Delta(\mu)}{\Delta_a}, \\ \text{and} \quad O \cdot u_{A(\mu)}(x, y, z) &= 0. \end{aligned} \right\} \quad (7)$$

Hence the operation applied to the first and third members of (4) yields a repetition of (4) with $s+1$ for s ; and (4) follows by repeated use of the operator starting from (5).

The formula of transference when $\alpha x + \gamma'y + \beta'z$ appears under the integral sign is got from (4) by use of the operator

$$O_x' = x \frac{d}{dA} + \frac{y}{2} \frac{d}{dC'} + \frac{z}{2} \frac{d}{dB'},$$

for which

$$\left. \begin{aligned} O_x' \cdot u_A &= l(lx + my + nz), \quad O_x' \cdot \Delta(\mu) = \frac{\Delta(\mu)}{\Delta_a} (\alpha x + \gamma'y + \beta'z), \\ O_x' \cdot u_{A(\mu)} &= 0. \end{aligned} \right\} \quad (8)$$

This gives

$$\frac{1}{4\pi} \int \frac{l(lx + my + nz)^{2s-1} d\omega}{(u_A + \mu u_p)^{s+3/2}} = \frac{u_a^{s-1} (\alpha x + \gamma'y + \beta'z)}{(2s+1) \Delta_a^s \sqrt{\Delta(\mu)}}. \quad (9)$$

Integrate with regard to μ from μ to ∞ and raise s by 1, then

$$\frac{\Delta_a^{s+1}}{2\pi} \int \frac{l(lx + my + nz)^{2s+1} d\omega}{u_p(u_A + \mu u_p)^{s+3/2}} = \int_{\mu}^{\infty} \frac{u_a^s (\alpha x + \gamma'y + \beta'z) d\mu}{\sqrt{\Delta(\mu)}}, \quad (10)$$

and in particular

$$\frac{\Delta_a^{s+1}}{2\pi} \int \frac{l(lx+my+nz)^{2s+1} d\omega}{u_p u_A^{s+3/2}} = \int_0^\infty \frac{u_a^s (\alpha x + \gamma' y + \beta' z) d\mu}{\sqrt{\Delta(\mu)}} \quad (10b)$$

§ 3. The fundamental integral for moments of inertia in an ellipsoid is

$$\frac{1}{\tau_0} \int (lx+my+nz)^{2s} d\tau = \frac{3}{(2s+1)(2s+3)} \left\{ \frac{u_A(l, m, n)}{\Delta_a} \right\}^s \quad (11)$$

The coefficients of like power arrangements in lmn are equal on the two sides, so that all moments and products of inertia of degree $2s$ are comprised in the formula. A way of expressing the results for individual terms, which also applies to (4), will be given later; but for our main purpose the form (11) and some collateral results are required. In immediate connexion with (11) are

$$\frac{1}{\tau_0} \int (lx+my+nz)^{2s} u_a^{s'} d\tau = \frac{3}{(2s+1)(2s+2s'+3)} \left(\frac{u_A}{\Delta_a} \right)^s \quad (11b)$$

and the surface integral

$$\frac{1}{\tau_0} \int (lx+my+nz)^{2s} \varpi dS = \frac{3}{2s+1} \left(\frac{u_A}{\Delta_a} \right)^s, \quad (11c)$$

where ϖ is a perpendicular from the centre on a tangent plane. Write $x, y, z = r(\lambda, \mu, \nu)$, use a polar element of volume $r^2 dr d\omega$ and integrate to the surface value

$$1 = R^2 u_a(\lambda, \mu, \nu).$$

For (11)

$$\begin{aligned} \int (lx+my+nz)^{2s} d\tau &= \iint r^{2s+2} (l\lambda + m\mu + n\nu)^{2s} dr d\omega \\ &= \frac{1}{2s+3} \int R^{2s+3} (l\lambda + m\mu + n\nu)^{2s} d\omega; \end{aligned}$$

for (11b) $\int (lx+my+nz)^{2s} u_a^{s'} d\tau$

$$\begin{aligned} &= \frac{1}{2s+2s'+3} \int R^{2s+2s'+3} (l\lambda + m\mu + n\nu)^{2s} u_a^{s'}(\lambda, \mu, \nu) d\omega \\ &= \frac{1}{2s+2s'+3} \int R^{2s+3} (l\lambda + m\mu + n\nu)^{2s} d\omega; \end{aligned}$$

and for (11c)

$$\int (lx+my+nz)^{2s} \varpi dS = \int R^{2s+3} (l\lambda + m\mu + n\nu)^{2s} d\omega,$$

since $\varpi dS = R^3 d\omega$.

Thus the three forms all depend on one angular integral which, in view of the value of R , and $\tau_0 = 4\pi/3\sqrt{\Delta_a}$, is

$$\frac{1}{4\pi} \int \frac{(l\lambda + m\mu + n\nu)^{2s} d\omega}{\{u_a(\lambda, \mu, \nu)\}^{s+3/2}} = \frac{\{u_A(l, m, n)\}^s}{(2s+1)\Delta_a^{s+1/2}}.$$

This integral is a variant of (4 *b*) got by writing a for A .

