On the Representation of Imaginary Points by Real Points on a Plane (Continued)

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ON THE REPRESENTATION OF IMAGINARY POINTS BY REAL POINTS ON A PLANE.

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I wish to express my great indebtedness to Mr. P. J. Heawood, of Durham University, for his careful and suggestive criticism of the contents of this paper, without, however, making him responsible for any statement in it with which he may disagree.—
A. LODGE.

For the sake of those readers who have not seen the first part of this paper, and to whom the whole subject is new, I may state that the whole argument is based on the fact that any imaginary point such as $(6+i, 4+3i)$ has a simple vector $x+iy=(6+i)+i(4+3i)=3+5i$, and therefore may be vectorially plotted on to, or represented by, the real point [in this case $(3, 5)$] which has the same vector. This real point is thus the *representative* of all the imaginary points which have the same vector. The paper deals with the consequent relations between the representatives of conjugate imaginaries, especially when the imaginaries satisfy the equation of some given curve: in this case the two real points representing such a pair of conjugate imaginaries are called images of each other in the curve.

Every point P in the plane has n such images in a curve of the n^{th} degree. If two images of P coincide, P is a focus—unless the two images coincide with P itself, in which case it is a double point on the curve.

THERE are a number of interesting theorems relating to the pair of representatives of a pair of conjugate imaginary points.

The first thing to do is to establish certain fundamental equations by means of which when any two of the four points are given the others can be found.

Let P, Q be the real points, with coordinates (α, β) and (γ, δ) respectively; and let I, J be the corresponding imaginary points with coordinates (x_1, y_1) and (x_2, y_2) respectively.

Then, since by supposition an imaginary point and its representative have the same *vectors*, we have the two equations

$$x_1+iy_1=\alpha+i\beta, \dots\dots\dots(1)$$

$$\text{and} \qquad x_2+iy_2=\gamma+i\delta. \dots\dots\dots(2)$$

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Two other important equations can be obtained by considering the values of what may be called the anti-vectors of the points, *i.e.* the values of $x - iy$.

It is obvious that $x_1 + iy_1$ and $x_2 - iy_2$ differ throughout in the sign of i , and in that only; for if the coordinates of I are of the form $m + ip$, $n + iq$, those of J will be of the form $m - ip$, $n - iq$.

Hence,
$$x_2 - iy_2 = \alpha - i\beta, \dots\dots\dots(3)$$

and, similarly,
$$x_1 - iy_1 = \gamma - i\delta. \dots\dots\dots(4)$$

These four equations are sufficient to find the other two points when two of them are given.

They can be illustrated graphically as shown below; for though the imaginary points themselves cannot be shown (unless they are considered as coinciding with the representative points), their *coordinates* can be so represented, since x and iy are merely vectors whose addition leads us to the point (x, y) .

Let

P be the point (α, β) ,
 P' „ „ $(\alpha, -\beta)$,
 Q „ „ (γ, δ) ,
 Q' „ „ $(\gamma, -\delta)$.

Then, if M is the mid-point of PQ' , we have, by addition of vectors,

$$OM + MP = OP,$$

$$OM - MP = OQ'.$$

These are the equations:

$$x_1 + iy_1 = \alpha + i\beta, \dots\dots\dots(1)$$

$$x_1 - iy_1 = \gamma - i\delta. \dots\dots\dots(4)$$

Hence

$$x_1 \text{ is the vector } OM,$$

and

$$iy_1 \text{ „ „ } MP.$$

Similarly, if M' is the mid-point of QP' , we see, from equations (2) and (3), that

$$x_2 \text{ is the vector } OM',$$

and

$$iy_2 \text{ „ „ } M'Q.$$

[In the diagram, P is at $(3, 5)$, and Q is at $(9, 3)$; and the coordinates of I are $(6 + i, 4 + 3i)$; while those of J are $(6 - i, 4 - 3i)$. So that the vector of

$$OI = 6 + i + i(4 + 3i) = 3 + 5i = OP,$$

and the vector of

$$OJ = 6 - i + i(4 - 3i) = 9 + 3i = OQ.]$$

It follows, from the above, that any pair of points P, Q in the plane can represent some one pair of conjugate imaginary points, and also that to any point P correspond an infinite number of imaginary points given by the equation (1), the conjugates of such points having representative points Q , which cover the whole plane.

Further, there will be one imaginary point given by equation (1) whose coordinates will be infinite, and the conjugate of that point will be situated at infinity. These points, which may be called Ω and Ω' , are the circular points at infinity. All the points given by equation (1), including Ω , appear to be piled on P , and Ω itself having infinite coordinates may be considered as being the meeting point of all the lines $x + iy = \text{constant}$.

