a limited straight line whose length is equal to the distance between the centres of two equal circles, moves with an extremity on each, the locus of any point rigidly connected with the line will consist of a circle and a bicircular quartic with a third node.

This construction is mechanically more convenient than the one discussed in Mr. Roberts' paper "On the Pedals of Conics," and the two constructions are intimately connected with each other, and belong to the same theory.

Prof. Peirce, of Harvard University, laid before the Society some details of the methods made use of by him in his work on "Linear Associative Algebra."

The President conveyed to the Author the thanks of the Society for his interesting communication; and, at the request of Dr. Hirst, Prof. Peirce presented a copy of his work to the Library of the Society.

The following presents were received:—

"Linear Associative Algebra," by Benjamin Peirce, LL.D., Perkins Professor of Mathematics and Astronomy in Harvard University: from the author.

"Transactions of the Connecticut Academy of Arts and Sciences," vol. I., part 1; vol. II., part 1: from the Connecticut Academy.

"Nautical Almanac for 1874:" from the Nautical Almanac Office.

"Monatsbericht" for August, September, and October, 1870.

"Crelle," 72 Band, 4es Heft.

Feb. 9th, 1871.

W. SPOTTISWOODE, Esq., F.R.S., President, in the Chair.

Mr. C. R. Hodgson, B.A. Lond., was proposed for election, and the Rev. J. Wolstenholme, M.A., and Mr. R. B. Hayward, M.A., were elected Members.

Prof. Cayley made a communication

On an Analytical Theorem from a New Point of View.

The theorem is a well known one, derived from the equation

$$(az^2+2bz+c)\,w^2+2\,(a'z^2+2b'z+c')\,w+a''z^2+2b''z+c''=0;$$

viz., considering this equation as establishing a relation between the variables z and w, and writing it in the forms

$$2u=\mathrm{A}w^2+2\mathrm{B}w+\mathrm{C}=\mathrm{A}'z^2+2\mathrm{B}'z+\mathrm{C}'=0,$$

(where, of course, A, B, C are quadric functions of z, and A′, B′, C′ quadric functions of w,) we have

$$0=\frac{du}{dw}\,dw+\frac{du}{dz}\,dz=(\mathrm{A}w+\mathrm{B})\,dw+(\mathrm{A}'z+\mathrm{B}')\,dz;.$$

but in virtue of the equation $u=0$, we have $\mathrm{A}w+\mathrm{B} = \sqrt{\mathrm{B}^2-\mathrm{AC}}$, and $\mathrm{A}'z+\mathrm{B}' = \sqrt{\mathrm{B}'^2-\mathrm{A}'\mathrm{C}'}$, and the differential equation thus becomes

$$\frac{dw}{\sqrt{\mathrm{B}'^2-\mathrm{A}'\mathrm{C}'}}+\frac{dz}{\sqrt{\mathrm{B}^2-\mathrm{AC}}}=0,$$

where $\mathrm{B}'^2-\mathrm{A}'\mathrm{C}'$ and $\mathrm{B}^2-\mathrm{AC}$ are quartic functions of w and z respectively. This is, of course, integrable (viz., the integral is the original equation $u=0$); and it follows, from the theory of elliptic functions, that the two quartic functions must be linearly transformable into each other; viz., they must have the same absolute invariant $\mathrm{I}^3 \div \mathrm{J}^2$. It is, in fact, easy to verify, not only that this is so, but that the two functions have the same quadrinvariant I, and the same cubinvariant J.

The new point of view is, that we take the coefficients a, b, &c., to be homogeneous functions of (x, y), their degrees being such that the equation $u=0$ is a quartic equation $(\ *\ )(x, y, z, w)^4=0$; viz., this equation now represents a quartic surface having a node (conical point) at the point $(x=0,\ y=0,\ z=0)$, and also a node at the point $(x=0,\ y=0,\ w=0)$, say, these points are O, O′ respectively. The equation $\mathrm{B}'^2-\mathrm{A}'\mathrm{C}=0$ gives the circumscribed sextic cone having O for its vertex, and the equation $\mathrm{B}^2-\mathrm{AC}=0$ the circumscribed sextic cone having O′ for its vertex; each of these cones has the line OO′ $(x=0,\ y=0)$ for a nodal line, as appears geometrically, and also by the equations containing z, w respectively in the degree 4. Considering $\mathrm{B}'^2-\mathrm{A}'\mathrm{C}'$ as a quartic function of z, its quadrinvariant is a function $(x, y)^8$, and its cubinvariant a function $(x, y)^{12}$; and similarly, considering $\mathrm{B}^2-\mathrm{AC}$ as a quartic function of w, its invariants are functions $(x, y)^8$ and $(x, y)^{12}$. We have thus, between the two cones, a geometrical relation answering to the analytical one of the identity of the invariants; but the nature of this geometrical relation is not obvious; and it presents itself as an interesting subject of investigation.

Prof. Cayley made also a communication

On a Problem in the Calculus of Variations.