A modification of (11) by a factor $ax + c'y + b'z$ is required for the χ functions. Operate with $a \frac{d}{dl} + c' \frac{d}{dm} + b' \frac{d}{dn}$ on (11), and raise s by 1, thus

$$\begin{aligned} \frac{1}{\tau_0} \int (ax + c'y + b'z) (lx + my + nz)^{2s+1} d\tau \\ = \frac{3l}{(2s+3)(2s+5)} \left(\frac{u_A}{\Delta_a}\right)^s; \quad \dots \quad (12) \end{aligned}$$

$$\begin{aligned} \text{then} \quad \frac{1}{\tau_0} \int (ax + c'y + b'z) (lx + my + nz)^{2s+1} u_a^{s'} d\tau \\ = \frac{3l}{(2s+3)(2s+2s'+5)} \left(\frac{u_A}{\Delta_a}\right)^s, \quad (12 \text{ } b) \end{aligned}$$

$$\begin{aligned} \text{and} \quad \frac{1}{\tau_0} \int (ax + c'y + b'z) (lx + my + nz)^{2s+1} \varpi dS \\ = \frac{3l}{2s+3} \left(\frac{u_A}{\Delta_a}\right), \quad \dots \quad (12 \text{ } c) \end{aligned}$$

are connected with (12) in the manner shown above for (11 *b*) and (11 *c*).

§ 4. The evaluation of energy now follows by combining the typical integrals, thus:—

$$\begin{aligned} \frac{1}{\tau_0} \int u_a^{s'} d\tau \int_0^\infty \frac{u_a^s d\lambda}{\sqrt{J}} &= \frac{1}{\tau_0} \int u_a^{s'} d\tau \int \frac{\Delta_a^s (lx + my + nz)^{2s} d\omega}{2\pi u_p u_A^{s+1/2}} \text{ by (3 } b) \\ &= \frac{3}{(2s+1)(2s+2s'+3)} \int \frac{d\omega}{2\pi u_p u_A^{1/2}} \text{ by (11 } b) \\ &= \frac{3\phi_0}{(2s+1)(2s+2s'+3)} \cdot \dots \quad (13) \end{aligned}$$

using ϕ_0 as in $\mathcal{A}$. (135) for

$$\int_0^\infty \frac{d\lambda}{\sqrt{J}} = \int_0^\infty \frac{d\mu}{\sqrt{\Delta(\mu)}} = \int \frac{d\omega}{2\pi u_p u_A^{1/2}}.$$

We may also write (13) as

$$\int u_a^{s'} \psi_s d\tau = \frac{3\rho\tau_0\phi_0}{4(2s+1)(2s+2s'+3)}, \quad \dots \quad (13 \text{ } b)$$

and the surface integral is

$$\int \psi_s \varpi dS = \frac{3\rho\tau_0\phi_0}{4(2s+1)} \quad (13c)$$

Now the potential $\psi_{s'}$ has a surface-density $\rho\varpi/2$, and a volume-density $-s'\rho u_a^{s'-1}$. The term in energy due to the interaction of ψ_s and $\psi_{s'}$ is therefore

$$\begin{aligned} E(s, s') &= \frac{1}{2} \int \rho\varpi\psi dS - \int s'\rho u_a^{s'-1} \psi_s d\tau \\ &= \frac{3\rho^2\tau_0}{8} \left(\frac{1}{2s+1} - \frac{2s'}{2s+1} \frac{2s'}{2s+2s'+1} \right) = \frac{3\rho^2\tau_0\phi_0}{8(2s+2s'+1)} \quad (14) \end{aligned}$$

The energy due to ψ_s alone is $3\rho^2\tau_0\phi_0/16(4s+1)$, *i. e.* we write $s'=s$ and halve the result. The energy belonging to $k_s\psi_s + k_{s'}\psi_{s'}$ is

$$\frac{3\rho^2\tau_0\phi_0}{16} \left[\frac{k_s^2}{4s+1} + \frac{2k_s k_{s'}}{2s+2s'+1} + \frac{k_{s'}^2}{4s'+1} \right],$$

but it is clearly sufficient to give as in (14) a single composite term. The total charge attaching to ψ_s is

$$\frac{1}{2} \int \rho\varpi dS - \int s\rho u_a^{s-1} d\tau = \frac{3\rho\tau_0}{2} - \frac{3s\rho\tau_0}{2s+1} = \frac{3\rho\tau_0}{2(2s+1)}.$$

For normal distribution on a conductor $s=0$, $e=3\rho\tau_0/2$, and the energy is $e^2\phi_0/12\tau_0$.

From (14) may be derived the composite term in the energy of $k_s\psi_s + k_{s'}\psi_{s'}$, or of $k_s(\psi_s - \psi_{s+1}) + k_{s'}(\psi_{s'} - \psi_{s'+1})$. For this term we are concerned with the products

$$\rho_{s'}\psi_s - \rho_{s'+1}\psi_s - \rho_{s'}\psi_{s+1} + \rho_{s'+1}\psi_{s+1},$$

and the application of (14) yields

$$\begin{aligned} &\frac{3\rho^2\tau_0\phi_0}{8} \left[\frac{1}{2s+2s'+1} - \frac{2}{2s+2s'+3} + \frac{1}{2s+2s'+5} \right] \\ &= \frac{3\rho^2\tau_0\phi_0}{2s+2s'+1} \frac{1}{2s+2s'+3} \frac{1}{2s+2s'+5}, \quad (15) \end{aligned}$$

which multiplied by $k_s k_{s'}$ is the composite term required.