If any pair of conjugate imaginary points I, J are points of a curve $f(x, y) = 0$, their representative points, P, Q , will, as a rule, not be situated on the curve, but they will have in many cases interesting relations with the curve. I propose to call them images of each other in the curve. In a curve of the n^{th} degree we can show that there will be n imaginary points of the curve, $I_1, I_2, \dots, I_n$, corresponding to any given representative point P which is not itself a point on the curve, and these points will have n conjugate points $J_1, \dots, J_n$, whose representative points $Q_1, \dots, Q_n$ will be scattered over the plane, i.e. every point P will have n images in a curve of the n^{th} degree.

For, if $\alpha + i\beta$ be the vector common to any given point P and to all the I points represented by P , then if we denote the vector of one of the conjugate J points by z and its anti-vector by ω , we shall have, for any of these J points

$$x + iy = z,$$

$$x - iy = \alpha - i\beta = \omega;$$

ω being given, and z being required.

From these we obtain

$$2x = z + \omega,$$

$$2iy = z - \omega;$$

$$\therefore 2y = i(\omega - z).$$

Substituting these values in $f(x, y) = 0$, we obtain the required equation in z , viz.:

$$f\left\{\frac{\omega + z}{2}, \frac{i(\omega - z)}{2}\right\} = 0,$$

the solution of which gives the vectors of the images of P , since J and Q have the same vector.

In this equation, the highest power of z involved is z^n , consequently the equation will be of the n^{th} degree unless the highest terms cancel. In such exceptional case, one or more of the values of z must be infinite, and consequently the points Ω and Ω' will be imaginary points of the curve.

Including the infinitely distant points, we may therefore say, generally, that to each point P correspond n images $Q_1, \dots, Q_n$. These points represent n imaginary points $J_1, \dots, J_n$ of the curve, the conjugates of which are $I_1, \dots, I_n$, all of which have the same vector as P .

[The above equation between z and ω may be called the z, ω equation of the curve, z and ω being respectively the vector and anti-vector of any point of the curve. It is important to notice, however, in the case of an imaginary point, that whereas z is also the vector of the real representative of the imaginary point, ω is the anti-vector of the real representative of the conjugate imaginary point.]

Various interesting special cases suggest themselves. Thus, if P is on the curve, one of the images will coincide with it; if it is a double point on the curve, two of its images will coincide with it, and so on. It is possible, however, for two of the images to coincide with each other without

coinciding with P , whether P is on the curve or not; in such cases P will be a focus of the curve, for the definition of a focus is that two of its I points are coincident,* which necessitates the corresponding Q points being also coincident. If one or more images of every point be at infinity, the point, if any, which has *another* infinitely distant image will be a focus.†

Again, considering special curves, I showed in my first paper that if $f(x, y) = 0$ is a straight line, any pair of images, P and Q , are perpendicularly equidistant from the line on opposite sides of it; i.e. they are, as it were, 'optical' images in the line, the distance of each from the line being equal to the modulus of the purely imaginary part of the coordinates of the corresponding imaginary points I and J . This line may be called the axis of the pair of conjugate points I and J . Hence every imaginary point I has a representative point P and an axis, the axis being the perpendicular line midway between P and its image Q (Q being the representative of the conjugate imaginary point J).

If $f(x, y) = 0$ is a circle, we can show, as follows, that P and Q are inverse points with regard to the circle. For, taking the equation $x^2 + y^2 = r^2$, we find that its z, ω equation is $z\omega = r^2$.

Hence, if P is at (α, β) , and ω is its anti-vector $\alpha - i\beta$, z will be the vector of Q , and is given by the equation

$$z = \frac{r^2}{\omega} = \frac{r^2}{\alpha - i\beta} = \frac{r^2}{\alpha^2 + \beta^2} (\alpha + i\beta).$$

Therefore Q is at $\left(\frac{r^2\alpha}{\alpha^2 + \beta^2}, \frac{r^2\beta}{\alpha^2 + \beta^2} \right)$, which is obviously the inverse of (α, β) .

It is important to notice that although a circle is a curve of the second degree, each point in the plane has only one finite image, the equation in z being of the first degree. The other image is at an infinite distance. The reason is that though x^2 and y^2 both contain z^2 , $x^2 + y^2$ does not, being equal to $z\omega$, the z^2 terms cancelling out. Thus Ω and Ω' are imaginary points of the circle.

In the same way, any curve for which every point in the plane has an infinitely distant image must have $x^2 + y^2$ as a factor of the terms of the n^{th} degree, where n is the degree of the curve, and every point will have two infinitely distant images if $(x^2 + y^2)^2$ is a factor of the n^{th} degree terms, and if either the terms of the $(n-1)^{\text{th}}$ degree are missing, or $x^2 + y^2$ is a factor of them. The first set of curves are called circular, since Ω and Ω' are imaginary points of them; and the second set are called bicircular, Ω and Ω' being imaginary double points of them. Similarly, we may have tricircular curves, and so on. In any of these cases, if any special point has *another* infinitely distant image, that point will be a focus, as already stated.