The problem is, $z=\frac{1}{3}(3x-y^2)\,y$, to find y a function of x such that $\int z\,dx=$ max. or min., subject to a given condition $\int y\,dx=c$ (the limits of each integral being x_1, x_0, where these quantities are each positive, and $x_1 > x_0$). The ordinary method of solution gives $y^2=x+\lambda$, where $(x_1+\lambda)^{\frac{3}{2}}-(x_0+\lambda)^{\frac{3}{2}}=\frac{3}{2}c$; so long as c is not less than $(x_1-x_0)^{\frac{3}{2}}$, there is a real value of λ, but for a smaller value of c there is no real value. The difficulty arising in this last case is somewhat illustrated by replacing the original problem by a like problem of ordinary maxima and minima; viz., $x_1, x_2 \ldots x_n$ being given positive values of x, in the order of increasing magnitude; and if, in general, $z_i=(3x_i-y_i^2)\,y_i$, then the problem is to find y_i a function of x_i, such that $\Sigma z_i =$ max. or min., subject to the condition $\Sigma y_i = c$. We have here $y_i^2 = x_i + \lambda$, where λ is to be determined by the condition $\Sigma y_i = c$; the remainder of the investigation turns on the question of the sign $y_i = +\sqrt{x_i+\lambda}$ or $y_i = -\sqrt{x_i+\lambda}$, to be taken for the several values of i respectively.

Prof. Henrici exhibited a plaster model of a Tubular Surface of the 6th order, for the generation of which the two following modes may be employed. Either a sphere of constant radius moves with its centre on a parabola, or it rolls along the same parabola, always touching both its branches. The two envelopes thus produced differ in position only. The second mode of generation shows that the surface has a nodal curve, which is a parabola congruent to that on which the centre of the sphere moves, but in a plane perpendicular to it. Through a part of it only real sheets of the surface pass. There is also a cuspidal curve of the 6th order which has two cusps. The nodal curve passes through them, and has, at the cusps, the same tangents. The equation to the surface is

$$(27py^2+9xK-x^3)^2 = (x^2+3K)^3,$$

where

$$K = (x+2p)^2+y^2+z^2-r^2,$$

and r is the radius of the sphere and $4p$ the parameter of the parabola. The equations to the parabola on which the centre of the sphere moves are

$$y^2 = 4p\,(x+2p), \quad z = 0;$$

those to the nodal curve

$$y = 0, \quad z^2 = -4px+r^2-4p^2;$$

whilst those to the cuspidal curve are

$$27py^2-4x^3 = 0, \quad x^2+3K = 0,$$

the first denoting a cylinder which cuts the plane $z = 0$ in the evolute of the parabola, and the second representing an ellipsoid of revolution.

The model was constructed to the scale $p = \frac{1}{20}$ inch, $r = 2$ inches.

At the suggestion of the Treasurer, Prof. Henrici was requested, by the members present, to order a second model to be cast for the Society's collection.

Mr. Merrifield, F.R.S., communicated the following property of Conical and Cylindrical Surfaces, which he thought to be new :—

If the equation of a surface be

$$z = F(x, y) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1),$$

it is very well known that the condition that it should be a ruled surface is that

$$\left(\lambda \frac{d}{dx} + \mu \frac{d}{dy}\right)^2 z \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2)$$

and

$$\left(\lambda \frac{d}{dx} + \mu \frac{d}{dy}\right)^3 z \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (3)$$

should have a common factor of the form $A\lambda + B\mu$, and also that the condition of its being devolopable is that (2) should have two equal factors of that form.

I have found, upon actual trial, that for a conical surface (3) will have two equal factors, and for a cylindrical surface three equal factors, that is to say, if we write $\alpha = \frac{d^3z}{dx^3}$, $\beta = \frac{d^3z}{dx^2dy}$, &c., we have for a conical surface

$$(\alpha\delta-\beta\gamma)^2 = (\alpha\gamma-\beta^2)(\beta\delta-\gamma^2),$$

and for a cylindrical surface we have separately

$$\alpha\delta-\beta\gamma = 0,\ \ \alpha\gamma-\beta^2 = 0,\ \ \beta\delta-\gamma^2 = 0.$$

In fact, if we use the equation of a cone

$$z-c = (x-a)\ \mathrm{F}\left(\frac{y-b}{x-a}\right),$$

we find, by differentiation,

$$\alpha\gamma-\beta^2 = -\frac{(y-b)^2}{(x-a)^6}\mathrm{F}''^2,$$

$$\beta\delta-\gamma^2 = -\frac{1}{(x-a)^4}\mathrm{F}''^2,$$

$$\alpha\delta-\beta\gamma = \frac{2\,(y-b)}{(x-a)^5}\mathrm{F}''^2,$$

whence

$$(\alpha\delta-\beta\gamma)^2 = 4\,(\alpha\gamma-\beta^2)(\beta\delta-\gamma^2).$$

If we use the equation of a cylinder $ny-mz = \mathrm{F}\,(mx-ly)$, we obtain

$$m\alpha = -m^3\mathrm{F}''',$$
$$m\beta = lm^2\mathrm{F}''',$$
$$m\gamma = -l^2m\mathrm{F}''',$$
$$m\delta = l^3\mathrm{F}''',$$

whence $\left(\lambda\frac{d}{dz}+\mu\frac{d}{dy}\right)^3 z$ takes the form

$$\frac{\mathrm{F}'''}{m}(l\mu-m\lambda)^3.$$

I have not yet had time to look into the questiou, whether the converse of the proposition is true, namely, whether the introduction of the condition of developability ($rt = s^2$) necessarily reduces the surface, in which two or three of the roots of (3) are equal, to a cone or cylinder.

Dr. Hirst made some remarks on the connection between the correlation of two planes, as described in his late communication to the Society, and Sturm's solution of the problem of Projectivity, as given by him in his memoir on the subject published in the "Mathematische Annalen," vol. I., p. 533.

The following presents were received :—

Nos. 1, 2, 3 of the Journal of the London Institution.

"On Hills and Dales," by J. Clerk Maxwell, F.R.S.; and an Address to the Mathematical and Physical Section of the British Association, Sept. 15th, 1870, by the same: both from the Author.

The Annual of the Royal School of Naval Architecture and Marine Engineering: from Mr. Merrifield (the Principal).