To deal with the function V_s or $\frac{\rho}{4} \int \frac{d\lambda}{\sqrt{\lambda}} (1-u_a)^s$, the two summations for the indices of ρ and ψ may be taken

separately. The first is

$$\begin{aligned} \frac{3\rho^2\tau_0\phi_0}{8} \left[1 - \frac{s'C_1}{3} + \frac{s'C_2}{5} - \dots \frac{-1^{s'}}{2s'+1} \right] \\ = \frac{3\rho^2\tau_0\phi_0}{8} \cdot \frac{2 \cdot 4 \dots 2s'}{3 \cdot 5 \dots 2s'+3}; \end{aligned}$$

and the second is

$$\begin{aligned} \frac{3\rho^2\tau_0\phi_0}{8} \times 2 \cdot 4 \dots 2s' \left[\frac{1}{1 \cdot 3 \dots 2s'+1} - \frac{sC_1}{3 \cdot 5 \dots 2s'+3} \right. \\ \left. + \dots \frac{-1^s}{2s+1 \cdot 2s+3 \dots 2s+2s'+1} \right] \\ = \frac{3\rho^2\tau_0\phi_0}{8} \cdot \frac{2 \cdot 4 \dots 2s+2s'}{3 \cdot 5 \dots 2s+2s'+1}; \quad (16) \end{aligned}$$

and this quantity multiplied by $k_s k$ is the composite term in the energy of $k_s V_s + k_s V_{s'}$. The summations for (16) are made by a series of steps in which the combination formula $C = C_{s-1} + C_{s-1}$ is used, and the difference of adjoining terms

taken. The densities attaching to U_s and V_s are respectively $\rho(s+1)u_a^s - s u_a^{s-1}$ and $s\rho(1-u_a)^{s-1}$, and the corresponding total charges $\frac{3\rho\tau_0}{2s+1} \frac{1}{2s+3}$ and $\frac{3\rho\tau_0}{2} \cdot \frac{2 \cdot 4 \dots 2s}{3 \cdot 5 \dots 2s+1}$; they

follow from those of ψ_s , or may be got by independent work. The energy formula for U_s may be got directly, viz. for the composite term

$$\begin{aligned} E(s, s') &= \int \rho_{s'} U_s d\tau \\ &= \int \rho(s'+1)u_a^{s'} - s' u_a^{s'-1} d\tau \cdot \frac{\rho}{4} \int_0^\infty \frac{u_a^s(1-u_a)d\lambda}{\sqrt{j}}, \text{ which by (13)} \\ &= \frac{3\rho^2\tau_0\phi_0}{4} \left[(s'+1) \left\{ \frac{1}{2s+1} \frac{1}{2s+2s'+3} - \frac{1}{2s+3} \frac{1}{2s+2s'+5} \right\} \right. \\ &\quad \left. - s' \left\{ \frac{1}{2s+1} \frac{1}{2s+2s'+1} - \frac{1}{2s+3} \frac{1}{2s+2s'+3} \right\} \right] \\ &= 3\rho^2\tau_0\phi_0/(2s+2s'+1)(2s+2s'+3)(2s+2s'+5), \end{aligned}$$

in agreement with (15). Moreover, the energy of ψ_s may be calculated from that of U_s by treating ψ_s as $U_s + U_{s+1} + \dots$ *ad inf.*, and making a double summation of the formula just obtained. Either U_s or V_s for $s=0$ gives the case of uniform volume distribution $e=\rho\tau_0$, and energy $=e^2\phi_0/10\tau_0$.

§ 5. We now give corresponding results for χ_s . Using the integrals, (10 *b*) for the first step, and (12 *b*) for the second,

$$\begin{aligned} \frac{1}{\tau_0} \int u_a^{s'}(ax + c'y + b'z) d\tau \int_0^\infty \frac{u_a^s(ax + \gamma'y + \beta'z) d\lambda}{\sqrt{J}} \\ = \frac{1}{\tau_0} \int u_a^{s'}(ax + c'y + b'z) d\tau \int \frac{\Delta_a^{s+1} l (lx + my + nz)^{2s+1} d\omega}{2\pi u_p u_A^{s+3/2}} \\ = \frac{3}{(2s+3)(2s+2s'+5)} \int \frac{\Delta_a l^2 d\omega}{2\pi u_p u_A^{3/2}} \\ = \frac{3L_0}{(2s+3)(2s+2s'+5)} \quad \dots \dots \dots (17) \end{aligned}$$

where L_0 as in $\mathcal{AE}$. (135) stands for

$$\int_0^\infty \frac{\alpha d\lambda}{\sqrt{J}} = \int \frac{\Delta_a l^2 d\omega}{2\pi u_p u_A^{3/2}}.$$

This and the corresponding surface integral may be written

$$\int u_a^{s'}(ax + c'y + b'z) \chi_s d\tau = \frac{3\rho\tau_0 L_0}{4(2s+3)(2s+2s'+5)} \quad \dots (17 \text{ } b)$$

and

$$\int (ax + c'y + b'z) \chi_s \varpi dS = \frac{3\rho\tau_0 L_0}{4(2s+3)} \quad \dots \dots \dots (17 \text{ } c)$$