The conditions for foci may be illustrated by finding the vector equation of the images of any point (α, β) in the ellipse $b^2x^2 + a^2y^2 = a^2b^2$, and showing that they will be coincident if (α, β) is a focus, and in no other case.

The z, ω equation of the ellipse is

$$b^2(z + \omega)^2 - a^2(z - \omega)^2 = 4a^2b^2,$$

which reduces to

$$z^2 - 2 \frac{a^2 + b^2}{a^2 - b^2} z\omega + \omega^2 + \frac{4a^2b^2}{a^2 - b^2} = 0,$$

whence

$$z = \frac{a^2 + b^2}{a^2 - b^2} \omega \pm \frac{2ab}{a^2 - b^2} \sqrt{(\omega^2 - a^2e^2)}.$$

* In other words, (α, β) is a focus if $x + iy = \alpha + i\beta$ is a tangent, which necessitates $x - iy = \alpha - i\beta$ being also a tangent.

† In this case $x + iy = \alpha + i\beta$ is an asymptote in the direction of Ω , and $x - iy = \alpha - i\beta$ is an asymptote in the direction of Ω' .

This equation gives the required images, and shows that they will be coincident if $\omega = \pm ae$; i.e. if $a = \pm ae$, $\beta = 0$, and in no other case: which proves the proposition.

In finding the points of intersection of two curves, if any of the points are imaginary they will occur in pairs of conjugate points, whose real representatives will be images in each of the curves, as above defined. If we work with vectors, these real points will be given by the same equation as if they were points on the curves.

The method will be to eliminate x and y between the equation $z = x + iy$ and the equations of the two curves. The resulting equation in z will give the vectors of all the points of intersection, real and imaginary.

We will take as an important illustration of this method the following problem:

To find the pair of images common to an ellipse and one of its directrices, and to show that the director circle of the ellipse has the same two points as images.

Taking the three equations $z = x + iy$,

$$b^2x^2 + a^2y^2 = a^2b^2,$$

$$ex = a,$$

we easily find $z = \frac{a}{e} \mp \frac{b^2}{ae} = ae$, or $\frac{a^2 + b^2}{ae}$;

$\therefore$ the required pair of images are $(ae, 0)$ and $\left(\frac{a^2 + b^2}{ae}, 0\right)$.

It is obvious that the equation $x^2 + y^2 = a^2 + b^2$; i.e. $z\omega = a^2 + b^2$, is satisfied by the vector and anti-vector of each of the imaginary points represented by these same points.

This problem has been introduced because it connects the foci of the ellipse with its director circle.

We have seen that the foci of a curve are points which have two coincident images in the curve. They are distinguishable from double points on the curve by the fact that the double images are not coincident with the points themselves. We will now find the point or points which have two coincident images in any curve $f(x, y) = 0$.

In this case it will be more convenient to denote the vector of the point itself by z , and to use ω to denote the anti-vector of its image.

The z, ω equation of the curve is

$$f\left\{\frac{\omega + z}{2}, \frac{i(\omega - z)}{2}\right\} = 0.$$

Denoting this, for brevity, as $u = 0$, we must have, as the condition for two equal values of ω , the two simultaneous equations,

$$u = 0, \quad \frac{du}{d\omega} = 0.$$

Now $\frac{du}{d\omega} = \frac{du}{dx} \cdot \frac{dx}{d\omega} + \frac{du}{dy} \cdot \frac{dy}{d\omega}$, z being constant,

where $2x = z + \omega$

and $2iy = z - \omega$;

$$\therefore \frac{du}{d\omega} = \frac{1}{2} \left(\frac{du}{dx} + i \frac{du}{dy} \right).$$

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Hence we may write the two simultaneous equations as

$$\left. \begin{aligned} u &= 0, \\ \frac{du}{dx} + i \frac{du}{dy} &= 0. \end{aligned} \right\}$$

We may, if we please, solve these as they stand and then plot the required points by the values of the vector $z = x + iy$, or we may use their z , ω equations, and eliminate ω between them. The resulting equation in z will, in either case, give the required points, which will include the foci and any double points the curve may have.

N.B.—Since $\frac{du}{dx} + i \frac{du}{dy} = 0$ is satisfied by the coordinates of the imaginary points on the curve corresponding to the foci, it follows that $\frac{du}{dx} - i \frac{du}{dy} = 0$ is satisfied by the coordinates of the conjugate imaginary points. Hence, both the foci and their images are given by

$$\left. \begin{aligned} u &= 0, \\ \left(\frac{du}{dx}\right)^2 + \left(\frac{du}{dy}\right)^2 &= 0, \\ z &= x + iy, \end{aligned} \right\}$$

and any curve of the form

$$\left(\frac{du}{dx}\right)^2 + \left(\frac{du}{dy}\right)^2 = A_1 u + A_2 u^2 + \dots$$

will have the same points as images.