The surface-density due to $\chi_{s'}$ is $\frac{\rho\varpi}{2} (ax + c'y + b'z)$, and the volume-density $-s'\rho u_a^{s'-1}(ax + c'y + b'z)$; thus applying (17 *b* and *c*) the composite term in energy attaching to indices s and s' is

$$\mathcal{E}(s, s') = \frac{3\rho^2\tau_0 L_0}{8(2s+3)} \left[1 - \frac{2s'}{2s+2s'+3} \right] = \frac{3\rho^2\tau_0 L_0}{8(2s+2s'+3)}. \quad (18)$$

If functions with $\gamma'x + \beta'y + \alpha'z$ and $\beta'x + \alpha'y + \gamma z$ are also under consideration, the three functions may be distinguished as $\chi_s(x)$, $\chi_s(y)$, $\chi_s(z)$. In the composite term for $\chi_s(y)$ and $\chi_{s'}(y)$, M_0 appears for L_0 above, but if a composite term for $\chi_s(x)$ and $\chi_{s'}(y)$ is in question N_0' appears, viz.,

$$N_0' = \int \frac{\Delta_a l m d\omega}{2\pi u_p u_A^{3/2}} = \int_0^\infty \frac{\gamma' d\lambda}{\sqrt{J}}.$$

For the density of $\chi_{s'}(y)$ has the factor $c'x + by + \alpha'z$; and if this is taken with $\chi_s(x)$ a factor m is introduced in place of l in the second step towards (17), where (12 *b*) is quoted; while the contrary arrangement, density of $\chi_s(x)$ with potential

$\chi_s(y)$, introduces the factor m in the first step where (10b) is quoted, with an identical result.

For the volume potentials of magnetic type

$$\frac{\rho}{4} \int_{\lambda}^{\infty} u_{\alpha}^s (1 - u_{\alpha}) (\alpha x + \gamma' y + \beta' z) \frac{d\lambda}{\sqrt{J}}$$

and

$$\frac{\rho}{4} \int_{\lambda}^{\infty} (1 - u_{\alpha})^s (\alpha x + \gamma' y + \beta' z) \frac{d\lambda}{\sqrt{J}},$$

which we may call $U_s'(x)$, $V_s'(x)$ the composite terms in energy are respectively

$$3\rho^2\tau_0 L_0 / (2s + 2s' + 3)(2s + 2s' + 5)(2s + 2s' + 7) \quad . \quad (19)$$

and

$$\frac{3\rho^2\tau_0 L_0}{8} \cdot \frac{2.4 \dots 2s + 2s'}{3.5 \dots 2s + 2s' + 3} \cdot \dots \quad (20)$$

In lieu of total charge in connexion with ψ , we may here take a magnetic moment* defined for $\chi_s(x)$ by

$$m_x = \int x \rho_s d\tau = \frac{3\rho\tau_0}{2(2s + 3)};$$

for the function $U_s'(x)$ it is $\frac{3\rho\tau_0}{2s + 3 \cdot 2s + 5}$. Thus an exact

correspondence is given between the two types of potential.

§ 6. We now consider with reference to a potential

$$P = k_1 U_0 + k_2 U_1 + \dots k_n U_{n-1}$$

$$= \frac{\rho}{4} \int_{\lambda}^{\infty} (1 - u_{\alpha}) (k_1 + k_2 u_{\alpha} + k_3 u_{\alpha}^2 + \dots k_n u_{\alpha}^{n-1}) \frac{d\lambda}{\sqrt{J}},$$

the problem of determining the k 's so as to make the energy E a minimum subject to the constancy of total charge e , E and e being given by

$$\left. \begin{aligned} E &= \frac{3\rho^2\tau_0\phi_0}{2} \left[\frac{k_1^2}{1 \cdot 3 \cdot 5} + \frac{2k_1 k_2}{3 \cdot 5 \cdot 7} + \frac{k_2^2 + 2k_1 k_3}{5 \cdot 7 \cdot 9} + \dots \right] \\ e &= 3\rho\tau_0 \left[\frac{k_1}{1 \cdot 3} + \frac{k_2}{3 \cdot 5} + \dots \right] \end{aligned} \right\} \dots (21)$$

* In the case of $\mathcal{AE}$. (117) where these functions are used, it is, however, a question of total charge, and $e = 3\rho\tau_0/2$ for the conductor ($s=0$), $e = \rho\tau_0$ for the uniform volume distribution ($s=1$). The energies for χ_0 and for χ_1 are then respectively $e^2 L_0 / 36\tau_0$ and $e^2 L_0 / 70\tau_0$; thus if we seek the energy of convection currents due to surface charge instead of body charge, we must write in $\mathcal{AE}$. (117) $e^2 / 36V^2$ outside the square bracket.

If a similar problem is stated for

$$Q = \frac{\rho}{4} \int_{\lambda}^{\infty} (1 - u_a) (k_1 + k_2 u_a + \dots k_n u_a^{n-1}) (\alpha x + \gamma' y + \beta' z) \frac{d\lambda}{\sqrt{J}},$$

ϕ_0 is replaced by L_0 and e by m_x , and the coefficients open with $\frac{1}{3.5.7}$ and $\frac{1}{3.5}$ instead of $\frac{1}{1.3.5}$ and $\frac{1}{1.3}$. By writing

$k = \frac{e\xi}{3\rho\tau_0}$ for each subscript the minimum problem is presented with pure numbers, and we may generalize it by making the series of coefficients start at a different place. Thus using

$$a_p = 1/(2p-1)(2p+1)(2p+3) \quad \text{and} \quad b_p = 1/(2p-1)(2p+1),$$

it is required to find ω the minimum value of

$$\left. \begin{aligned} \Omega &= \xi_1 \eta_1 + \xi_2 \eta_2 + \dots \quad \xi_n \eta_n \\ \eta_1 &= a_p \xi_1 + a_{p+1} \xi_2 + \dots \quad a_{p+n-1} \xi_n \\ \eta_2 &= a_{p+1} \xi_1 + a_{p+2} \xi_2 + \dots \quad a_{p+n} \xi_n \\ &\dots \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \\ \eta_n &= a_{p+n-1} \xi_1 + \dots \quad a_{p+2n-2} \xi_n \end{aligned} \right\} \cdot \quad (22)$$

is an equation of condition. From the point of view of potential theory, interest attaches to the value of the minimum, which proves to be

$$\omega = \frac{2p-1}{2} \cdot \frac{n+1}{n} \frac{2n+2p-1}{2n+2p+1} \cdot \cdot \cdot \quad (23)$$

in the general case; and to the nature of the distribution which gives the minimum. For $p=1$ the case of (21),

$$\omega = (n+1)(2n+1)/2n(2n+3) \text{ and } E = e^2 \phi_0 \omega / 6\tau_0 ;$$

while the density for the potential P is

$$\frac{e}{3\tau_0} \left[\xi_1 + \xi_2(-1 + 2u_a) + \xi_3(-2u_a + 3u_a^2) \dots + \xi_n(-\overline{n-1} u_a^{n-2} + nu_a^{n-1}) \right]$$

or

$$\left. \begin{aligned} & \frac{e}{3\tau_0} [\xi_1 - \xi_2 + 2(\xi_2 - \xi_3)u_a + 3(\xi_3 - \xi_4)u_a^2 + \dots + n\xi_n u_a^{n-1}] \\ \text{or say} \\ & \frac{e}{3\tau_0} [x_1 + x_2 u_a + x_3 u_a^2 + \dots + x_n u_a^{n-1}]. \end{aligned} \right\} \quad (24)$$

The conditions of no variation are

$$\eta_1 = \omega b_1, \eta_2 = \omega b_2, \dots \text{with } 1 = b_p \xi_1 + \dots b_{p+n-1} \xi_n;$$

$$i. e. \quad \left. \begin{array}{l} a_p \xi_1 + \dots a_{p+n-1} \xi_n = \omega b_p \\ \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ a_{p+n-1} \xi_1 + \dots a_{p+2n-2} \xi_n = \omega b_{p+n-1} \\ b_p \xi_1 + \dots b_{p+n-1} \xi_n = 1 \end{array} \right\} \cdot \cdot \cdot \quad (25)$$

Write the first as

$$\omega b_p = (\xi_1 - \xi_2) a_p + (\xi_2 - \xi_3) (a_p + a_{p+1}) \\ + (\xi_3 - \xi_4) (a_p + a_{p+1} + a_{p+2}) + \dots + \xi_n (a_p + \dots a_{p+n-1}),$$

and note that

$$a_p + a_{p+1} + \dots a_{p+n-1} = \frac{1}{4} (b_p - b_{p+n}) \\ = \frac{1}{4} \left\{ \frac{1}{2p-1} \frac{1}{2p+1} - \frac{1}{2p+2n-1} \frac{1}{2p+2n+1} \right\} \\ = \frac{1}{8} \left\{ \frac{1}{2p-1} - \frac{1}{2p+1} - \frac{1}{2p+2n-1} + \frac{1}{2p+2n+1} \right\} \\ = \frac{n}{4} \left\{ \frac{1}{2p-1} \frac{1}{2p+2n-1} - \frac{1}{2p+1} \frac{1}{2p+2n+1} \right\}.$$

The first equation of (25) is then

$$\frac{1}{2p-1} \left[\frac{x_1}{2p+1} + \frac{x_2}{2p+3} + \dots \frac{x_n}{2p+2n-1} \right] - \frac{1}{2p+1} \left[\frac{x_1}{2p+3} \right. \\ \left. + \dots \frac{x_n}{2p+2n+1} \right] = 4\omega b_p = 2\omega \left(\frac{1}{2p-1} - \frac{1}{2p+1} \right),$$

and others of this type are got by increasing p by 1, 2, 3...

Since $b_p + b_{p+1} + \dots b_{p+n-1} = \frac{n}{2p-1 \cdot 2p+2n-1}$, the equation of condition treated in this way becomes

$$\frac{x_1}{2p+1} + \frac{x_2}{2p+3} + \dots \frac{x_n}{2p+2n-1} = 2p-1. \quad (26)$$

The group of equations is then equivalent to

$$\frac{x_1}{2p+1} + \dots \frac{x_n}{2p+2n-1} = 2\omega + C(2p-1), \\ \frac{x_1}{2p+3} + \dots \frac{x_n}{2p+2n+1} = 2\omega + C(2p+1) \dots,$$

together with (26). The first with (26) gives $2\omega + C(2p-1) = 2p-1$, and the right hand of the $q+1$ th equation

$$= 2\omega + C(2p+2q-1) = 2p+2q-1 - \frac{4q\omega}{2p-1},$$

which if the stated minimum (23) is correct

$$= (2p-1) \left[1 - \frac{2q}{n(2p+2n+1)} \right].$$

The group of equations is then

$$\begin{aligned} \frac{x_1}{2p+2q+1} + \frac{x_2}{2p+2q+3} + \dots \frac{x_n}{2p+2q+2n-1} \\ = \frac{2p-1}{n(2p+2n+1)} [n(2p+2n+1)-2q] \quad (27) \end{aligned}$$

where q has in succession the values 0, 1, 2 .. n ; if these are satisfied the original conditions are satisfied and ω has the value in (23).