We will apply this method to find the director circle and directrices, and also the foci and focal images of the conic

$$2x^2 - 2xy + 2y^2 - 2x - 8y + 11 = 0; \dots\dots\dots(1)$$

$$\left(\frac{du}{dx}\right)^2 + \left(\frac{du}{dy}\right)^2 = 0 \text{ gives}$$

$$(2x - y - 1)^2 + (x - 2y + 4)^2 = 0. \dots\dots\dots(2)$$

If we eliminate the xy term between (1) and (2) we obtain for the director circle

$$x^2 + y^2 - 4x - 6y + 9 = 0. \dots\dots\dots(3)$$

If from (1) and (2) or from (1) and (3) we make the second degree terms a perfect square, which can be done in two ways, we obtain the directrices $(x+y-8)(x+y-2)=0$, and a pair of imaginary directrices (parallel to the major axis) $x-y+1=\pm i$.

Solving these simultaneously, we obtain four points whose vectors are

$$z = i, \quad z = 1 + 2i, \quad z = 3 + 4i, \quad z = 4 + 5i.$$

Hence the two inner points, viz. (1, 2) and (3, 4), are the foci; and their images are (0, 1) and (4, 5).

The equation of the major axis is $x-y+1=0$, and that of the minor axis is $x+y-5=0$, these being the lines midway between the two imaginary and the two real directrices respectively.

Applying the same method to the general conic

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$$

we obtain for the director circle

$$\left(\frac{du}{dx}\right)^2 + \left(\frac{du}{dy}\right)^2 = 4(a+b)u,$$

which reduces to the normal form

$$C(x^2 + y^2) - 2Gx - 2Fy + A + B = 0.$$

The real and imaginary directrices are given by

$$\left(\frac{du}{dx}\right)^2 + \left(\frac{du}{dy}\right)^2 = 4\lambda u,$$

where λ has one or other of the values given by the quadratic equation

$$(a - \lambda)(b - \lambda) = h^2.$$

If we wish to obtain the foci *without* their images, we eliminate x, y between

$$\begin{aligned} u &= 0, \\ \frac{du}{dx} + i \frac{du}{dy} &= 0, \\ x + iy &= z, \end{aligned}$$

and find for the vector equation of the foci

$$Cz^2 - 2(G + iF)z + (A + 2iH - B) = 0.$$

[NOTE.—In the above, the capital letters $A, B, \dots$ denote the first minors of $a, b, \dots$ in the discriminant of the conic.]

In the foregoing work we have tacitly assumed that the position of the point P which represents an imaginary point I of any curve is fixed relatively to the curve, *i.e.* is independent of any particular system of rectangular axes. It is perhaps desirable to prove this formally. This we shall do by showing that the position of the representative point is not affected by (1) changing the origin, (2) rotating the axes.

(1) *Move the origin to (h, k) .*

Let (α, β) be the original coordinates of P , and (x, y) the original coordinates of the corresponding imaginary point I , so that $x + iy = \alpha + i\beta$.

Then the new coordinates are respectively $(\alpha - h, \beta - k)$ and $(x - h, y - k)$. We have to show that these continue to possess equal vectors.

Now, the new vector of I is $(x - h) + i(y - k)$ which

$$= (\alpha - h) + i(\beta - k) \text{ since } x + iy = \alpha + i\beta.$$

Therefore the new vector of P is equal to the new vector of I .

Hence P is still the representative of I .

(2) *Rotate the axes through the angle ϕ .*

Let $(r \cos \theta, r \sin \theta)$ be the original coordinates of P . Then its new coordinates will be

$$\{r \cos(\theta - \phi), r \sin(\theta - \phi)\}.$$

Now if (x, y) are the original coordinates of I , and (X, Y) its new coordinates, we have

$$X = x \cos \phi + y \sin \phi,$$

$$Y = y \cos \phi - x \sin \phi;$$

also $x + iy = r \cos \theta + ir \sin \theta$.

$$\text{Now } X + iY = x \cos \phi + y \sin \phi + i(y \cos \phi - x \sin \phi)$$

$$= (x + iy)(\cos \phi - i \sin \phi)$$

$$= r(\cos \theta + i \sin \theta)(\cos \phi - i \sin \phi)$$

$$= r \cos(\theta - \phi) + ir \sin(\theta - \phi).$$

Hence, in this case also, the point P is still the representative of I .

Q.E.D.

A. LODGE.