If $P_n^r(\mu)$ is used for the r th differential coefficient of the zonal harmonic $P_n(\mu)$, the solution of (27) is comprised in the statement of the following as an identity, viz. :

$$\begin{aligned} x_1 + x_2\mu^2 \cdot \frac{2p+1}{1} + x_3\mu^4 \cdot \frac{2p+1}{1} \frac{2p+3}{3} + x_4\mu^6 \cdot \frac{2p+1}{1} \frac{2p+3}{3} \frac{2p+5}{5} + \dots \\ = \frac{(2p-1)P_{2n+p}^{2+p}}{n(2p+2n+1)1.3\dots 2p-1}. \quad (28) \end{aligned}$$

The numerical coefficients on the left hand are cleared by an integration $\int_0^\mu f d\mu$, followed by $p-1$ integrations

$$I.f = \int_0^\mu \mu f d\mu,$$

and the result is

$$x_1\mu^{2p-1} + x_2\mu^{2p+1} + \dots x_n\mu^{2p+2n-1} = \frac{(2p-1)}{n(2p+2n+1)} I^{p-1}.P_{2n+p}^{1+p}; \quad (29)$$

$$\text{for } p=1, \quad x_1\mu + x_2\mu^3 + \dots = P_{2n+1}''/n(2n+3), \quad \dots \quad (29b)$$

$$\text{and for } p=2, \quad x_1\mu^3 + x_2\mu^5 + \dots = 3(\mu P_{2n+2}'' - P_{2n+2}')/n(2n+5). \quad (29c)$$

The left-hand member of (27) is got from the left-hand member of (29) by multiplying by μ^{2q+1} and integrating from

$\mu=0$ to 1, *i. e.* we require

$$\left. \begin{aligned} \int_0^1 P''_{2n+1} \mu^{2q+1} d\mu &= n(2n+3) - 2q, \quad \text{for } p=1, \\ \int_0^1 (\mu P''_{2n+2} - P'_{2n+2}) \mu^{2q+1} d\mu &= n(2n+5) - 2q, \quad \text{for } p=2, \\ \text{and} \\ \int_0^1 \mu^{2q+1} (I^{p-1} \cdot P^{1+p}_{2n+p}) d\mu &= n(2p+2n+1) - 2q \\ \text{for the general case.} \end{aligned} \right\} \quad (30)$$

Now

$$\int_0^1 P''_{2n+1} \mu^{2q+1} d\mu = P'_{2n+1}(1) - (2q+1) \int_0^1 \mu^{2q} P'_{2n+1} d\mu,$$

and

$$\int_0^1 \mu^{2q} P'_{2n+1} d\mu = P_{2n+1}(1) - 2q \int_0^1 \mu^{2q-1} P_{2n+1} d\mu;$$

the last term vanishes for $q=0, 1, 2, \dots, n$, but no further, and therefore

$$\int_0^1 P''_{2n+1} \mu^{2q+1} d\mu = (n+1)(2n+1) - (2q+1) = n(2n+3) - 2q.$$

Again

$$\int_0^1 (\mu P''_{2n+2} - P'_{2n+2}) \mu^{2q+1} d\mu = P'_{2n+2}(1) - (2q+3) \int_0^1 \mu^{2q+1} P'_{2n+2} d\mu$$

while

$$\int_0^1 \mu^{2q+1} P'_{2n+2} d\mu = P_{2n+2}(1) - (2q+1) \int_0^1 \mu^{2q} P_{2n+2} d\mu.$$

The last integral vanishes for $q=0, 1, 2, \dots, n$, but no further, and therefore

$$\int_0^1 (\mu P''_{2n+2} - P'_{2n+2}) \mu^{2q+1} d\mu = (n+1)(2n+3) - (2q+3) = n(2n+5) - 2q.$$

The solution is therefore fully verified for the cases $p=1$ and 2, which belong to the two problems stated, the cases of potentials P and Q . For the general case a mode of proof will be briefly indicated. Carry out the operation I by using $P'_{s+1} - \mu P'_s = (s+1)P_s$ differentiated any number of times, then in the result $I^{p-1} \cdot P^{p+1}_{2n+p}$ yields

$$P''_{2n+2p-1} - (p-1)(2n+2p+1)P'_{2n+2p-2} + a \text{ residue.}$$

The residue can be expressed as a linear function of the P 's of odd degree from P_{2n+3} to $P_{2n+2p+1}$, and the multiplication of these by μ^{2q+1} and integration from 0 to 1 yields a zero result from $q=0$ to $q=n$. The integral on the left hand of (30) is then

$$\int_0^1 \mu^{2q+1} [P''_{2n+2p-1} - (p-1)(2n+2p+1)P'_{2n+2p-2}] d\mu = (n+p)(2n+2p-1) \\ - (2q+1) - (p-1)(2n+2p+1) = n(2n+2p+1) - 2q \text{ as required.}$$

A comparison of (29*b*) with (24) shows that the solution gives directly the density. Now a zonal harmonic has a point of inflexion between each maximum and minimum, therefore $\mu^{-1} P''_{2n+1}$ vanishes for $n-1$ values of μ^2 ; hence the density vanishes for these values of u_a . The distribution is one in which there are n concentric ellipsoidal shells of alternate positive and negative charges. When n is indefinitely increased the charges on all but those for which u_a is nearly $=1$ are indefinitely small, and the case becomes that of a conductor. A simple independent proof for the case of n infinite shows the character of the limit. Since

$$a_p + a_{p+1} + \dots \text{ad inf.} = \frac{b_p}{4}$$

the equation of condition may be written

$$\frac{1}{4} = \xi_1(a_p + a_{p+1} + \dots) + \xi_2(a_{p+1} + a_{p+2} + \dots) + \dots = \eta_1 + \eta_2 + \dots$$

i. e. it can be expressed in terms of the η 's. But

$$\delta\Omega = \xi_1\delta\eta_1 + \dots \text{ subject to } 0 = \delta\eta_1 + \delta\eta_2 + \dots$$

makes the ξ 's all equal; and it is then easy to show that each $= 2(2p-1)$, and that $\omega = (2p-1)/2$. The potential P (case $p=1$) is then

$$\frac{e}{6\tau_0} \int_{\lambda}^{\infty} (1-u_a)(1+u_a+u_a^2+\dots) \frac{d\lambda}{\sqrt{J}} = \frac{e}{6\tau_0} \int_{\lambda}^{\infty} \frac{d\lambda}{\sqrt{J}},$$

and the potential Q (case $p=2$) is

$$\frac{m_x}{2\tau_0} \int_{\lambda}^{\infty} (\alpha x + \gamma' y + \beta' z) (1-u_a)(1+u_a+u_a^2+\dots) \frac{d\lambda}{\sqrt{J}} \\ = \frac{m_x}{2\tau_0} \int_{\lambda}^{\infty} \frac{(\alpha x + \gamma' y + \beta' z)}{\sqrt{J}}.$$

The higher derived functions which would correspond to higher values of p in the minimum problem are not here constructed.

§ 7. In conclusion we return to the general theorem of moments, give an independent proof, and then show how to

deal with individual terms. We set out from the sphere, for which

$$\begin{aligned} & \frac{1}{\tau_0} \int x^{2\nu_1} y^{2\nu_2} z^{2\nu_3} d\tau \\ &= \frac{1}{\tau_0} \int_0^R \int_0^\pi \int_0^{2\pi} r^2 \sin \theta \, dr \, d\theta \, d\phi \, (r \sin \theta \cos \phi)^{2\nu_1} (r \sin \theta \sin \phi)^{2\nu_2} (r \cos \theta)^{2\nu_3} \\ &= 3R^{2\nu_1+2\nu_2+2\nu_3} \times \frac{1 \cdot 3 \dots \overline{2\nu_1-1} \cdot 1 \cdot 3 \dots \overline{2\nu_2-1} \cdot 1 \cdot 3 \dots \overline{2\nu_3-1}}{1 \cdot 3 \dots 2\nu_1+2\nu_2+2\nu_3+3}; \end{aligned}$$

hence for an ellipsoid referred to principal axes

$$ax^2 + by^2 + cz^2 = 1, \text{ with } A=bc, \dots$$

$$\begin{aligned} & \frac{1}{\tau_0} \int x^{2\nu_1} y^{2\nu_2} z^{2\nu_3} d\tau \\ &= 3 \left(\frac{A}{\Delta_a} \right)^{\nu_1} \left(\frac{B}{\Delta_a} \right)^{\nu_2} \left(\frac{C}{\Delta_a} \right)^{\nu_3} \times \frac{1 \cdot 3 \dots \overline{2\nu_1-1} \cdot 1 \cdot 3 \dots \overline{2\nu_2-1} \cdot 1 \cdot 3 \dots \overline{2\nu_3-1}}{1 \cdot 3 \dots 2\nu_1+2\nu_2+2\nu_3+3}. \end{aligned}$$

Therefore, if $2s = 2\nu_1 + 2\nu_2 + 2\nu_3$,

$$\begin{aligned} \frac{1}{\tau_0} \int (lx + my + nz)^{2s} d\tau &= \Sigma \frac{2s \, l^{2\nu_1} m^{2\nu_2} n^{2\nu_3}}{[2\nu_1][2\nu_2][2\nu_3] \tau_0} \int x^{2\nu_1} y^{2\nu_2} z^{2\nu_3} d\tau \\ &= \frac{3}{(2s+1)(2s+3)} \Sigma \frac{|s|}{[\nu_1][\nu_2][\nu_3]} \left(\frac{Al^2}{\Delta_a} \right)^{\nu_1} \left(\frac{Bm^2}{\Delta_a} \right)^{\nu_2} \left(\frac{Cn^2}{\Delta_a} \right)^{\nu_3} \\ &= \frac{3}{(2s+1)(2s+3)} \left(\frac{Al^2 + Bm^2 + Cn^2}{\Delta_a} \right)^s. \end{aligned}$$

Now suppose $x y z$ transformed by a linear substitution to other axes $x_1 y_1 z_1$ which are not principal axes of the ellipsoid, and $l m n$ transformed by the inverse substitution to $l_1 m_1 n_1$ (*cf.* Salmon, Higher Algebra, p. 102). The transformation makes $lx + my + nz = l_1 x_1 + m_1 y_1 + n_1 z_1$, leaves Δ_a unaltered, and gives $A_1 l_1^2 + \dots + 2A_1' m_1 n_1 \dots$ for $Al^2 + Bm^2 + Cn^2$.

Hence dropping the subscripts we have the general form

$$\frac{1}{\tau_0} \int (lx + my + nz)^{2s} d\tau = \frac{3}{(2s+1)(2s+3)} \left(\frac{u_A}{\Delta_a} \right)^s. \quad (11)$$

In order to deal with specified products of inertia, a mode of writing the coefficients in the expansion of a ternary quadric is required, say

$$\left. \begin{aligned} \{u_a(x, y, z)\}^s &= \Sigma x^{\nu_1} y^{\nu_2} z^{\nu_3} K(\nu_1, \nu_2, \nu_3; a) \\ \text{with} \quad 2s &= \nu_1 + \nu_2 + \nu_3 \end{aligned} \right\}. \quad (31)$$

Then comparing corresponding terms in the expansion of (11),

$$\frac{1}{\tau_0} \int x \nu_1 y \nu_2 z \nu_3 d\tau = \frac{3}{(2s+3)} \frac{|\nu_1| |\nu_2| |\nu_3|}{|2s+1|} \cdot \frac{K(\nu_1, \nu_2, \nu_3; \Lambda)}{\Delta_a^s} \quad (32)$$

The particular cases $s=1$ and 2 were given in *Æ.* p. 443 without proof. Making a similar application to the angular integral (3), the type of an individual term is

$$\frac{1}{2\pi} \int \frac{l \nu_1 m \nu_2 n \nu_3 d\omega}{u_p(u_\Lambda + \mu u_p)^{s+\frac{1}{2}}} = \frac{|\nu_1| |\nu_2| |\nu_3|}{|2s \cdot \Delta_a^s|} \int_\mu^\infty \frac{K(\nu_1, \nu_2, \nu_3; \alpha) d\mu}{\sqrt{\Delta(\mu)}} \quad (33)$$

The special cases $s=1$ and 2 were also given in *Æ.* p. 445 and p. 459.

The calculation of the coefficients K is facilitated by the use of sequence equations, obtained by differentiation of (31). Thus differentiation with regard to x gives the first of

$$\begin{aligned} \nu_1 K(\nu_1, \nu_2, \nu_3; a) &= (\nu_1 + \nu_2 + \nu_3) [a K(\nu_1 - 2, \nu_2, \nu_3) + c' K(\nu_1 - 1, \nu_2 - 1, \nu_3) \\ &\quad + b' K(\nu_1 - 1, \nu_2, \nu_3 - 1)] \\ \nu_2 K(\nu_1, \nu_2, \nu_3; a) &= (\nu_1 + \nu_2 + \nu_3) [c' K(\nu_1 - 1, \nu_2 - 1, \nu_3) + b K(\nu_1, \nu_2 - 2, \nu_3) \\ &\quad + a' K(\nu_1, \nu_2 - 1, \nu_3 - 1)] \\ \nu_3 K(\nu_1, \nu_2, \nu_3; a) &= (\nu_1 + \nu_2 + \nu_3) [b' K(\nu_1 - 1, \nu_2, \nu_3 - 1) + a' K(\nu_1, \nu_2 - 1, \nu_3 - 1) \\ &\quad + c K(\nu_1, \nu_2, \nu_3 - 2)] \end{aligned}$$

If in $K(\nu_1, \nu_2, \nu_3)$ one of the indices is negative $K=0$.

There is a second group of relations of which the type is

$$\begin{aligned} A(\nu_1 + 1)K(\nu_1 + 1, \nu_2, \nu_3; a) &+ C'(\nu_2 + 1)K(\nu_1, \nu_2 + 1, \nu_3; a) \\ &+ B'(\nu_3 + 1)K(\nu_1, \nu_2, \nu_3 + 1; a) \\ &= (\nu_1 + \nu_2 + \nu_3 + 1)K(\nu_1 - 1, \nu_2, \nu_3; a), \end{aligned}$$

got either by inverting the first group, or independently by the use of

$$\left(A \frac{d}{dx} + C' \frac{d}{dy} + B' \frac{d}{dz} \right) u_a^s = 2s \Delta_a \cdot x u_a^{s-1}.$$

XLIX. The Absorption of the γ Rays of Radioactive Substances.

By A. S. EVE, M.A., Lecturer in Mathematics, McGill University, Montreal*.

THE following investigations were made in order to ascertain whether the γ rays could be taken as an accurate measure of the total amount of radioactive matter present in a given substance. It was, therefore, necessary to ascertain whether the γ rays of various substances were absorbed to an equal degree under the same conditions.

* Communicated by Prof. E. Rutherford, F.R.S.